

Cutting Planes
and
Maximal Lattice-Free Convex Sets

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Mixed Integer Linear Programming

$$\begin{array}{ll} \min & cx \\ \text{s.t.} & Ax = b \\ & x_j \in \mathbb{Z} \quad \text{for } j = 1, \dots, p \\ & x_j \geq 0 \quad \text{for } j = 1, \dots, n. \end{array}$$

CUTTING PLANE approach to solving MILPs:

- First solve the LP relaxation. Basic optimal solution:

$$x_i = f_i + \sum_{j \in N} r^j x_j \quad \text{for } i \in B.$$

- If $f_i \notin \mathbb{Z}$ for some $i \in B \cap \{1, \dots, p\}$, add cuts:

Gomory's 1963 Mixed Integer Cuts, Marchand and Wolsey's 2001 MIR inequalities, Balas, Ceria and Cornuéjols' 1993 lift-and-project cuts, for instance, are used in current MILP codes.

These cuts can be generated from a single row. What about cuts that need multiple rows?

References

Related work

Corner polyhedron

Gomory LAA 1969

Gomory and Johnson 70s

Intersection cuts Balas OR 1971

Maximal lattice-free convex sets

Lovász MPS 1988

Cuts generated from two or more rows of the simplex tableau

Andersen, Louveaux, Weismantel and Wolsey IPCO 2007

Dey and Richard MOR 2008

Borozan and Cornuéjols MOR 2009

Espinoza IPCO 2008

Dey and Wolsey IPCO 2008

Cornuéjols and Margot MP 2009

Zambelli ORL 2009

More recent contributions by Andersen, Basu, Bonami, Conforti, Cornuéjols, Dey, Fukasawa, Gomory, Günlük, Louveaux, Margot, Wagner, Weismantel, Wolsey, Zambelli

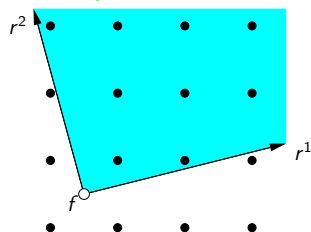
Corner Polyhedron [Gomory 1969]

Relax nonnegativity on basic variables x_j .

In addition, Andersen, Louveaux, Weismantel, Wolsey 2007 also relax integrality on nonbasic variables.

$$\begin{aligned}x &= f + \sum_{j=1}^k r^j s_j \\x &\in \mathbb{Z}^q, \quad s \geq 0\end{aligned}$$

Example



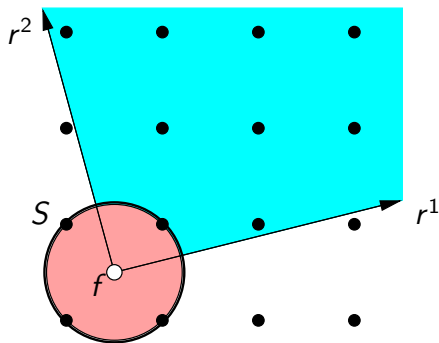
Feasible set $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{Z}^2 : \right.$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = f + r^1 s_1 + r^2 s_2$$

where $s_1 \geq 0, s_2 \geq 0$

Intersection Cuts [Balas 1971]

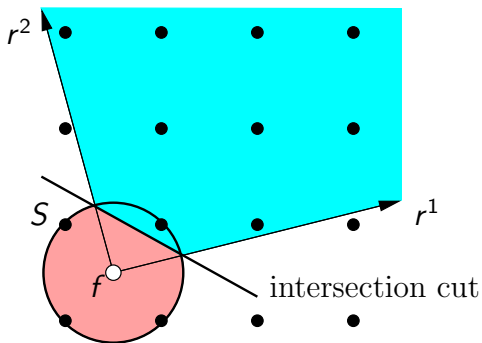
Assume $f \notin \mathbb{Z}^q$. Want to cut off the basic solution $s = 0, x = f$.



Any convex set S with $f \in \text{int}(S)$ and no integer point in $\text{int}(S)$.

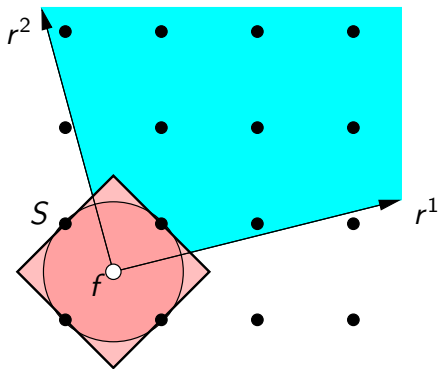
Intersection Cuts [Balas 1971]

Assume $f \notin \mathbb{Z}^q$. Want to cut off the basic solution $s = 0, x = f$.



Any convex set S with $f \in \text{int}(S)$ and no integer point in $\text{int}(S)$.
Compute intersection of the rays with the boundary of S .
Cut defined by these points is valid: $\psi(r^1)s_1 + \psi(r^2)s_2 \geq 1$.

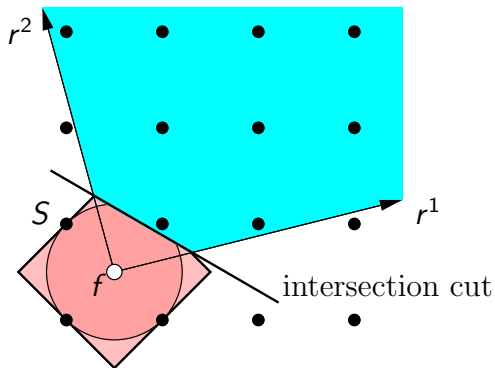
A Better Intersection Cut [Balas 1971]



Bigger convex set:

Octahedron $f \in \text{int}(S)$ with no integral point in $\text{int}(S)$.

A Better Intersection Cut [Balas 1971]



Bigger convex set:

Octahedron $f \in \text{int}(S)$ with no integral point in $\text{int}(S)$.

Better cut: $\psi(r^1)s_1 + \psi(r^2)s_2 \geq 1$.

How to compute an intersection cut

Let $S \subseteq \mathbb{R}^q$ be a convex set with $f \in \text{int}(S)$ and no integer point in $\text{int}(S)$.

Define the function $\psi_S : \mathbb{R}^q \rightarrow \mathbb{R}$ as follows.

- Set $\psi_S(r) = 0$ for any vector $r \in \mathbb{R}^q$ in the recession cone of S .
- For any $r \in \mathbb{R}^q$ that is not in the recession cone of S , set $\psi_S(r) = \frac{1}{\lambda}$ where $\lambda > 0$ is the scalar for which the point $f + \lambda r$ is on the boundary of S .

Infinite Relaxation

Gomory and Johnson 1972 suggest relaxing the k -dimensional space $s = (s_1, \dots, s_k)$ to an infinite-dimensional space, where the variables s_r are defined for any $r \in \mathbb{R}^q$.

$$R_f \quad \left\{ \begin{array}{l} x = f + \sum r s_r \\ x \in \mathbb{Z}^q \\ s \geq 0 \text{ has a finite support.} \end{array} \right.$$

Assume $f \notin \mathbb{Z}^q$. Want to cut off the basic solution $s = 0, x = f$.

Valid inequality for R_f : $\sum \psi(r) s_r \geq 1$.

Valid inequalities for R_f are valid for MILP.

Our main interest is in **minimal** valid inequalities for R_f , namely valid inequalities $\sum \psi(r) s_r \geq 1$ such that there is no valid inequality $\sum \psi'(r) s_r \geq 1$ where $\psi' \leq \psi$ and $\psi'(r) < \psi(r)$ for at least one $r \in \mathbb{R}^q$.

Main Theorem (Borožan and Cornuéjols MOR 2009)

Let $f \in \mathbb{Q}^q \setminus \mathbb{Z}^q$ and $\psi : \mathbb{Q}^q \rightarrow \mathbb{R} \cup \{+\infty\}$.

If $\sum \psi(r) s_r \geq 1$ is a minimal valid inequality for R_f , then ψ is

- nonnegative
- piecewise linear
- positively homogeneous
- and convex.

Furthermore the closure of $B_\psi := \{x \in \mathbb{Q}^q : \psi(x - f) \leq 1\}$ is a full-dimensional maximal lattice-free convex set containing f .

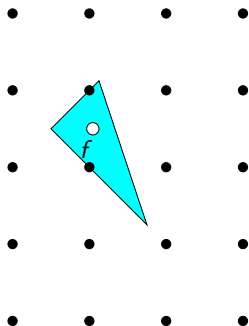
Conversely, any full-dimensional maximal lattice-free convex set B containing f gives rise to a minimal valid function ψ .

Positively homogeneous means $\psi(\lambda r) = \lambda \psi(r)$ for any scalar $\lambda \in \mathbb{Q}_+$ and $r \in \mathbb{Q}^q$.

A convex set is **lattice-free** if it does not have any integral point in its interior. However, it may have integral points on its boundary.

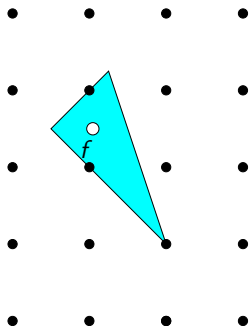
Maximal Lattice-Free Convex Set

- ▶ Lattice-free convex set contains no integral point in its interior



Maximal Lattice-Free Convex Set

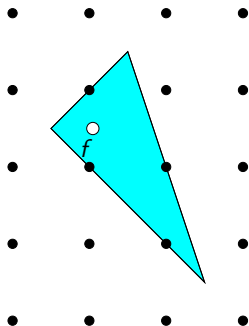
- ▶ **Lattice-free convex set** contains no integral point in its interior



Maximal: each edge contains an integral point in its relative interior.

Maximal Lattice-Free Convex Set

- ▶ **Lattice-free convex set** contains no integral point in its interior

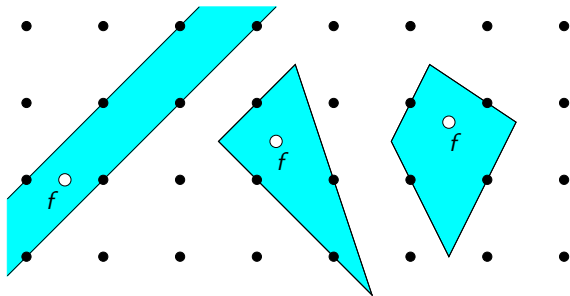


Maximal: each edge contains an integral point in its relative interior.

In the plane: it is a strip, a triangle or a quadrilateral.

Maximal Lattice-Free Sets in the Plane

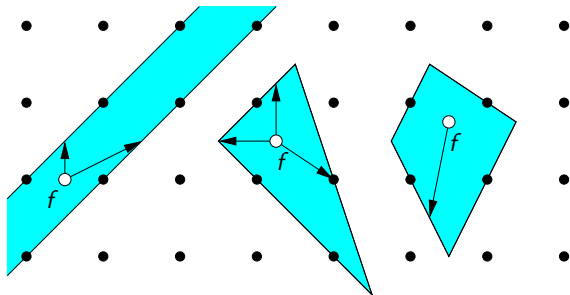
Split, triangles and quadrilaterals



generate split, triangle and quadrilateral inequalities $\sum \psi(r) s_r \geq 1$.

Maximal Lattice-Free Sets in the Plane

Split, triangles and quadrilaterals



generate split, triangle and quadrilateral inequalities $\sum \psi(r) s_r \geq 1$.

Graphical representation and homogeneity define ψ .

Maximal Lattice-Free Convex Sets and Minimal Valid Inequalities

Let B be a full-dimensional maximal lattice-free convex set in \mathbb{R}^q and let $f \in \text{int}(B)$.

Define the function $\psi_B : \mathbb{R}^q \rightarrow \mathbb{R}$ as follows.

- Set $\psi_B(r) = 0$ for any vector r in the recession cone of B .
- For any r that is not in the recession cone of B , set $\psi_B(r) = \frac{1}{\lambda}$ where $\lambda > 0$ is the scalar for which the point $f + \lambda r$ is on the boundary of B .

COROLLARY The function ψ_B is a minimal valid function for R_f , and satisfies $B_{\psi_B} = B \cap \mathbb{Q}^q$.

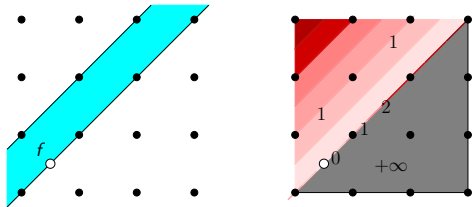
Conversely, all **finite** minimal valid functions for R_f are of this form.

Maximal Lattice-Free Convex Sets and Minimal Valid Inequalities

Zambelli 2009 showed that every minimal valid inequality for the finite model (k nonbasic variables) can be derived from a minimal valid function ψ for the infinite model R_f where f is in the interior of $\text{cl}(B_\psi)$.

Therefore, in practice, the above corollary suffices.

A maximal lattice-free convex set B where f is on the boundary can give rise to a minimal valid inequality for R_f . The corresponding function ψ is not finite everywhere!



How do we prove
the main theorem?

Theorem [Lovász 1989]

A maximal lattice-free convex set in the plane (x_1, x_2) is one of the following:

- Irrational line $ax_1 + bx_2 = c$ with a/b irrational, $c \notin a\mathbb{Z} + b\mathbb{Z}$;
- A strip $c \leq ax_1 + bx_2 \leq c + 1$
with a, b coprime integers, c integer;
- A triangle with an integral point in the relative interior of each edge;
- A quadrilateral containing exactly four integral points, one in the relative interior of each edge; The four integral points are vertices of a parallelogram of area 1.

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Definitions

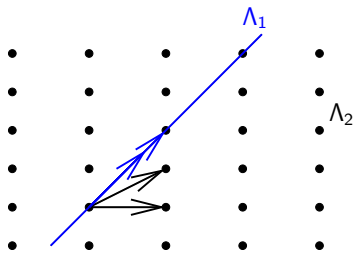
Let $a_1, \dots, a_m \in \mathbb{R}^n$. The set $\Lambda = \{\lambda_1 a_1 + \dots + \lambda_m a_m \mid \lambda_1, \dots, \lambda_m \in \mathbb{Z}\}$ is called a **finitely generated additive group** of \mathbb{R}^n .

If a_1, \dots, a_m are linearly independent, Λ is called a **lattice** of the linear space $\langle a_1, \dots, a_m \rangle$.

EXAMPLES

$\Lambda_1 = \{\lambda_1 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} \mid \lambda_1, \lambda_2 \in \mathbb{Z}\}$ is a finitely generated additive group but **is not a lattice**.

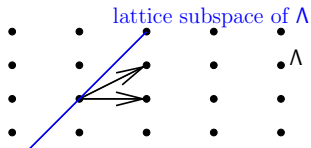
$\Lambda_2 = \{\lambda_1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \mid \lambda_1, \lambda_2 \in \mathbb{Z}\}$ **is a lattice**.



Lovász' theorem in \mathbb{R}^n

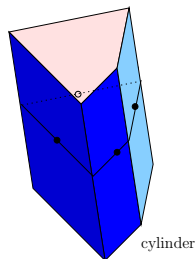
Let Λ be a lattice of a linear space V .

DEFINITION A linear subspace $L \subseteq V$ is a **lattice subspace** if there exists a basis of L contained in Λ .



THEOREM Lovász 1989

Every unbounded maximal lattice-free convex set is a cylinder above a polytope $P + L$ where L is a lattice subspace.



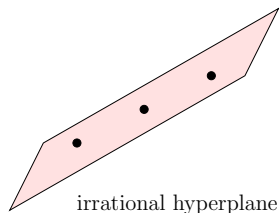
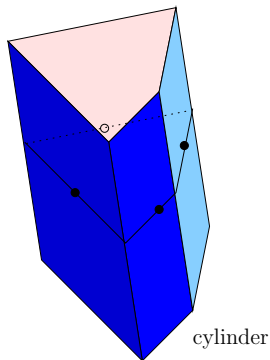
Lovász' Theorem

Actually, the correct statement is the following.

THEOREM

Every unbounded maximal lattice-free convex set S is either

- a full-dimensional cylinder above a polytope $P + L$ where L is a lattice subspace, or
- an affine hyperplane $v + L$ where L is not a lattice subspace of V .



Ingredients of the proof of Lovász' theorem

Basu, Conforti, Cornuéjols, Zambelli 2009

THEOREM

Let Λ be an additive group generated by $a_1, \dots, a_m \in \mathbb{R}^n$.
Then Λ is a lattice of the linear space $\langle a_1, \dots, a_m \rangle$ if and only if
there exists $\epsilon > 0$ such that $\|y\| \geq \epsilon$ for every $y \in \Lambda \setminus \{0\}$.

(See, for example, [Barvinok 2002](#): A Course in Convexity.)

LEMMA Let Λ be a lattice of linear space V . If $S \subset V$ is a
full-dimensional **bounded** maximal Λ -free convex set, then S is a
polytope with a point of Λ in the relative interior of each facet.

Ingredients of the proof of Lovász' theorem

THEOREM Dirichlet

Given any real numbers $\alpha_1, \dots, \alpha_n, \epsilon$ with $0 < \epsilon < 1$, there exist integer p_1, \dots, p_n, q such that

$$|\alpha_i - \frac{p_i}{q}| < \frac{\epsilon}{q}, \text{ for } i = 1, \dots, n, \text{ and } 1 \leq q \leq \frac{1}{\epsilon}.$$

LEMMA Let L be a linear subspace of V .

If L is not a lattice-subspace of V then for every $\epsilon > 0$, there exists $y \in \Lambda \setminus L$ at distance less than ϵ from L .

Ingredients of the proof of Lovász' theorem

LEMMA Let L be a linear subspace of V with $\dim L = \dim V - 1$, and let $v \in V$. Then $H = v + L$ is a maximal Λ -free convex set if and only if L is not a lattice subspace of V .

Let S be a maximal Λ -free convex set. If $\dim S < \dim V$ then S is contained in some affine hyperplane $H = v + L$ and by maximality of S we have $S = H$. The above lemma shows that L is not a lattice subspace of V .

Therefore we assume $\dim S = \dim V$.

Let C be the recession cone of S and L its lineality space. We first show that $C = L$ and then that L is a lattice subspace of V ...

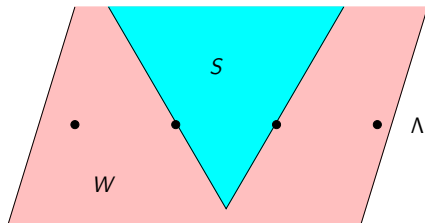
A Generalization to Linear Subspaces

THEOREM Basu, Conforti, Cornuéjols, Zambelli 2009

Let Λ be a lattice of a linear space $V \subseteq \mathbb{R}^n$,
and let $W \subseteq \mathbb{R}^n$ be a linear space containing V .

A maximal Λ -free convex set S of W is either

- a polyhedron in W such that $S \cap V$ is a maximal Λ -free convex set P of V and, for every facet F of S , $F \cap V$ is a facet of P .
- an affine hyperplane $v + L$ where $v \in S$ and L is not a lattice-subspace of V .
- a halfspace of W that contains V on its boundary.



Proof sketch of the Borozan-Cornuéjols theorem

THEOREM A minimal valid function ψ for R_f is nonnegative, piecewise linear, positively homogeneous and convex.

IDEA OF PROOF Nonnegativity, positive homogeneity and convexity are proved directly.

To prove that ψ is piecewise linear, consider B_ψ .

First we prove that $\text{cl}B_\psi$ is a maximal lattice-free convex set. Thus by Lovász' theorem, $\text{cl}B_\psi$ is a polyhedron.

Suppose $\text{cl}B_\psi$ is a hyperplane. Since $B_\psi \subset \mathbb{Q}^q$ this implies that $\text{cl}B_\psi$ is a rational hyperplane, contradicting Lovász' theorem.

So $\text{cl}B_\psi$ is a full-dimensional polyhedron. If f is in its interior, we can use Lovász' theorem directly to argue that ψ is piecewise linear. When f is on the boundary of $\text{cl}B_\psi$, we need the generalization of Lovász' theorem.

The Dey-Wolsey Model

Let $S \subset \mathbb{R}^q$ be the set of integral points in a rational polyhedron.
Assume $\dim(S) = q$.

$$R_f(S) \quad \left\{ \begin{array}{l} x = f + \sum rs_r \\ x \in S \\ s \geq 0 \text{ has a finite support.} \end{array} \right.$$

THEOREM

Dey, Wolsey 2009 and Basu, Conforti, Cornuéjols, Zambelli 2009

Every valid inequality for $R_f(S)$ of the form $\sum_{r \in \mathbb{R}^q} \psi(r)s_r \geq 1$ is dominated by a minimal valid inequality for $R_f(S)$ of the form

$$\sum_{r \in \mathbb{R}^q} \psi_B(r)s_r \geq 1,$$

where B is a maximal S -free convex set in \mathbb{R}^q with f in its interior such that $B \cap \text{conv}(S)$ has nonempty interior.

Theorem Basu, Conforti, Cornuéjols, Zambelli 2009

A set $B \subset \mathbb{R}^n$ is a maximal S -free convex set if and only if one of the following holds:

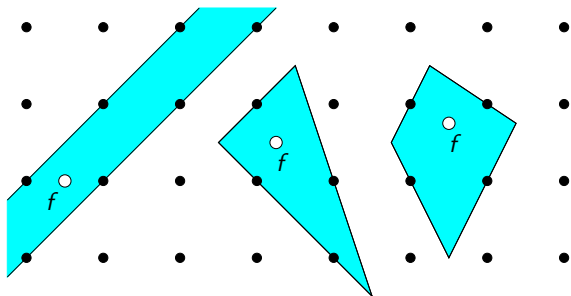
- B is a polyhedron such that $B \cap \text{conv}(S)$ has nonempty interior, B does not contain any point of S in its interior and there is a point of S in the relative interior of each of its facets. The recession cone of $B \cap \text{conv}(S)$ is rational and it is contained in the lineality space of B .
- B is a half-space of \mathbb{R}^n such that $B \cap \text{conv}(S)$ has empty interior and the boundary of B is a supporting hyperplane of $\text{conv}(S)$.
- B is a hyperplane of \mathbb{R}^n such that $\text{lin}(B) \cap \text{rec}(\text{conv}(S))$ is not rational.

Comparing the strength of inequalities

Basu, Bonami, Cornuéjols, Margot 2009

$$\begin{aligned}x &= f + \sum_{j=1}^k r^j s_j \\x &\in \mathbb{Z}^q \\s &\geq 0\end{aligned}$$

Maximal lattice-free sets **in the plane**:



Split, triangles and quadrilaterals.

Split, triangle and quad closures

Split closure := the intersection of all split inequalities.

Triangle closure := all ineq. arising from maximal lattice-free triangles.

Quad closure := ineq. arising from maximal lattice-free quadrilaterals.

Andersen, Louveaux, Weismantel, Wolsey 2007 show that

$$\text{Conv}(S) = \text{Split closure} \cap \text{Triangle closure} \cap \text{Quad closure}$$

THEOREM Basu, Bonami, Cornuéjols, Margot 2009

Triangle closure \subseteq Split closure and

Quad closure \subseteq Split closure.

A theorem of Goemans 1995

Let $Q := \{x : a^i x \geq b_i \text{ for } i = 1, \dots, m\} \subseteq \mathbb{R}_+^n \setminus \{0\}$

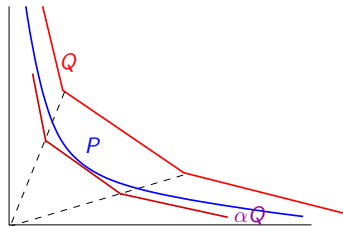
where $a^i \geq 0$ and $b_i \geq 0$ for all i .

For $\alpha > 0$ let $\alpha Q := \{x : \alpha a^i x \geq b_i \text{ for } i = 1, \dots, m\}$.

THEOREM If convex set $P \subseteq \mathbb{R}_+^n$ contains Q , then the smallest value of $\alpha \geq 1$ such that $P \subseteq \alpha Q$ is

$$\max_{i=1, \dots, m} \left\{ \frac{b_i}{\inf\{a^i x : x \in P\}} : b_i > 0 \right\}.$$

In other words, the only directions that need to be considered to compute α are those defined by the nontrivial facets of Q .



Relative strength of closures Basu, Bonami, Cornuéjols, Margot 2009

Both the triangle closure and the quad closure are good approximations of the integer hull:

THEOREM

$\text{Conv}(S) \subseteq \text{Triangle closure} \subseteq 2 \text{Conv}(S)$ and
 $\text{Conv}(S) \subseteq \text{Quad closure} \subseteq 2 \text{Conv}(S)$

But the split closure may not be a good approximation of the integer hull:

THEOREM

For any $\alpha > 1$, there is a choice of f, r^1, \dots, r^k such that
 $\text{Split closure} \not\subseteq \alpha \text{Conv}(S)$.

REMARK Related results of Andersen, Wagner, Weismantel.

Papers available on <http://integer.tepper.cmu.edu/>