Cooperative provision of indivisible public goods

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Abstract

A community faces the obligation of providing an indivisible public good that each of its members is able to provide at a certain cost. The solution is to rely on the member who can provide the public good at the lowest cost, with a due compensation from the other members. This problem has been studied in a non-cooperative setting by Kleindorfer and Sertel (1994). They propose an auction mechanism that results in an interval of possible individual contributions whose lower bound is the equal division. Here, instead we take a cooperative stand point by modelling this problem as a cost sharing game that turns out to be a "reverse" airport game whose core is shown to have a regular structure. This enables an easy calculation of the nucleolus that happens to define the upper bound of the Kleindorfer-Sertel interval. The Shapley value instead is not an appropriate solution in this context because it may imply compensations to non-providers.

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1. Introduction

A community faces the obligation of providing a public good (or service) for the benefit of its members. It is assumed that the public good is *indivisible* and that each member is capable of providing it at a certain cost. The quality of the public good does not depend on who provides it and individual costs are publicly known. The question is *not* to know whether the public good should be provided, nor in which quantity. The question is to know who should provide the public good and how much each member should contribute. The division of labour within a household or the location of a noxious facility[1] are two illustrations of such a problem.

This problem was first studied by Kunreuther et al. (1987) in a mechanism design perspective and later by Kleindorfer and Sertel (1994) who proposed an *auction* procedure whereby each agent makes a bid specifying the compensation he or she would ask for in exchange of being the provider. The public good is then provided by the agent having made the lowest bid and the required compensation is equally divided among the non-providers. This defines a non-cooperative game whose Nash equilibria induce a solution satisfying the following properties: (i) *efficiency*: the public good is provided by the lowest cost agent; (ii) *fairness*: the non-providers contribute an identical amount; and (iii) *incentive compatibility*: the non-providers have no interest in providing the public good and the provider has no interest in non-providing the public good.[2]

These three properties define an *interval* of individual contributions, henceforth called "*KS solutions*". To be more precise, let c_i denote the cost associated to agent i ($i = 1,\ldots,n$) and assume that c_1 and c_2 are the lowest and next lowest cost i.e. $0 \le c_1 < c_2$ and $c_2 \le c_i$ for all $i \ne 1, 2$. Efficiency requires that the public good be provided by agent 1. Fairness and incentive compatibility lead to an identical contribution $t \ge 0$ from each non-provider satisfying the following inequalities: $c_1 - (n-1)t \le t$ and $c_i - (n-1)t \ge t$ for all $i \ne 1$. Hence, individual contributions must lie within the interval $T(c) = [c_1/n, c_2/n]$. The lower bound is the *equal division*: every agent supports the same share in the lowest cost. A first question concerns the nature of the upper bound. We observe that the provider's net contribution may fall below the contributions of the non-providers, depending upon the extend of the provider's competitive advantage measured by the cost difference $c_2 - c_1$. It may even be *negative*: providing the public good may be a source of revenue. A second question concerns the choice of a particular contribution within the interval $T(c)$.

We approach these questions from a *cooperative* stand point using games with transferable utility. Given a cost vector $c = (c_1,...,c_n)$, we define a cost sharing game henceforth called "*compensation game*". The *core* of a compensation game has interesting properties. It is non-

[1] A problem known as LULU: location of a Locally Undesirable Land Use.

[2] Kleindorfer and Sertel use the term "non-envy". Incentive compatibility is a more appropriate denomination.

empty and includes the allocations associated to *KS* contributions. Like the *KS* interval, it depends exclusively on the two lowest cost components. Furthermore, it has a regular structure. More precisely, the core is a *regular simplex*. As a consequence, the *nucleolus* coincides with the centre of gravity of the core that, in this case, is simply the average of its vertices. It happens to define a particular *KS* contribution, namely its upper bound c_2/n.

A compensation game is a "reverse" *airport game* and its *Shapley value* is defined by a simple formula equivalent to the solution of the airport game. While core allocations only compensate the lowest cost player, the Shapley value may compensate other players as well, rendering it inadequate as a compensation rule in this context. The Shapley value is indeed based on players' marginal costs and it therefore takes the entire cost distribution into account. We show that the Shapley value belongs to the core *if and only if* no more than one player is compensated.

The analysis is easily extended to the provision of *several* public goods. The resulting compensation game is the sum of the compensation games associated to individual public goods. We show that the core and the nucleolus are *additive* on the class of compensation games.

Dehez and Tellone (2012) study compensation mechanisms in a problem of exchange of "data" between firms, knowing the dataset owned by each firm. It that framework, the Shapley value is a compensation rule that appears more appropriate than the nucleolus. It turns out that the particular case where the datasets are *nested* gives rise to the compensation games studied here.

The paper is organized as follows. The auction procedure proposed by Kleindorfer and Sertel is described in Section 2. Cost games and airport games are introduced in Section 3. The transferable utility games associated to the provision of an indivisible public good are defined in Section 4. They are shown to be decreasing and subadditive. The regularity of the core of a compensation game is proven in Section 5. The subsequent two sections are devoted to the nucleolus and Shapley value. The extension to the simultaneous provision of several indivisible public goods is treated in Section 8. The concluding section includes a discussion on the nature of the cost components.

2. Auctioning an indivisible public good

Each player has the capacity to provide the public good at a given cost, c_i for player i. Without loss of generality, we order players in terms of their cost:

$$0 \leq c_1 \leq c_2 \leq \ldots \leq c_n \quad (1)$$

Kleindorfer and Sertel (1994) have proposed an auction procedure whereby each player submits a sealed bid stipulating the compensation he or she requires for providing the public good. For a given bid profile $b = (b_1, ..., b_n) \in \mathbb{R}^n$, the provider is identified as the *first* lowest bidder:

$$i(b) = Min\{ j \mid b_j = \ell(b)\}$$

where the lowest bid $\ell(b) = Min_{i \in N} b_i$ is equally divided among the other players.[3] The utility for player *i* associated to a bid profile $b = (b_1, ..., b_n)$ is then given by:

$$u_i(b_1, ..., b_n) = \begin{cases} \ell(b) - c_i & \text{if } i = i(b) \\ \dfrac{-1}{n-1}\ell(b) & \text{if } i \neq i(b) \end{cases}$$

This defines a game in normal form. Kleindorfer and Sertel show that any *Nash equilibrium* $\bar{b} = (\bar{b}_1, ..., \bar{b}_n)$ of this game identifies player 1 as the provider, $i(\bar{b}) = 1$, and that player receives a compensation $\ell(\bar{b}) = \bar{b}_1$ satisfying

$$\rho_1(c) \leq \ell(\bar{b}) \leq \rho_2(c) \text{ where } \rho_i(c) = \frac{n-1}{n} c_i$$

The interval $[\rho_1(c), \rho_2(c)]$ corresponds to the interval $T(c) = \left[c_1/n, c_2/n\right]$ of individual contributions defined in the introduction. Moreover, any compensation in the interval $[\rho_1(c), \rho_2(c)]$ can be associated to a Nash equilibrium i.e. compensations in $[\rho_1(c), \rho_2(c)]$ are *fully Nash implementable* through that simple auction procedure.[4] Hence $\rho_2(c)$ is the maximum equilibrium bid by player 1.

Incidentally, $\rho_i(c)$ is the *maximin* (prudent) bid by player *i*. In the case where $c_1 = c_2$ there is a *unique* Nash equilibrium since the interval then reduces to a single point: $T(c) = \{c_1/n\}$.

Incentive compatibility implies *individual rationality*. Indeed the inequalities

$$c_1 - (n-1)t \leq c_1 \text{ and } t \leq c_i \text{ for all } i \neq 1$$

are obviously satisfied by any $t \in T(c)$. Actually, no *coalition* of players can object to a *KS* solution. That problem will be formalized as a transferable utility game – a cost sharing game – to which standard allocation concepts will be applied namely the core, the nucleolus and the Shapley value.

[3] Ties are arbitrarily broken by refering to the natural order of the players.

[4] Actually Kleindorfer and Sertel actually show that compensation in $T(c)$ are Nash implementable by an auction procedure of the "k^{th} lowest bidder" type ($2 \leq k \leq n$), with *full* implementation if the number of players having a cost equal to the minimum cost does not exceed $n - k + 1$.

3. Cost games and airport games

A set $N=\{1,...,n\}$ of players, $n\geq 2,$ have a common project and face the problem of dividing its cost. The cost of realizing the project to the benefit of the members of any given coalition is also known. This defines a real-valued function *C* – a cost function – on the subsets of *N*. By convention, it satisfies $C(\varnothing)=0.$ The pair (N,C) is a *cost game.*[5] A *sharing rule* φ associates a cost allocation $y=\varphi(N,C)$ to any cost game (N,C) such that

$$\sum_{i=1}^{n}\varphi_i(N,C)=C(N).$$

Airport games form a particular class of cost games.[6] The project is to build a facility (e.g. a runway or an elevator) capable of meeting all players' needs. The cost of a facility meeting player *i*'s needs is denoted by $c_i, c_i\geq 0.$ It is assumed that the facility whose cost is c_i also covers the needs of players with a *lower* cost. The associated cost game (N,C) is then defined by:

$$C(S)=Max_{i\in S}\ c_i$$

It is an *increasing* and *concave* (and thereby subadditive) game:

$$S\subset T\ \Rightarrow\ C(S)\leq C(T)$$

$$S,T\subset N\ \Rightarrow\ C(S)+C(T)\geq C(S\cup T)+C(S\cap T)$$

Notation: The letters *n, s, t,…* denote the size of the sets *N, S, T,…* For a vector *x*, $x(S)$ denotes the sum over *S* of its coordinates, with $x(\varnothing)=0$ by convention. Coalitions are identified as *ijk…* instead of $\{i,j,k,...\}$. For any set *S*, $S\backslash i$ denotes the coalition from which player *i* has been removed.

4. Compensation games

Consider an *ordered* cost vector $c=(c_1,...,c_n)$ satisfying (1). If a coalition forms, it relies on the player with the lowest cost. The associated compensation game (N,C) is then defined by:

$$C(S)=Min_{i\in S}\ c_i\ \text{ for all } S\subset N, S\neq\varnothing \qquad (2)$$

with $C(\varnothing)=0.$ In particular $C(N)=c_1.$ The cost function *C* can alternatively be written as:

$$C(S)=c_n-C_0(S)\ \text{ where }\ C_0(S)=Max_{i\in S}(c_n-c_i) \qquad (3)$$

where the cost function C_0 defines an airport game on *N*.

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[5] For an introduction to the theory of cost allocation, see Young (1985).

[6] Airport games were introduced by Littlechild and Owen (1973). See Thomson (2007) for a complete survey.

Example 1 The 3-player game associated to the cost vector $c = (3,9,15)$ is defined by:

$C(1) = C(12) = C(13) = C(123) = 3$

$C(2) = C(23) = 9$

$C(3) = 15$

Here the cost to be divided is $c_1 = 3$.

Example 2 The 4-player game associated to the cost vector $c = (0,3,9,12)$ is defined by:

$C(S) = 0$ for all coalition S containing player 1

$C(2) = C(23) = C(24) = C(234) = 3$

$C(3) = C(34) = 9$

$C(4) = 12$

Here the cost to be divided is $c_1 = 0$.

Lemma 1 A compensation game is *decreasing* and *subadditive*. It is *essential* if $c_n > 0$.

Proof Consider a compensation game (N,C) as defined by (2). For any two coalitions S and T in N, we have successively:

$$S \subset T \;\Rightarrow\; C(T) - C(S) = Min_{i\in T} c_i - Min_{i\in S} c_i \le 0$$

$$S \bigcap T = \varnothing \;\Rightarrow\; C(S) + C(T) = Min_{i\in T} c_i + \; Min_{i\in S} c_i \ge Min_{i\in S\cup T} c_i = C(S \bigcup T)$$

Hence the game (N,C) is monotonically decreasing and subadditive.

Furthermore,

$$c_n > 0 \;\Rightarrow\; \sum_{i\in N} C(i) = \sum_{i\in N} c_i > Min_{i\in N} c_i = C(N)$$

i.e. the game (N,C) is essential if $c_n > 0$. ●

Compensation games are generally not concave. This is the case of both Example 1 and Example 2. They are however concave in the special cases where the n–1 last cost components are equal.

Lemma 2 The compensation game defined by an ordered cost vector $c \in \mathbb{R}^n_+$ such that $0 \le c_1 \le c_2 = \ldots = c_n$ is *concave*.

Proof Consider any two coalitions S and T in N and define

$$\Delta = C(S) + C(T) - C(S \cup T) - C(S \cap T).$$

To check for concavity, we have to verify that $\Delta \geq 0$. If $S \bigcap T = \varnothing$, $\Delta = c_2 > 0$. If instead $S \bigcap T \neq \varnothing$, there are four possible cases:

$1 \in S \setminus T$ or $1 \in T \setminus S$
$1 \in S \cap T$
$1 \notin S \cup T$

It is easily verified that $\Delta = 0$ in all cases. •

5. Imputations and the core

An *imputation y* is an individually rational cost allocation:

$$y(N) = C(N) \text{ and } y(i) \leq C(i) \text{ for all } i \in N$$

The set of imputations is denoted by $I(N,C)$. For essential and subadditive cost games, $I(N,C)$ is a non-empty subset of $\mathbb{R}^n$ of dimension $n-1$. The *core* (Gillies 1953) is the set of imputations against which no coalition can object:

$$\mathbb{C}(N,C) = \{y \in \mathbb{R}^n \mid y(N) = C(N), \, y(S) \leq C(S) \text{ for all } S \subset N\}$$

i.e. no coalition pays more that its stand alone cost. Equivalently, an imputation belongs to the core *if and only if* there is *no cross-subsidization* (Faulhaber 1975) in the sense that every coalition pays at least its marginal cost:

$$\mathbb{C}(N,C) = \{y \in \mathbb{R}^n \mid y(N) = C(N), \, y(S) \geq C(N) - C(N \setminus S) \text{ for all } S \subset N\} \qquad (4)$$

As intersection of a finite number of half-spaces, the core is a *convex polyhedron*, possibly empty, of maximal dimension $n-1$.

The core of a compensation game is non-empty: it always contains the no-compensation allocation $\bar{y} = (c_1, 0, ..., 0)$. Indeed, we have:

$$\bar{y}(N) = c_1 = C(N) \text{ and } \bar{y}(S) \leq c_1 \leq C(S) \text{ for all } S \subset N$$

The following proposition establishes that the core of a compensation game is a *regular simplex*, with full dimension if $c_2 > 0$, that reduces to $\{0\}$ if $c_1 = c_2 = 0$.[7]

[7] A *simplex* in $\mathbb{R}^n$ is the convex hull of *n* affinely independent vectors. A simplex is a polyhedral set. A *facet* is a maximal proper face of a polyhedral set (Grünbaum 2003). Cost games whose core are regular simplices are 1-*concave* games (Driessen 1985).

Proposition 1 The core of the compensation game defined by the ordered cost vector $c \in \mathbb{R}^n_+$ such that $c_2 > 0$ is a regular simplex of full dimension whose *n* vertices are:

$$\begin{aligned} v^1 &= (c_1, 0, ..., 0) \\ v^2 &= (c_1 - c_2, c_2, ..., 0) \\ v^3 &= (c_1 - c_2, 0, c_2, 0, ..., 0) \\ &... \\ v^n &= (c_1 - c_2, 0, ...0, c_2) \end{aligned} \qquad (5)$$

If instead $c_2 = 0$, $\mathbb{C}(N, C) = \{0\}$.

Proof Using the definition of the core given by (4), the core of a compensation game can be simply written as

$$\mathbb{C}(N,c) = \{y \in \mathbb{R}^n \mid y(N) = c_1,\ y_1 \geq c_1 - c_2 \text{ and } y_i \geq 0 \text{ for all } i \neq 1\} \qquad (6)$$

Indeed, if $y \in \mathbb{C}(N,c)$ we have successively:

$$\begin{aligned} y(N \backslash 1) &\leq C(N \backslash 1) = c_2 \\ y(N \backslash i) &\leq C(N \backslash i) = c_1 \text{ for all } i \neq 1 \end{aligned} \qquad (7)$$

(6) then follows from $y(N) = c_1$. If now *y* satisfies (6) and $S \subset N$, we have successively:

$$\text{if } 1 \in S:\ y(N \backslash S) \geq 0 \qquad \Rightarrow\ y(S) \leq c_1 = C(S)$$

$$\text{if } 1 \notin S:\ y(N \backslash S) \geq c_1 - c_2 \quad \Rightarrow\ y(S) \leq c_2 \leq C(S)$$

Hence, $y \in \mathbb{C}(N,c)$. Translating the core by adding the vector $(c_2 - c_1, 0, ..., 0)$, we get the standard simplex $\{y \in \mathbb{R}^n_+ \mid y(N) = c_2\}$.[8] It has full dimension if $c_2 > 0$ and $\mathbb{C}(N,c) = \{0\}$ if $c_1 = c_2 = 0$. •

Hence the core of a compensation game is an equilateral triangle for $n = 3$, a regular tetrahedron for $n = 4,...$ and the distance between its vertices is equal to $c_2\sqrt{2}$. This is illustrated by Figure 1 for the case of three players.

We observe that the core of the compensation game (N, c) only depends on the two lowest cost components c_1 and c_2. Furthermore it coincides with the core of the concave cost game $(N, \overline{C})$ defined by the ordered cost vector $\overline{c} \in \mathbb{R}^n_+$ where $\overline{c}_1 = c_1$ and $\overline{c}_i = c_2$ for all $i \neq 1$. The marginal cost vectors associated to the game $(N, \overline{C})$ are then precisely the core's vertices (5).

[8] If $c_2 > 0$, the unit simplex $\Delta_n = \{y \in \mathbb{R}^n_+ \mid y(N) = 1\}$ is obtained by dividing by c_2.

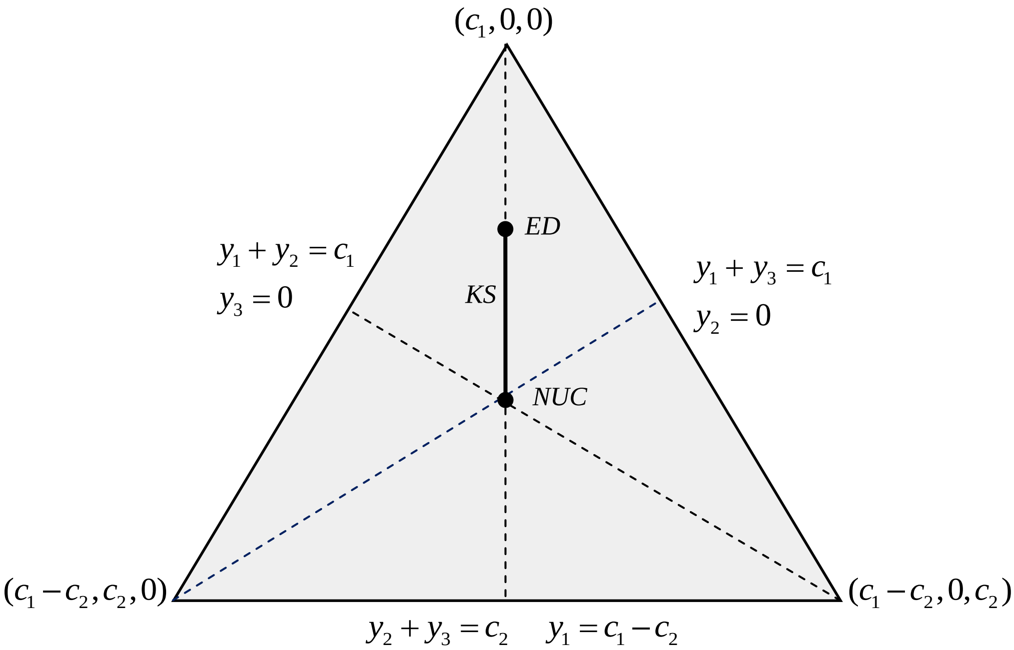

Figure 1: The core of a 3-player compensation game

Referring to (6), we observe that the *equal division* (*ED*) allocation $(c_1/n, ..., c_1/n)$ also belongs to the core. We actually have the following proposition:

Proposition 2 *KS* contributions define core allocations.

Proof We have to prove that $KS(N,c) \subset \mathbb{C}(N,c)$ where

$$KS(N,c) = \{y \in \mathbb{R}^n \mid y = (c_1 - (n-1)t, t, ..., t),\ t \in T(c)\}.$$

is the set of allocations corresponding to the *KS* contributions. For any $y \in KS(N,c)$, we have $y(N) = c_1$ and

$$y(S) = s\,t \le s\frac{c_2}{n} \le c_2 \le C(S) \quad \text{if } 1 \notin S$$

$$y(S) = c_1 - \frac{s-1}{n-1}t \le c_1 = C(S) \quad \text{if } 1 \in S$$

This completes the proof of Proposition 2. •

We notice that the allocation

$$y^0 = \left(0, \frac{c_1}{n-1}, ..., \frac{c_1}{n-1}\right)$$

that gives the provider exactly its cost belongs to the core if $c_2 > 0$. It may however fall outside the set of *KS* allocations: $y^0 \in KS(N,c)$ if $nc_1 \le (n-1)c_2$. That inequality is verified if the cost advantage of the provider exceeds c_2/n.

6. The nucleolus

For an imputation $y \in I(N,C)$ and a coalition $S \subset N$ we define the gain of coalition *S* at the imputation *y* by $e(y,S) = C(S) - y(S)$. It is the difference between the cost of coalition *S* and what it contributes under *y*. An imputation *y* belongs to the core if $e(y,S) \ge 0$ for all $S \subset N$. The *least core* (*LC*) and the *nucleolus* (*NUC*) are solution concepts that are concerned with the maximization of these gains. The least core (Mashler, Peleg and Shapley 1979) is the set of imputations that maximizes the lowest gain:

$$Max_{y \in I(N,C)}\ Min_{\substack{S \subset N \\ S \neq \varnothing, N}}\ e(y,S)$$

The least core is a nonempty set of dimension at most $n-2$ that is obviously a subset of the core if the latter is nonempty. The nucleolus (Schmeidler 1969) goes further by comparing excesses lexicographically, so as to eventually retain a *unique* allocation.

Computing the nucleolus of a compensation game is simple and only requires the identification of the two lowest cost components.

Proposition 3 The nucleolus of the compensation game (N,C) defined by the ordered cost vector $c \in \mathbb{R}^n_+$ coincides with the least core. It is given by:

$$NUC(N,c) = \left(c_1 - \frac{n-1}{n}c_2, \frac{c_2}{n}, ..., \frac{c_2}{n}\right) \quad (8)$$

Proof Using (7), the *n* facets of the core of a compensation game are given by:

$$F_i = \{y \in \mathbb{R}^n \mid y(N) = c_1,\ y(N \backslash i) = C(N \backslash i)\} \quad i = 1,...n$$

The definition of the least core then simplifies to:

$$Max_{y \in I(N,C)}\ Min_{i \in N}\ e(y, N \backslash i)$$

By regularity of the core, the least core is the set of allocations *y* such that $y(N) = c_1$ and

$$e(y, N \setminus i) = a \text{ for all } i \in N$$

for some real *a*. Using (7), we get:

$$e(y, N \backslash 1) = y(N \backslash 1) - c_2 = (c_1 - y_1) - c_2$$

$$e(y, N \backslash i) = y(N \backslash i) - c_1 = (c_1 - y_i) - c_1 = -y_i \quad (i = 2,...,n)$$

The solution is given by:

$$y_1 = c_1 - c_2 - a \qquad \text{where} \quad a = -\frac{c_2}{n}.$$
$$y_i = -a$$

The least core being uniquely defined, it coincides with the nucleolus. •

Given core's regularity, the nucleolus coincides with the centre of gravity of the core: it is located at equal distance from the facets and at equal distance from the vertices.[9] It depends only on the two lowest cost components, c_1 and c_2, and it corresponds to the upper bound of the *KS* interval. The lowest cost player supports the cost of producing the public good and receives a compensation from the other players. It is proportional to the second lowest cost and all contribute the same amount.

The nucleolus (8) can alternatively be written as:

$$NUC(N,c) = \left(\frac{c_2}{n} - (c_2 - c_1), \frac{c_2}{n}, ..., \frac{c_2}{n} \right)$$

showing that all players pay c_2/n while the provider receives its cost advantage $c_2 - c_1$. Applied successively to the examples 1 and 2, we obtain the allocations $(-3, 3, 3)$ and $(-2.25, 0.75, 0.75, 0.75)$ respectively.

Referring to Figure 1, the *equal division* allocation is located on the segment joining the nucleolus to the no-compensation allocation $v^1 = (c_1, 0, ..., 0)$, depending on the value of c_1:

$$ED(N,c) = 0 \qquad \text{if } c_1 = 0$$
$$ED(N,c) = NUC(N,c) \quad \text{if } c_1 = c_2.$$

7. The Shapley value

Consider a cost game (N,C) and the set of player's permutations Π_N. To each permutation $\pi = (i_1, ..., i_n) \in \Pi_N$ we associate a marginal cost vector that is the allocation $t(\pi)$ whose elements are given by:

$$t_{i_1}(\pi) = C(i_1)$$
$$t_{i_k}(\pi) = C(i_1, ..., i_k) - C(i_1, ..., i_{k-1}) \quad \text{for } k = 2, ..., n$$

The Shapley value (Shapley 1953) is then simply the average marginal cost vector

[9] The center of gravity of the core has been introduced as a core selection by Gonzáles-Díaz and Sánchez-Rodríguez (2007).

$$SV_i(N,C) = \frac{1}{n!}\sum_{\pi\in\Pi_N} t_i(\pi)$$

It is the unique *additive* sharing rule that satisfies *symmetry* and *dummy*.[10] Alternative axioms have been proposed.[11] The Shapley value is individually rational for subadditive cost games. It does however not necessarily belong to the core except for some classes of games like for instance concave cost games.

Proposition 4 The Shapley value $y = SV(N,c)$ of the compensation game defined by the ordered cost vector $c \in \mathbb{R}^n_+$ is given by:

$$\begin{aligned}
y_n &= \frac{c_n}{n} \\
y_{n-1} &= \frac{c_n}{n} + \frac{c_{n-1}-c_n}{n-1} \\
&\dots \\
y_1 &= \frac{c_n}{n} + \frac{c_{n-1}-c_n}{n-1} + \dots + \frac{c_2-c_3}{2} + c_1 - c_2
\end{aligned} \tag{9}$$

Proof Using the linearity of the Shapley value and the alternative definition (3) of a compensation game (N,C) we get:

$$SV_i(N,C) = \frac{c_n}{n} - SV_i(N,C_0)$$

where C_0 is the airport game defined by the cost vector $(k_1,\dots,k_n)$ with $k_i = c_n - c_i$. The cost components satisfy the inequalities $k_n \le k_{n-1} \le \dots \le k_1$ with $k_n = 0$.

Hence, the Shapley value $z = SV(N,C_0)$ has the following "triangular" form (Littlechild and Owen 1973):

$$\begin{aligned}
z_n &= \frac{k_n}{n} = 0 \\
z_{n-1} &= \frac{k_n}{n} + \frac{k_{n-1}-k_n}{n-1} = \frac{c_n - c_{n-1}}{n-1} \\
&\dots \\
z_1 &= \frac{k_n}{n} + \frac{k_{n-1}-k_n}{n-1} + \dots + \frac{k_2-k_3}{2} + k_1 - k_2 = \frac{c_n - c_{n-1}}{n-1} + \dots + \frac{c_3-c_2}{2} + c_2 - c_1
\end{aligned}$$

We then obtain exactly (9). •

[10] These are the original axioms used by Shapley (1953): players with identical marginal costs pay the same amount (symmetry) and players with zero marginal costs pay nothing (dummy). The nucleolus satisfies symmetry and dummy but not additivity.

[11] See for instance Young (1985). In the context of cost sharing it is shown that the Shapley sharing rule is the unique rule which allocates fixed costs uniformly (Dehez 2011).

We observe that the allocation defined by (9) is actually the Shapley value of the airport game defined by a cost vector $(c_1,...,c_n)$ satisfying the reversed inequalities $c_n \le c_{n-1} \le ... \le c_1$. Furthermore, the Shapley value and the nucleolus coincide in the particular case where $c_i = c_2$ for all $i \ge 3$. It is an immediate consequence of the concavity of the game (Lemma 2) and the regularity of the core (Proposition 1).

The recursive structure of the formula allows the Shapley value to be written in matrix form as *y = A.c* where *A* is a $n \times n$ *triangular* matrix whose elements are:

$$a_{ij} = \frac{-1}{j(j-1)} \text{ for all } j > i \text{ and } a_{ii} = \frac{1}{i} \text{ for all } i$$

with $a_{ij} = 0$ otherwise. Applying (9) to the examples 1 and 2, we obtain the allocations $(-4, 2, 5)$ and $(-4, -1, 2, 3)$ respectively. Only the first allocation belongs to the core because in the second, more than player is compensated. The core of the game given by Example 1 is illustrated by Figure 2.

Proposition 5 The allocation y derived from the Shapley value of a compensation game belongs to the core *if and only if* none of the last $n-1$ players are compensated: $y_i \ge 0$ for all $i \ne 1$.

Proof Let $y = SV(N,c)$ be the allocation derived from the Shapley value (9). If *y* belongs to the core, we know from (6) that $y_i \ge 0$ for all $i \ne 1$. To prove the "if" part, we observe that $y_2 \ge 0$ implies $y_1 = y_2 + (c_1 - c_2) \ge c_1 - c_2$. Hence (6) is verified if $y_i \ge 0$ for all $i \ne 1$. •

Actually, it suffices to check that $y_2 \ge 0$. Indeed, the allocation *y* derived from the Shapley value satisfies $y_n \ge y_{n-1} \ge ... y_1$. Hence $y_2 \ge 0$ implies $y_i \ge 0$ for all $i \ne 1$.

This gives a single the condition on the cost components c_i's under which the Shapley value belongs to the core:

$$\frac{c_2}{2} \ge \sum_{j=3}^{n} \frac{1}{j(j-1)} c_j$$

We observe that this inequality is independent of the lowest cost c_1. For $n = 3$, it reduces to $c_3 \le 3c_2$. For $n = 4$, the inequality becomes $c_4 + 2c_3 \le 6c_2$.

Graphically, the Shapley values are located along the line segment starting at the nucleolus for $c_3 = c_2$ and passing through the lower-right vertex $(c_1 - c_2, 0, c_2)$ for $c_3 = 3c_2$, depending on the value of c_3. This is illustrated in Figure 2.

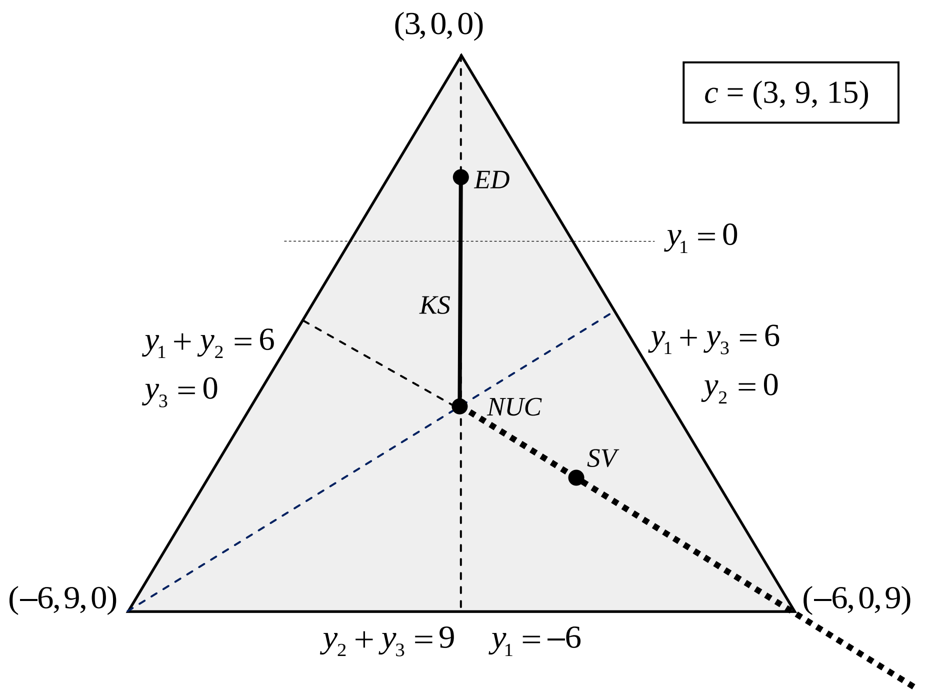

Figure 2: The core of the compensation game given by Example 1

8. Extension to several public goods

Let $M = \{1, ..., m\}$ be the set of public goods to be supplied. A cost vector is associated to each public good, $a(h) \in \mathbb{R}_+^n$ for good $h \in M$. The resulting compensation game is denoted by (N, C_h). The cost game (N, C) associated to the simultaneous provision of the m public goods is the sum of the individual compensation games:

$$C(S) = \sum_{h \in M} C_h(S) \quad \text{where } C_h(S) = \text{Min}_{i \in S}\; a_i(h)$$

Its core has a simple relationship to the core of the individual compensation games (N, C_h).

Proposition 6 The core of the aggregate compensation game is the sum of the cores of the individual compensation games:

$$\mathbb{C}(N, C) = \sum_{h \in M} \mathbb{C}(N, C_h)$$

It is a regular simplex whose vertices are the sum of the vertices of the cores of the individual games.

Proof We denote by $c_1(h)$ and $c_2(h)$ the two lowest costs of producing public good *h*, with $0 \le c_1(h) \le c_2(h)$. We shall first show that the core can be written as:

$$\mathbb{C}(N,C) = \left\{ y \in \mathbb{R}^n \,\middle|\, y(N) = \sum_{h \in M} c_1(h),\ \ y_i \ge \sum_{h \in M_i} \big(c_1(h) - c_2(h)\big) \text{ for all } i \right\} \quad (10)$$

where $M_i = \{h \in M \mid c_i(h) = c_1(h) < c_2(h)\}$ is the subset of public goods for which player *i* has the strictly lowest cost (recalling that sums over empty sets are zero.) The sets $M_0, M_1, ..., M_n$ form a partition of *M* where $M_0 = \{h \in M \mid c_1(h) = c_2(h)\}$.

(i) If $y \in \mathbb{C}(N,C)$ we have:

$$y(N \setminus j) \le \sum_{h \in M} C(N \setminus j) = \sum_{h \in M_j} c_2(h) + \sum_{h \in M \setminus M_j} c_1(h) \text{ for all } j \in N$$

Since $y(N) = \sum_{h \in M} c_1(h)$ we get:

$$y_j \ge \sum_{h \in M_j} c_1(h) - \sum_{h \in M_j} c_2(h) \text{ for all } j \in N$$

(ii) If *y* satisfies (10) and $S \subset N$, we have successively:

$$C(S) = \sum_{h \in M} C_h(S) \ge \sum_{i \in S} \sum_{h \in M_i} c_1(h) + \sum_{i \in N \setminus S} \sum_{h \in M_i} c_2(h) + \sum_{h \in M_0} c_1(h)$$

and

$$y(N \setminus S) \ge \sum_{i \in N \setminus S} \sum_{h \in M_i} (c_1(h) - c_2(h))$$

That implies

$$y(S) \le \sum_{h \in M} c_1(h) - \sum_{i \in N \setminus S} \sum_{h \in M_i} (c_1(h) - c_2(h)) = \sum_{h \in M_0} c_1(h) + \sum_{i \in S} \sum_{h \in M_i} c_1(h) + \sum_{i \in N \setminus S} \sum_{h \in M_i} c_2(h)$$

Hence $y(S) \le C(S)$ for all $S \subset N$ and $y \in \mathbb{C}(N,C)$.

(iii) Translating (10) by adding the vector $b \in \mathbb{R}^n$ defined by

$$b_i = \sum_{h \in M_i} (c_2(h) - c_1(h)) \quad i = 1, ..., n$$

we obtain the standard simplex $\{y \in \mathbb{R}^n_+ \mid y(N) = \sum_{h \in M} c_2(h)\}$. Indeed, recalling that $c_1(h) = c_2(h)$ for all $h \in M_0$ we get:

$$y \in \mathbb{C}(N,C) \text{ and } z = y + b$$

$$\Rightarrow \quad z_i \ge 0 \text{ and } z(N) = \sum_{h \in M_0} c_1(h) + \sum_{i \in N} \sum_{h \in M_i} c_2(h) = \sum_{h \in M} c_2(h)$$

This confirms that $\mathbb{C}(N,C)$ is a simplex.

(iv) The *n* vertices $v^1,...,v^n$ of $\mathbb{C}(N,C)$ are obtained by subtracting *b* from the vertices of the simplex $\{y \in \mathbb{R}^n_+ \mid y(N) = \sum_{h\in M} c_2(h)\}$. Using (5), they are defined by:

$$\begin{aligned} v_i^j &= \sum_{h\in M_i} c_1(h) + \sum_{h\in M\setminus M_i} c_2(h) = \sum_{h\in M} v_i^j(h) \quad \text{if } i = j \\ &= \sum_{h\in M_i} (c_1(h) - c_2(h)) = \sum_{h\in M} v_i^j(h) \quad \text{if } i \neq j \end{aligned} \tag{11}$$

where $v^1(h),...,v^n(h)$ are the vertices of the individual game (N, C_h) given by:

$$\begin{aligned} &\text{for all } h \in M \setminus M_0 \text{ and } i = i(h) \quad & v_i^j(h) &= c_1(h) & &\text{if } j = i \\ & & &= c_1(h) - c_2(h) & &\text{if } j \neq i \\ &\text{for all } h \in M \setminus M_0 \text{ and } i \neq i^h \quad & v_i^j(h) &= c_2(h) & &\text{if } j = i \\ & & &= 0 & &\text{if } j \neq i \\ &\text{for all } h \in M_0 \text{ and for all } i \quad & v_i^j(h) &= c_2(h) & &\text{if } j = i \\ & & &= 0 & &\text{if } j \neq i \end{aligned}$$

where for all $h \in M \setminus M_0$, $i(h)$ is the lowest cost player i.e. $a_{i(h)} = c_1(h)$. This concludes the proof of Proposition 6. •

The core of a general compensation game being a regular simplex, the nucleolus coincides with the centre of gravity that is easily characterized using (11). Furthermore, it is the sum of the nucleoli associated to the individual compensation games.

Corollary The nucleolus of the aggregate compensation game is given by:

$$NUC_i(N,C) = \frac{1}{n}\sum_{h\in M} c_2(h) - \sum_{h\in M_i} (c_2(h) - c_1(h)) = \sum_{h\in M} NUC_i(N, C_h)$$

Hence, the core and the nucleolus (like the Shapley value) are additive on the set of all general compensation games.

Example 3 Consider the combination of Example 1 with the game associated to the cost vector $a(2) = (18,12,15)$. Given the cost vector $a(1) = (3,9,15)$, the resulting compensation game is given by:

C(1) = 21, *C*(2) = 21, *C*(3) = 30

C(12) = 15, *C*(13) = 18, *C*(23) = 21

C(123) = 15

Its nucleolus is the sum of the nucleoli: $(2,5,8) = (-3,3,3) + (5,2,5)$. The public goods are provided by player 1 and player 2 respectively. Incidentally, we observe that in the second game, the provider (player 2) does not make an income.

9. Concluding remarks

Kleindorfer and Sertel provide non-cooperative foundations to the interval $T(c)$ of contributions resulting from the properties of efficiency, fairness and incentive compatibility. The present paper provides cooperative foundations. In both approaches, the question of the nature of the cost components emerges naturally. The answer depends on the context.

If the problem concerns a chore within a household, like for instance moving the garbage, some may be willing to pay more than others to avoid that unpleasant task. This is reflected in the individual cost components that measure the disutility associated to that task. The individual cost components may as well reflect the capacity – physical and/or mental – of each player to carry a particular task within a community.

If the problem concerns the location of a noxious facility, like for instance a waste incinerator, c_i is understood as the *nuisance value* for player i expressed in monetary terms, to which the cost of building the facility could be added. This is however not necessary if this cost is independent of who host the facility. This is then a fixed cost that should be divided equally among the players. Indeed, if $\tilde{c}$ is the cost vector defined by $\tilde{c}_i = c_i + F$ for some fixed cost $F > 0$,

$$t \in T(c) \quad \text{if and only if} \quad t + \frac{F}{n} \in T(\tilde{c})$$

In particular, the nucleolus is given by:

$$NUC_i(N,\tilde{c}) = NUC_i(N,c) + \frac{F}{n} \quad \text{for all } i = 1,\dots,n$$

The cost component may also include a possible benefit related to being provider. This is the case for instance if the problem concerns the location of a *desirable* facility.[12] Our analysis covers such cases if the benefit never exceeds the cost, ensuring that the c_i are all non-negative. This excludes the case where some players would be willing to pay to be provider. The auction procedure proposed by Kleindorfer and Sertel still applies in this case but the stability of *KS* contributions is not ensured: some (or all) associated cost allocations may not belong to the core. *KS* contributions may even fail to be individually rational. The problem is that the compensation game is not subadditive, resulting in an empty core. There may even be no imputations.

[12] Kleindorfer and Sertel suggest the location of European institutions or scientific meetings as illustrations.

This is for instance the case for the cost vector $c = (-6, -3, 2)$ for which we have:

$$T(c) = [-2, -1] \text{ and } KS(c) = \{ y \in \mathbb{R}^3 \mid y = (-6 - 2t, t, t), -2 \le t \le -1 \}$$

Obviously no *KS* allocation is individually rational. This indicates that the non-negativity of the cost components is an essential assumption. This also suggests that the way the problem is modelled here is not appropriate to a situation where some players are willing to pay to be provider while some others are willing to be compensated.

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