Fair and Efficient Rules in School Choice Problem: Maximal Domain of Preference Profiles

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First draft: March 15, 2011
This version: October 22, 2011

Abstract

We study the school choice problem. There is a set of (public) schools, whose seats are to be distributed to students. Each school has a "priority," a strict ordering over students. Each student has a strict preference over schools and should be assigned to one and only one school. For each priority profile of schools and each preference profile reported by students, a rule assigns school seats to students. "Efficiency" and "stability" are two central requirements in this problem. Two competing rules have been studied and used in practice: the student-proposing deferred acceptance (DA) rule and the top-trading cycles (TTC) rule. The former satisfies *stability* but not *efficiency*, and the converse holds for the latter. Unfortunately, no rule satisfies these two requirements.

Given this baseline impossibility, we look for restrictions on preference profiles that guarantee the compatibility of *efficiency* and *stability*. In particular, we focus on restoring *stability* of the TTC rule (equivalently, it turns out to be the coincidence of the DA and TTC rules). Our main result is the identification of a "maximal" domain of preference profiles on which, for *each* priority profile, the resulting TTC assignment is *fair*. This maximal domain is unique. We also present several subdomains of preference profiles on which the DA rule is *efficient*.

JEL classification Numbers: C78; C79; D61; D78; I20

Keywords: the deferred acceptance rule; the top-trading cycles rule; the immediate acceptance rule; maximal preference profile; fairness; efficiency.

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[†]I appreciate William Thomson for his support and guidance. I benefited from discussions with Wonki Cho, Bettina Klaus, Youngwoo Koh, Vikram Manjunath, and Kyoungwon Seo. I am also grateful to seminar audiences at University of Rochester and Rice University for useful comments and suggestions. All remaining errors are my own responsibility.

1. Introduction

We study the problem of assigning schools to students when each school has a strict priority order over students by local or state law, depending on residence proximity, siblings, a tie-breaking rule,[1] etc. Each student has a strict preference over schools and should be assigned to one and only one school. For each priority profile and each preference profile reported by students,[2] a rule assigns schools to students.

There have been three central properties that a rule should satisfy in this context. Among those is an incentive property: a rule should provide each student an incentive to report his true preferences.[3] Examples of rules satisfying this property are the "sequential priority rules".[4] Another property is "efficiency": a rule should choose an assignment at which no student can be made better off without some other student being made worse off. The last property is that a rule should "respect" the priority profile: consider a priority profile, a preference profile, and an assignment. If a student, say i, is assigned to a school, say a, but he prefers some other school, say school b, then the students who are assigned to b should have higher priorities than student i at b. If this holds for each student, we say that the assignment is fair at this priority profile and preference profile. A rule is fair if at each priority profile and preference profile, the assignment chosen by the rule is fair.

Unfortunately, no rule is efficient and fair (Balinski and Sönmez, 1999). Among those satisfying the incentive property, two rules have been proposed and studied extensively: the student-proposing deferred acceptance (DA) rule (Gale and Shapley, 1962) and the top-trading cycles (TTC) rule (Abdulkadiroğlu and Sönmez, 2003). The DA rule is fair but not efficient. By contrast, the TTC rule is efficient but not fair.

Positive results can be obtained by restricting the domain of priority profiles. On the domain of "weakly acyclic" priority profiles, the DA rule is efficient. Conversely, if a priority profile is not weakly acyclic, there is a preference profile at which the DA assignment is not efficient (Ergin, 2002). On the domain of "acyclic" priority profiles, the

[1] In Boston public school assignment procedure, there are only five categories of students, depending on "guaranteed priority", walk-zone, and sibling (Abdulkadiroğlu et al., 2006). Ties in each category are broken by a tie-breaking rule, for example, by a random tie-breaking rule. See Erdil and Ergin (2008) and Abdulkadiroğlu et al (2009) for detailed discussion of tie-breaking.

[2] The college admission problem is closely related to our problem, but the difference is that colleges have *preferences* that they have to report.

[3] This property is referred to as "strategy-proofness".

[4] To each strict order over students is associated a sequential priority rule. The rule assigns to the first student in the order his most preferred school. Next, it assigns to the second student in the order his most preferred school among the available schools. Then, it assigns the third student in the order his most preferred school among the available schools, and so on.

TTC rule is fair, equivalently, the TTC rule and the DA rule coincide. Conversely, if a priority profile is not acyclic, there is a preference profile at which the TTC assignment is not fair (Kesten, 2006). When schools' priorities are weak, the domain of priority profiles has been identified on which efficiency and fairness are compatible (Ehlers and Erdil, 2010).

In sum, by restricting priority profiles, we obtain the compatibility of the two properties as well as the coincidence of the DA rule and the TTC rule. However, recalling that a rule has two inputs, namely, priority profiles and preference profiles, one natural question follows from Ergin (2002) and Kesten (2006).

• If we restrict the domain of preference profiles, do we also obtain the compatibility of the two properties and the coincidence of the DA and TTC rules?

Restricting the preference profiles is not the same thing as restricting priority profiles in two respects. First, schools have "capacities" (namely, the number of available seats), but each student is assigned to one and only one school. Second, the DA and TTC rules are "student-proposing" rules. Thus, students and schools are in asymmetric positions. Thus, we cannot follow the approaches of Ergin (2002) and Kesten (2006) in answering our question.

Given a rule, we say that a domain of preference profiles is "maximal" for a certain requirement (or a list of requirements),[5] if the rule satisfies the requirement (or a list of requirements) on the domain, but any enlargement of the domain results in a violation of the requirement. Next, another natural question follows.

• If we obtain the compatibility of the two properties on some restricted domain, is there a maximal domain of preference profiles, containing this restricted domain, on which this compatibility is preserved?

The issue of maximality of domains of preferences for compatibility of certain properties is first studied by Barberà et al. (1991) in a public good problem. A number of papers address such issues for different models (Alcalde and Barberà (1994), Ching and Serizawa (1998), Berga and Serizawa (2000), Massó and Neme (2001), Kojima (2007) and so on). These papers, however, differ from ours: they look for maximal domains of *preferences*, whereas we look for maximal domains of *preference profiles*. In this respect, our approach is similar to Ergin (2002) and Kesten (2006): they too restrict the domain of *priority profiles*, not the domain of *priorities*.

By identifying a maximal domain of preferences, we do not have to be concerned about the requirement that we impose at least within the domain, even if the requirement is not

[5]Our requirement is that the TTC rule be fair, equivalently, the DA and TTC rules coincide.

satisfied on the entire domain. As we require that the TTC rule be fair, in particular, we obtain both efficiency and fairness on our maximal domain. Moreover, an authority that has to adopt a rule needs not have trouble choosing between the DA and TTC rules, as they indeed coincide.

There are two extreme examples of preference profiles on which coincidence is obtained. The first is the profile at which preferences over schools are the same across students. Then, the assignment will be determined in order of each school's priority by both rules.[6] The second is the profile at which students have "no conflict" on their most preferred schools: for each school, say a, the number of students whose most preferred school is a does not exceed the capacity of a. Then, each student will be assigned his most preferred school by the two rules. There are more preference profiles, other than these two extremes. We identify a maximal domain of preference profiles for our requirement and it is in fact unique (Theorem 1). On this domain, we obtain not only the coincidence of the DA and TTC rules, but also the coincidence of these rules with the immediate acceptance (IA) rule (usually called the Boston rule) (Corollary 1).

As noted above, our result holds for any priority profile. There are cases in which we have to consider all possible priority profiles. For example, it is often the case that the prioritizing criteria, such as walk zone and siblings, are not fine enough to order students strictly. A common practice is to break ties randomly to obtain a strict priority. In such cases, the priority profiles are not perfectly under the control of the authority as many priority profiles can possibly come out.

As in Ergin(2002), we could also seek to identify a larger domain of preference profiles to restore efficiency of the DA rule, instead of coincidence of the two rules. Such a domain would certainly include the profiles of Theorem 1. We introduce several structures of profiles and show that the DA rule is efficient on such profiles (Theorem 3).

On the other hand, we could restrict ourselves to weakly acyclic priority profiles (Ergin, 2002) and look for a domain of preference profiles on which the TTC rule is fair and, thereby, on which the two rules coincide (Theorem 4). Again, the domain certainly includes the profiles of Theorem 1. We note that there is no inclusion relation between the set of preference profiles on which the DA rule is efficient and the set of preference profiles on which the TTC is fair for any weakly acyclic priority profile.

This paper is organized as follows: Section 2 introduces the model, axioms, and the

[6] That is, each school first distributes a seat to the student with the highest priority at the school. If a student is given more than one school seat, he chooses most preferred school among them and declines the remaining. The schools declined by the student take the seat back. Next, each school distributes a seat to the student with the second highest priority at the school, if the school has a remaining seat. The same process repeats until each student is assigned to a school.

two rules. Section 3 contains our main result on maximality of preference profiles. Lastly, Section 4 deals with a larger domain than that in Section 3 on which fairness and efficiency are compatible. In the Appendix, we also address an impossibility result on maximal preference profiles in large economy.

2. Model

Let A be the set of schools. Each $a \in A$ has a capacity, that is, a number of available seats, $q_a \in \mathbb{N}_+$. Let $q \equiv (q_a)_{a \in A}$ be the capacity profile. Let $N \equiv \{1, 2, \cdots, n\}$ be the set of students. Each $a \in A$ has a strict order f_a over N by local or state law.[7] We call f_a the **priority at a**. It is represented by a function, $f_a : N \to \{1, 2, \cdots, n\}$ such that $f_a(i) < f_a(j)$ if and only if i has higher priority than j at a. Let $\mathcal{F}$ be the set of all priorities. Let $f \equiv (f_a)_{a \in A}$ be the priority profile. Let $\mathcal{F}^A$ be the set of all priority profiles.

Each $i \in N$ should be assigned to one and only one school. We assume that $n \leq \sum_{a \in A} q_a$.[8] Each $i \in N$ has a complete, transitive, and strict preference P_i over A. Let Π be the set of all such preferences. Let R_0 be the weak relation associated with $P_0 \in \Pi$.[9] Let $P \equiv (P_i)_{i \in N}$ be the preference profile and Π^N be the set of all preference profiles. For each $P \in \Pi^N$, each $B \subseteq A$, and each $a \in B$, let $\boldsymbol{N(a, P : B)} \equiv \{i \in N : \text{for each } b \in B,\ a\ R_i\ b\}$ be the set of students whose most preferred school in B is a. For each $B \subseteq A$ and each $P_0 \in \Pi$, let $\boldsymbol{P_0|_B}$ be the restriction of P_0 to B. For each $P \in \Pi^N$, let $P|_B \equiv (P_i|_B)_{i \in N}$. For each $k \in \{1, \cdots, |A|\}$, let $\boldsymbol{P_i|_1^k}$ be the preference P_i restricted from student i's most preferred school to his k-th most preferred school. For each $k \in \{1, \cdots, |A|\}$ and each $P_0 \in \Pi$, let $\boldsymbol{A^k(P_0)}$ be the k-th most preferred school at P_0. For each $P \in \Pi^N$, let $\boldsymbol{A^k(P)} \equiv \bigcup_{i \in N} A^k(P_i)$. Let $\boldsymbol{A_1^k(P)} \equiv \bigcup_{t=1}^k A^k(P)$. For simplicity, let $P|_1^0 \equiv P|_\emptyset$ and $A_1^0(P) \equiv \emptyset$.

An **economy** is a list (A, N, f, P, q). Unless otherwise specified, we fix A, N, and q. Then, an economy is a pair (f, P) and the set of economies is the Cartesian product $\mathcal{F}^A \times \Pi^N$. An **assignment** is a list $x \equiv (x_i)_{i \in N}$ such that (i) for each $i \in N$, $x_i \in A$ and (ii) for each $a \in A$, $|\{i \in N : x_i = a\}| \leq q_a$. Let $\boldsymbol{X}$ be the set of all assignments. A **rule** is a function $\varphi : \mathcal{F}^A \times \Pi^N \to X$.

[7]The criteria that determines the priority are walk-zone, siblings, a tie-breaking rule, etc.

[8]It is usual that every student is required to be enrolled in a school by law.

[9]That is, for each pair $a, b \in A$, if $a\ R_0\ b$, then $a = b$ or $a\ P_i\ b$.

2.1. Axioms

We focus on two properties that have been studied extensively in the context of matching. Let $P \in \Pi^N$. We say that an assignment $x \in X$ is **efficient at $\boldsymbol{P}$** if no student can be made better off without some other student being made worse off. Formally, there is no $x' \in X \setminus x$ such that for each $i \in N$, either $x'_i \; P_i \; x_i$ or $x'_i = x_i$. Let $\mathbf{Eff}(\boldsymbol{P}) \equiv \{x \in X : x \text{ is efficient at } P\}$. The corresponding property of a rule is that it selects for each economy, an efficient assignment.

Efficiency: for each $f \in \mathcal{F}^A$ and each $P \in \Pi^N$, $\varphi(f, P) \in \text{Eff}(P)$.

Consider $f \in \mathcal{F}^A$, $P \in \Pi^N$, and $x \in X$. Suppose that a student, say i, prefers some other school, say school a, to x_i. Then, it is natural to require that each of the students who are assigned to a should have higher priority than him at a. If this holds for each student, we say that x is **fair at $(\boldsymbol{f}, \boldsymbol{P})$**. Formally, for each pair $i, j \in N$, $x_j \; P_i \; x_i$ implies $f_{x_j}(j) < f_{x_j}(i)$. Let $\mathbf{Fair}(\boldsymbol{f}, \boldsymbol{P}) \equiv \{x \in X : x \text{ is fair at } (f, P)\}$. The corresponding property of a rule is that it selects for each economy, a fair assignment at (f, P).

Fairness: For each $f \in \mathcal{F}^A$ and each $P \in \Pi^N$, $\varphi(f, P) \in \text{Fair}(f, P)$.

Unfortunately, no rule is *efficient* and *fair* (Balinski and Sönmez, 1999). In the light of this negative result, we restrict the domain of preference profiles.

2.2. The (student-proposing) deferred-acceptance rule

For each priority profile and each preference profile, the student-proposing deferred-acceptance rule assigns schools as follows:

Step 1 Each student applies to his most preferred school. If the number of students applying to a school, say a, does not exceed q_a, then all of these students are temporarily accepted. Otherwise, among those, the students with the highest priority down to the q_a-th priority at a are temporarily accepted: the others are rejected.

Step $\boldsymbol{t}$ (≥ 2) Each student who is rejected at Step $(t-1)$ applies to his next most preferred school. For each school a, if the number of students who had been temporarily accepted by a at Step $(t-1)$, together with the students who are newly applying to a, does not exceed q_a, then all of these students are temporarily accepted. Otherwise, among those, the students with the highest priority down to the q_a-th priority at a are temporarily accepted: the others are rejected.

Last Step We stop when each student is accepted by a school.

We call this rule the **deferred acceptance rule**, or the DA rule for short. The DA rule is *fair*. Moreover, for each $f \in \mathcal{F}^A$ and each $P \in \Pi^N$, DA(f, P) *Pareto dominates* each other assignment that is fair at (f, P) (Balinski and Sönmez, 1999). However, the rule is not *efficient*.

To restore *efficiency* of the DA rule, Ergin (2002) restricts the domain of priority profiles. Let us describe these profiles. For each $a \in N$, each $f_a \in \mathcal{F}^A$, and each $i \in N$, let $U_a(i) \equiv \{j \in N : f_j(a) < f_i(a)\}$ and $L_a(i) \equiv \{j \in N : f_j(a) > f_i(a)\}$.

We say that $\boldsymbol{f}$ **contains a strong $\boldsymbol{q}$-cycle** if

(1) there are $i, j, k \in N$ and $a, b \in A$ such that $\begin{cases} f_a(i) < f_a(j) < f_a(k) \\ f_b(k) < f_b(i), \text{ and} \end{cases}$

(2) there are $N_a, N_b \subseteq N \backslash \{i, j, k\}$ such that $\begin{cases} N_a \subseteq U_a^f(j),\ N_b \subseteq U_b^f(i),\ N_a \cap N_b = \emptyset, \text{ and} \\ |N_a| = q_a - 1 \text{ and } |N_b| = q_b - 1. \end{cases}$

We say that f is **weakly $\boldsymbol{q}$-acyclic** if it does not contain a strong q-cycle. Let $\boldsymbol{\mathcal{F}}^{\textbf{weak } \boldsymbol{q}\textbf{-acy}}$ be the set of weakly q-acyclic priority profiles.

Theorem E. (Ergin, 2002) Let $f \in \mathcal{F}^A$. The following statements are equivalent.

(i) For each $P \in \Pi^N$, DA$(f, P) \in$ Eff(P),

(ii) $f \in \mathcal{F}^{\text{weak } q\text{-acy}}$.

Remark 1. Weak q-acyclicity of a priority profile is equivalent to the following condition (Ergin, 2002):

$$\text{Condition } (a)\text{:} \quad \text{for} \begin{pmatrix} \text{each pair } a, b \in A \\ \text{each } i \in N \end{pmatrix} \text{ s.t. } f_a(i) > (q_a + q_b),\ f_a(i) - f_b(i) \leq 1$$

2.3. Top-trading cycles rule

For each priority profile and each preference profile, the student-proposing top-trading cycles rule assigns school seats to student as follows:[10]

Step 1 Each student with the highest priority at a school, say a, is "endowed" with q_a seats at a. Each student, say i, points to the student who is endowed with the seat that i most prefers. There is at least one cycle. Each student in a cycle is assigned to the school to which he pointed. We remove those students and decrease the capacity of each school appearing in a cycle by one.

[10]Here we describe "the TTC rule with inheritance" that Kesten (2006) formulates. This is equivalent to the original TTC rule proposed by Abdulkadiroğlu and Sönmez (2003).

Step $t(\geq 2)$ We proceed with the remaining students and schools with remaining seats. Each student with the highest priority at a remaining school, say a, is endowed with the unassigned seats at a. Each student, say i, points to the student who is endowed with the seat that i most prefers. There is at least one cycle. Each student in a cycle is assigned to the school to which he pointed. We remove those students and decrease the capacity of each school appearing in a cycle by one.

Last Step We stop when no student remains.

We call this rule the **top trading cycles rule**, or TTC rule for short. The TTC rule is *efficient* but not *fair*. Kesten (2006) further restricts the domain of priority profiles that Ergin (2002) identifies. Then, the TTC rule is *fair* and the TTC and DA rules coincide (Kesten, 2006).

We say that $\boldsymbol{f}$ **contains a $\boldsymbol{q}$-cycle** if

(1) there are $i, j, k \in N$ and $a, b \in A$ such that $\begin{cases} f_a(i) < f_a(j) < f_a(k) \\ f_b(k) < f_b(i), f_b(j) \end{cases}$, and

(2) there is $N_a \subseteq N \setminus \{i, j, k\}$: $\begin{cases} N_a \subseteq U_a(i) \cup (U_a(j) \setminus U_b(k)) \\ |N_a| = q_a - 1. \end{cases}$

We say that f is **$\boldsymbol{q}$-acyclic** if f does not contain a q-cycle.
Let $\boldsymbol{\mathcal{F}^{q\text{-acy}}}$ be the set of q-acyclic priority profiles.

Theorem K. (Kesten, 2006) Let $f \in \mathcal{F}^A$. The following statements are equivalent.

(i) For each $P \in \Pi^N$, $\mathrm{TTC}(f, P) \in \mathrm{Fair}(f, P)$.

(ii) For each $P \in \Pi^N$, $\mathrm{TTC}(f, P)$=$\mathrm{DA}(f, P)$.

(iii) $f \in \mathcal{F}^{q\text{-acy}}$.

We provide a characterization of q-acyclic priority profiles that parallels Ergin (2002)'s characterization of weakly q-acyclic priority profiles (Remark 1).[11]

Proposition 1. q-acyclicity of a priority profile f is equivalent to the following:

$$\begin{cases} \text{Condition } (a) \text{ holds and} \\ \text{Condition } (b)\text{:} \quad \begin{array}{l} \text{for each pair } a, b \in A \text{ and each } i \in N \\ f_a(i) - f_b(i) \leq q_a. \end{array} \end{cases}$$

Proof.
($\Rightarrow$) Suppose that f is q-acyclic. We show that conditions (a) and (b) hold. First,

[11] Kesten (2006) also formulates a stronger notion of acyclicity, which we call "strong q-acyclicity". The TTC rule, properly adapted for cases with variable populations and variable schools, satisfies "consistency" on the domain of strongly q-acyclic priority profiles. We discuss this property in Section 3.1 in detail. We also provide a characterization of strong q-acyclicity in the Appendix.

if f is q-acyclic, then it is weakly q-acyclic (Kesten, 2006). Thus, condition (a) hold. Suppose that condition (b) does not hold. Then, there are $a, b \in A$ and $i \in N$ such that $f_a(i) - f_b(i) \geq q_a + 1$. Then, $|U_a(i) \setminus U_b(i)| \geq q_a + 1$ and there is $N' \subseteq U_a(i) \cap L_b(i)$ such that $|N'| = q_a + 1$. Let $j, k \in N'$. Then, f contains a q-cycle involving j, k, i, and $N' \setminus \{j, k\}$.

($\Leftarrow$) Suppose that conditions (a) and (b) hold. We show that f is q-acyclic. Suppose, by contradiction, that f contains a q-cycle: there are $i, j, k \in N$, $a, b \in A$, and $N' \subseteq U_a(j) \setminus (U_b(k) \cup \{i, j\})$ such that $f_a(i) < f_a(j) < f_a(k)$, $f_b(k) < f_b(i), f_b(j)$, and $|N'| = q_a - 1$. Let $s \equiv f_a(k)$ and $t \equiv f_b(k)$. Then, $|U_a(k)| = s - 1$ and $|U_a(k) \setminus U_b(k)| \geq |N'| + |\{i, j\}| = q_a + 1$. Thus, $|U_a(k) \cap U_b(k)| \leq s - q_a - 2$. Since $|U_b(k)| = t - 1$ and $|U_a(k) \cap U_b(k)| \leq s - q_a - 2$, we obtain $|U_b(k) \setminus U_a(k)| \geq t - s + q_a + 1$. Then, there is $m \in U_b(k) \setminus U_a(k)$ such that $f_a(m) \geq s + t - s + q_a + 1 = t + q_a + 1$. Since $m \in U_b(k)$, we have $f_b(m) \leq t - 1$. Thus, $f_a(m) - f_b(m) \geq q_a + 2$, a violation of condition (b). □

2.4. Immediate acceptance rule

The last rule we discuss is used in many school districts in the U.S. In recent literature, it is usually referred to as the"Boston rule". The student-proposing immediate acceptance rule assigns school seats to students as follows. Note that the acceptance at each step is final.

Step 1 Each student applies to his most preferred school. If the number of students applying to a school, say a, does not exceed q_a, then all of these students are accepted. Otherwise, among those, a is assigned to the students with the highest priority down to the q_a-th priority at a: the others are rejected. We decrease the capacity of each school by the number of students accepted to the school at this step and denote it by $q^1 \equiv (q^1_o)_{o \in A}$.

Step t(≥ 2) Each student who is rejected at Step ($t-1$) applies to his next most preferred school. For each school a, if the number of students who are newly applying to a does not exceed q_a^{t-1}, then all of these students are accepted. Otherwise, among those, a is assigned to the students from the highest priority down to the q_a^{t-1}-th priority at a: the others are rejected. We further decrease the capacity of each school by the number of students accepted to the school at this step and denote it by $q_t \equiv (q^t_o)_{o \in A}$.

Last Step We stop when each student is accepted by a school.

We call this rule the **immediate acceptance rule**, or IA rule for short (Thomson, 2010). The IA rule is *efficient* but not *fair*.

3. Maximal domain for TTC to be fair

In this section, we turn to the two questions that we raised in Introduction. For that purpose, we first introduce several structures of preference profiles. First is when preferences over schools are the same across students.

Identical preference profiles, $\mathbf{\Pi}^N_{\mathbf{iden}}$: for each pair $i, j \in N$, $P_i = P_j$.

Consider a list of schools in A (with possible repetitions). The list has **no conflict at *q*** if the number of times each school, say a, appears in the list does not exceed its capacity q_a. Otherwise, namely, if there is a school that appears more than q_a times in the list, we say that the list has conflict at q. For example, suppose that $A = \{a, b, c\}$ and $q = (2, 3, 2)$. The lists (a, b, b, b, c) and (a, a, c) has no conflict at q, but (c, c, c, b) does have conflict at q. Next are the preference profiles such that the list of students' most preferred schools at a profile has no conflict at q.

No-conflict preference profiles, $\mathbf{\Pi}^N_{\mathbf{nc}}$: for each $a \in A$, $|N(a, P : A)| \leq q_a$.

Next are composites of identical preference profiles and no-conflict preference profiles. They are defined inductively. Included is any profile such that the most preferred school is the same across students, but the list of the students' second most preferred schools has no conflict at q. Also included is any preference profile such that the preferences over the two most preferred schools are the same across students, but the list of the students' third most preferred schools has no conflict at q, and so on.

Other preference profiles have to be considered. Let $k \in \{2, \cdots, |A|\}$. Consider a profile such that the preferences over the $(k-1)$ most preferred schools are the same across students (equivalently, for each pair $i, j \in N$, $P_i|_1^{k-1} = P_j|_1^{k-1}$). Now, consider the list of the students' k-th most preferred schools. We require that the list of the students' k-th most preferred schools whose capacity is smaller than $|N| - \sum_{o \in A_1^{k-1}} q_o - 1$, has no conflict at q.

Composites of identical preference profiles and no-conflict preference profiles, $\mathbf{\Pi}^N_{\mathbf{iden\text{-}nc}}$: $\exists\ k \in \{2, \cdots, |A| - 1\}$ such that

(i) for each pair $i, j \in N$, $P_i|_1^{k-1} = P_j|_1^{k-1}$,

(ii) $|A^k(P)| \geq 2$,[12] and

(iii) for each $a \in A^k(P)$ with $q_a < n - \sum_{o \in A_1^{k-1}(P)} q_o - 1$,

$$|N(a, P : A^k(P))| \leq q_a.$$

[12]That is, the k-th most preferred schools are not the same across students.

Example 1. Preference profiles in Π^N_{iden}, Π^N_{nc}, and $\Pi^N_{\text{iden-nc}}$
Let $N \equiv \{1,2,3,4,5\}$, $A \equiv \{a,b,c,d,e\}$, and $q = (1,1,1,2,2)$.

P_1	P_2	P_3	P_4	P_5
b	a	a	a	a
a	b	b	b	b
c	c	c	c	c
d	d	d	d	d
e	e	e	e	e

Π^N_{iden}

P_1	P_2	P_3	P_4	P_5
a	b	c	d	d
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Π^N_{nc}

P_1	P_2	P_3	P_4	P_5
a	a	a	a	a
b	c	d	d	e
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

P_1	P_2	P_3	P_4	P_5
a	a	a	a	a
b	b	b	b	b
c	d	d	d	e
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

P_1	P_2	P_3	P_4	P_5
a	a	a	a	a
d	d	d	d	d
b	b	c	c	c
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$\Pi^N_{\text{iden-nc}}$

Our first result demonstrate the structure of preference profiles on which the TTC rule is *fair* at each priority profile, equivalently, on which the TTC and DA rules coincide.

Theorem 1. Let $P \in \Pi^N$. The following statements are equivalent.
(i) For each $f \in \mathcal{F}^A$, $\text{TTC}(f,P) \in \text{Fair}(f,P)$.
(ii) For each $f \in \mathcal{F}^A$, $\text{TTC}(f,P) = \text{DA}(f,P)$.
(iii) $P \in \Pi^N_{\text{iden}} \cup \Pi^N_{\text{iden-nc}} \cup \Pi^N_{\text{nc}}$.

Proof. (i)⇔(ii): This comes directly from the fact that the deferred acceptance rule Pareto dominates each other *fair* rule.
(i)⇒(iii): Suppose, by contradiction, that $P \notin \Pi^N_{\text{iden}} \cup \Pi^N_{\text{iden-nc}} \cup \Pi^N_{\text{nc}}$. Then, there are $k \in \{1,\cdots,|A|\}$ and $a \in A^k(P)$ such that [13]

(1) for each pair $i,j \in N$, $P_i|_1^{k-1} = P_j|_1^{k-1}$,
(2) $|A^k(P)| \geq 2$,
(3) $q_a < n - \sum_{o \in A_1^{k-1}(P)} q_o - 1$, and
(4) $|N(a,P : A^k(P))| > q_a$.

Without loss of generality, for each $t \in \{1,\cdots,k-1\}$, let $a_t \in A^t(P)$ and $\{a,b\} \subseteq A^k(P)$. For each $i \in \{1,\cdots,q_a+1\}$, let $A^k(P_i) = \{a\}$ and $A^k(P_{q_a+2}) = \{b\}$. Let $f \in \mathcal{F}^A$ be such that

[13]The second condition is the same as $|A^k(P)| \geq 2$.

$$\begin{cases} f_{a_1}(n) = 1 < f_{a_1}(n-1) = 2 < \cdots < f_{a_1}(n - q_{a_1} + 1) = q_{a_1} < \cdots, \\ f_{a_2}(n - q_{a_1}) = 1 < f_{a_2}(n - q_{a_1} - 1) = 2 < \cdots < f_{a_2}(n - q_{a_1} - q_{a_2} + 1) = q_{a_2} < \cdots, \\ \vdots \\ f_{a_{k-1}}(n - \sum_{t=1}^{k-2} q_{a_t}) = 1 < f_{a_{k-1}}(n - \sum_{t=1}^{k-2} q_{a_t} - 1) = 2 \\ \qquad < \cdots < f_{a_{k-1}}(n - \sum_{t=1}^{k-1} q_{a_t} + 1) = q_{a_{k-1}} < \cdots, \\ f_a(1) = 1 < f_a(2) = 2 < \cdots < f_a(q_a - 1) = q_a - 1 \\ \qquad < f_a(q_a + 2) = q_a < f_a(q_a) = q_a + 1 < f_a(q_a + 1) = q_a + 2 < \cdots, \\ f_b(q_a + 1) = 1 < \cdots. \end{cases}$$

Then, $\mathrm{TTC}_{q_a+1}(f, P) = a$, $\mathrm{TTC}_{q_a}(f, P) \neq a$, and $a \ P_{q_a} \ \mathrm{TTC}_{q_a}(f, P)$, a violation of *fairness*.

(iii)⇒(i): If $P \in \Pi^N_{\text{iden}} \cup \Pi^N_{\text{nc}}$, then it is easy to see that (i) holds. Let $P \in \Pi^N_{\text{iden-nc}}$. Then, there is $k \in \{2, \cdots, |A| - 1\}$ such that

$$\begin{cases} \text{(1) for each pair } i, j \in N,\ P_i|_1^{k-1} = P_j|_1^{k-1},\ |A^k(P)| \geq 2, \text{ and} \\ \text{(2) for each } a \in A^k(P) \text{ with } q_a < n - \sum_{o \in A_1^{k-1}(P)} q_o - 1,\ |N(a, P : A^k(P))| \leq q_a. \end{cases}$$

Let $x \in X$, $f \in \mathcal{F}^A$, and $a \in A$. We say that ***a* violates *f* at (*x*, *P*)** if there is a pair $i, j \in N$ such that $x_j = a$, $a \ P_i \ x_i$, and $f_a(i) < f_a(j)$.

For each $t \in \{1, \cdots, k-1\}$, without loss of generality, let $A^t(P) \equiv \{a_t\}$. It is easily checked that for each $f \in \mathcal{F}^A$, each school in $\{a_1, \cdots, a_{k-1}\}$ is assigned in order of priorities at the school by either the TTC rule or the DA rules to students.[14] Without loss of generality, let students n down to $n - \sum_{o \in A_1^{k-1}(P)} q_o + 1$ be assigned to $A_1^{k-1}(P)$.

Next, $n - \sum_{o \in A_1^{k-1}(P)} q_o$ students are assigned to a school in $A \setminus (A_1^{k-1}(P))$. Let $\bar{N}$ be the set of these students, and for each $o \in A^k(P)$, let $\bar{N}(o, P : A^k(P))$ be the set of students in $\bar{N}$ whose most preferred school in $A^k(P)$ is o. We prove that there is only one school, say a, in $A^k(P)$ such that $q_a \geq n - \sum_{o \in A_1^{k-1}(P)} q_o - 1$ and $|\bar{N}(a, P : A^k(P))| \geq q_a + 1$. Suppose otherwise, namely, there are two such schools, $a, a' \in A^k(P)$. Then,

$$\begin{aligned} n - \textstyle\sum_{o \in A_1^{k-1}(P)} q_o \quad &\leq \quad q_a + 1 \quad \leq \quad |\bar{N}(a, P : A^k(P))| \\ n - \textstyle\sum_{o \in A_1^{k-1}(P)} q_o \quad &\leq \quad q_{a'} + 1 \quad \leq \quad |\bar{N}(a', P : A^k(P))| \end{aligned}$$

[14] That is, each school in $\{a_1, \cdots, a_{k-1}\}$, first distributes a seat to the student with the highest priority at the school. If a student is given more than one school seat, he chooses most preferred school among them and declines the remaining. The schools declined by the student take the seat back. Next, each school in $\{a_1, \cdots, a_{k-1}\}$, distributes a seat to the student with the second highest priority at the school, if the school has a remaining seat. The same process repeats until either each student is assigned to a school or there is no remaining seats of those schools.

Summing the two equalities, we obtain

$$\begin{array}{rcl} 2(n - \sum_{o\in A_1^{k-1}(P)} q_o) & \leq & |\bar{N}(a, P : A^k(P))| + |\bar{N}(a', P : A^k(P))| \\ & \leq & |\bar{N}| = n - \sum_{o\in A_1^{k-1}(P)} q_o, \end{array}$$

which is a contradiction. There are two cases:

Case 1. The list of the k-th most preferred schools of $\bar{N}$ has no conflict at q. Then, it is easily checked that at $\mathrm{TTC}(f, P)$, each student in $\bar{N}$ is assigned his k-th most preferred school. Thus, $\mathrm{TTC}(f, P) \in \mathrm{Fair}(f, P)$.

Case 2. There is $a \in A^k(P)$ such that $|\bar{N}(a, P : A^k(P))| = q_a + 1 = n - \sum_{o\in A_1^{k-1}(P)} q_o$. Then, a is assigned by $\mathrm{TTC}(f, P)$ to the q_a students with the highest priorities at a, and the only remaining student is assigned his $(k + 1)$-th most preferred school. Note that this student, say i, has a lower priority at a than each other student in $\bar{N} \setminus \{i\}$. Thus, $\mathrm{TTC}(f, P) \in \mathrm{Fair}(f, P)$. □

We indeed have more than the coincidence of the two rules on the domain of preference profiles of Theorem 1. On this domain, the two rules coincide with the immediate acceptance rule.

Corollary 1. If $P \in \Pi^N_{\text{iden}} \cup \Pi^N_{\text{iden-nc}} \cup \Pi^N_{\text{nc}}$, then for each $f \in \mathcal{F}^A$, $\mathrm{IA}(f, P) = \mathrm{DA}(f, P) = \mathrm{TTC}(f, P)$.

Proof. Let $P \in \Pi^N_{\text{iden}} \cup \Pi^N_{\text{iden-nc}} \cup \Pi^N_{\text{nc}}$. We show that for each $f \in \mathcal{F}^A$, $\mathrm{IA}(f, P) = \mathrm{DA}(f, P)$. Then, by Theorem 1, for each $f \in \mathcal{F}^A$, $\mathrm{IA}(f, P) = \mathrm{DA}(f, P) = \mathrm{TTC}(f, P)$. As shown in the proof of Theorem 1, there is $k \in \{2, \cdots, |A|\}$ such that $|A^k(P)| \geq 2$ and for each pair $i, j \in N$, $P_i|_1^{k-1} = P_j|_1^{k-1}$. The most preferred school down to the $(k-1)$-th most preferred school are assigned to the students with the highest priorities at these school at $\mathrm{IA}(f, P)$, as in $\mathrm{DA}(f, P)$. For the remaining students, the same argument as in Cases 1 and 2 of the proof of Theorem 1 applies. □

3.1. Maximal preference profiles for TTC to be consistent

In this section, we consider a property pertaining to variable populations. Let A, N and q be given. We define a rule for all subpopulations of N. Let $N' \subseteq N$. Let $\mathcal{F}_{N'}$ be the set of priorities defined over N'. An economy is a list $e' \equiv (N', A, f', P', q')$ such that (i) $N' \subseteq N$, (ii) $f' \in \mathcal{F}^A_{N'}$, (iii) $P' \in \Pi^{N'}$, and (iv) $\sum_{a\in A} q'_a \geq |N'|$. An assignment is defined in the same way as in Section 2. Let $X(e')$ be the set of assignments at e'. A rule is a function $\widetilde{\varphi}$ from the set of such economies to the set of assignments defined for each population.

Let $f_0 \in \mathcal{F}_N$. For each $N' \subseteq N$, let $f_0|_{N'} \in \mathcal{F}_{N'}$ be the priority f, restricted to N'. For each $f \in \mathcal{F}_N^A$, let $f|_{N'} \equiv (f_a|_{N'})_{a\in A}$ be the priority profile restricted to N'. For each $q \in \mathbb{N}_+^{|A|}$, each $x \in X$, and each $N' \subseteq N$, with a slight abuse of notation, let $q - x_{N'} \equiv (q_a - |\{i \in N' : x_i = a\}|)_{a\in A}$. For each economy $e = (N, A, f, P, q)$, each $N' \subseteq N$, and each $x \in X$, define the **reduced economy of $\boldsymbol{e}$ at $\boldsymbol{(x, N')}$** to be $r^x_{N'}(e) \equiv (N', A, f|_{N'}, P_{N'}, q - x_{N'})$.

Consistency: For each $e \equiv (N, A, f, P, q)$ and each $N' \subseteq N$, $\widetilde{\varphi}_{N'}(e) = \widetilde{\varphi}(r^{\widetilde{\varphi}(e)}_{N'}(e))$.[15]

Let A, N and q be given. For each $P \in \Pi^N$, we say that a rule is **consistent at $\boldsymbol{P}$** if for each $f \in \mathcal{F}^A$ and each $N' \subseteq N$, $\widetilde{\varphi}_{N'}(N, A, f, P, q) = \widetilde{\varphi}(r^{\widetilde{\varphi}(N,A,f,P,q)}_{N'}(N, A, f, P, q))$. Next theorem says that the domain of profiles in Theorem 1 is also maximal for the $\widetilde{\text{TTC}}$ rule to be *consistent*.

Theorem 2. Let $P \in \Pi^N$. The following statements are equivalent.

(i) The $\widetilde{\text{TTC}}$ rule is *consistent* at P.

(ii) $P \in \Pi^N_{\text{iden}} \cup \Pi^N_{\text{iden-nc}} \cup \Pi^N_{\text{nc}}$.

Proof. Let N, A and q be fixed. Let $f \in \mathcal{F}^A$ and $P \in \Pi^N_{\text{iden}} \cup \Pi^N_{\text{iden-nc}} \cup \Pi^N_{\text{nc}}$. Let $e \equiv (N, A, f, P, q)$. It is easy to check that the $\widetilde{\text{TTC}}$ rule is *consistent* at P.

Conversely, let $P \notin \Pi^N_{\text{iden}} \cup \Pi^N_{\text{iden-nc}} \cup \Pi^N_{\text{nc}}$. Let $x \equiv \widetilde{\text{TTC}}(e)$. Then, we show that there is $f \in \mathcal{F}^A$ such that for some $N' \subseteq N$, $\widetilde{\text{TTC}}(r^x_{N'}(e)) \neq x_{N'}$. Suppose, by contradiction, that $P \notin \Pi^N_{\text{iden}} \cup \Pi^N_{\text{iden-nc}} \cup \Pi^N_{\text{nc}}$. Then, there are $k \in \{1, \cdots, |A|\}$ and $a \in A^k(P)$ such that

(1) for each pair $i, j \in N$, $P_i|_1^{k-1} = P_j|_1^{k-1}$,

(2) $|A^k(P)| \geq 2$,

(3) $q_a < n - \sum_{o\in A_1^{k-1}(P)} q_o - 1$, and

(4) $|N(a, P : A^k(P))| > q_a$.

Without loss of generality, for each $t \in \{1, \cdots, k-1\}$, let $a_t \in A^t(P)$ and $\{a, b\} \subseteq A^k(P)$. For each $i \in \{1, \cdots, q_a + 1\}$, let $A^k(P_i) = \{a\}$ and $A^k(P_{q_a+2}) = \{b\}$. Let $f \in \mathcal{F}^A$ be such that

[15] For a survey on consistency, see Thomson (2007).

$$\begin{cases} f_{a_1}(n) = 1 < f_{a_1}(n-1) = 2 < \cdots < f_{a_1}(n - q_{a_1} + 1) = q_{a_1} < \cdots, \\ f_{a_2}(n - q_{a_1}) = 1 < f_{a_2}(n - q_{a_1} - 1) = 2 < \cdots < f_{a_2}(n - q_{a_1} - q_{a_2} + 1) = q_{a_2} < \cdots, \\ \qquad\qquad\qquad\qquad \vdots \\ f_{a_{k-1}}(n - \sum_{t=1}^{k-2} q_{a_t}) = 1 < f_{a_{k-1}}(n - \sum_{t=1}^{k-2} q_{a_t} - 1) = 2 \\ \qquad\qquad < \cdots < f_{a_{k-1}}(n - \sum_{t=1}^{k-1} q_{a_t} + 1) = q_{a_{k-1}} < \cdots, \\ f_a(q_a + 2) = 1 < f_a(1) = 2 < f_a(2) = 3 \\ \qquad\qquad < \cdots < f_a(q_a - 1) = q_a < f_a(q_a) = q_a + 1 < f_a(q_a + 1) = q_a + 2 < \cdots, \\ f_b(q_a) = 1 < f_b(q_a + 1) = 2 < \cdots. \end{cases}$$

Then, $x_1 = \cdots = x_{q_a - 1} = a$, $x_{q_a+1} \neq a$, and $x_{q_a+2} = b$. Suppose that student q_a leaves with his assignment a. Let $x' \equiv \widetilde{\mathrm{TTC}}(r^x_{N \setminus \{q_a+2\}}(f, P))$. Then, $x'_1 = \cdots = x'_{q_a-2} = a$, $x'_{q_a+1} = a$, and $x_{q_a+2} = b$, a violation of *consistency*. $\square$

4. Larger domain to achieve efficiency and fairness

Our next question is whether there is a larger domain of preference profiles than that in Theorem 1, on which *fairness* and *efficiency* are still compatible. We identify several structures of preference profiles on which

(i) the DA rule is *efficient* for each priority profile, or

(ii) the TTC rule is *fair* for each weakly q-acyclic priority profile.

4.1. Domain for DA to be efficient

We introduce two classes of preference profiles. Define a bijection σ from A to $\{1, \cdots, |A|\}$: for each $o \in A$, $\sigma(o) \in \{1, \cdots, |A|\}$ and let $o \equiv a_{\sigma(o)}$. That is, σ is a function that relabels schools. Let Σ be the set of all such bijections. Given $\sigma \in \Sigma$, let $P_0 \in \Pi$ be such that for some $t \in \{1, 2, \cdots, |A|\}$, $a_t \; P_0 \; a_{(t+1)\mathrm{mod}(|A|)} \; P_0 \; a_{(t+2)\mathrm{mod}(|A|)} \; P_0 \; \cdots \; P_0 \; a_{(t+|A|)\mathrm{mod}(|A|)}$. Let $\Pi^N_{\sigma\text{-lin}}$ be the set of all profiles with such individual preferences.

Σ-linear preference profiles, $\boldsymbol{\Pi^N_{\Sigma\text{-lin}}}$: there is $\sigma \in \Sigma$ such that $P \in \Pi^N_{\sigma\text{-lin}}$.

Remark 2. Consider the algorithm used to define the DA rule when applied to each $P \in \Pi^N_{\boldsymbol{\Sigma}\text{-}\mathbf{lin}}$. There is no triple of schools $a, b, c \in A$ with $\sigma(a) \leq \sigma(b) \leq \sigma(c)$ and no pair of agents $i, j \in N$ such that student i is rejected from a and applies to b and student j is rejected from c and applies to a. This comes from the assumption that $\sum_{a \in A} q_a \geq |N|$.

Example 2. A preference profile in $\Pi^N_{\Sigma\text{-lin}}$
Let $N \equiv \{1,2,3,4,5\}$, $A \equiv \{a,b,c,d,e\}$, and $q = (1,1,2,2,2)$,

$$\begin{pmatrix} \sigma(a) = 1 \\ \sigma(b) = 2 \\ \sigma(c) = 3 \\ \sigma(d) = 4 \\ \sigma(e) = 5 \end{pmatrix}$$

P_1	P_2	P_3	P_4	P_5
a	b	b	e	c
b	c	c	a	d
c	d	d	b	e
d	e	e	c	a
e	a	a	d	b

The next preference profiles are defined by a sequential construction. First, consider each student's most preferred school. Except for exactly one school, the number of students who prefer each given school, say a, to each other school does not exceed q_a. Second, remove school a from the preference profile and consider each student's most preferred school in the resulting preference profile. Except for exactly one school, the number of students who prefer each given school, say b, to each other school does not exceed q_b. Third, remove school b from the previous preference profile and proceed as in the previous step. Stop when only one school remains. The remaining part of preferences that is not specified by this sequential process can be completed arbitrarily. Formally, there is $\sigma \in \Sigma$ such that (i) for each $o \in A$, $o \equiv a_{\sigma(o)}$, (ii) for each $k \geq 2$, $|N(a_k, P : A)| < q_{a_k}$, (iii) for each $k \geq 3$, $|N(a_k, P : A \setminus \{a_1\})| < q_{a_k}$, (iv) for each $k \geq 4$, $|N(a_k, P : A \setminus \{a_1, a_2\})| < q_{a_k}$, and so on. Let $\Pi^N_{\sigma\text{-seq}}$ be the set of all such preference profiles.

Σ-sequential dominance preference profiles, $\mathbf{\Pi^N_{\Sigma\text{-seq}}}$: there is $\sigma \in \Sigma$ such that $P \in \Pi^N_{\sigma\text{-seq}}$.

Example 3. Procedure for constructing a preference Profile in $\Pi^N_{\sigma\text{-seq}}$
Let $N \equiv \{1,2,3,4,5\}$, $A \equiv \{a,b,c,d,e\}$, and $q = (1,1,2,2,2)$.

P_1	P_2	P_3	P_4	P_5
a	a	a	a	c

P_1	P_2	P_3	P_4	P_5
a	a	a	a	c
b	b	b	e	

P_1	P_2	P_3	P_4	P_5
a	a	a	a	c
b	b	b	e	
c	c	d		

P_1	P_2	P_3	P_4	P_5
a	a	a	a	c
b	b	b	e	d
c	c	d		
d	d			

P_1	P_2	P_3	P_4	P_5
a	a	a	a	c
b	b	b	e	d
c	c	d		e
d	d	e		
e	e			

P_1	P_2	P_3	P_4	P_5
a	a	a	a	c
b	b	b	e	d
c	c	d	d	e
d	d	e	b	b
e	e	c	c	a

Theorem 3. If $P \in \Pi^N_{\text{iden}} \cup \Pi^N_{\text{iden-nc}} \cup \Pi^N_{\text{nc}} \cup \Pi^N_{\Sigma\text{-seq}} \cup \Pi^N_{\Sigma\text{-lin}}$, then

$$\text{for each } f \in \mathcal{F}^A,\ \text{DA}(f,P) \in \text{Eff}(P).$$

Proof. If $P \in \Pi^N_{\text{iden}} \cup \Pi^N_{\text{iden-nc}} \cup \Pi^N_{\text{nc}}$, then by Theorem 1, for each $f \in \mathcal{F}^A$, $\text{DA}(f,P) \in \text{Eff}(P)$. Let $P \in \Pi^N_{\Sigma\text{-seq}} \cup \Pi^N_{\Sigma\text{-lin}}$. We show that for each $f \in \mathcal{F}^A$, $\text{DA}(f,P) \in \text{Eff}(P)$. If $P \in \Pi^N_{\Sigma\text{-lin}}$, there is $\sigma \in \Sigma$ such that $P \in \Pi^N_{\sigma\text{-lin}}$. Let $(a_{\sigma(o)})_{o \in A} = (a_t)_{t=1}^{|A|}$. Suppose, by contradiction, that $\text{DA}(f,P) \notin \text{Eff}(P)$. There is $x \in X$ such that for each $i \in N$, $x_i\ R_i\ \text{DA}_i(f,P)$ and for some $i \in N$, $x_i\ P_i\ \text{DA}_i(f,P)$. That is, there is a sequence of students $(i_k)_{k=1}^m$ for some $m \in \{2, \cdots, |A|\}$ such that for each $k \in \{1, \cdots, m\}$, $\text{DA}_{i_{k+1}}(f,P)\ P_{i_k}\ \text{DA}_{i_k}(f,P)$ and $i_{m+1} = i_1$. As $P \in \Pi^N_{\Sigma\text{-lin}}$, whenever each student is rejected from a school at each step, he applies to another school in order of σ. Note that for each $k = \{1, \cdots, m\}$, student i_k is rejected from $\text{DA}_{i_{k+1}}(f,P)$ and applies to $\text{DA}_{i_k}(f,P)$. Thus, without loss of generality, we may let $\sigma(\text{DA}_{i_{m+1}}(f,P)) < \sigma(\text{DA}_{i_m}(f,P)) < \sigma(\text{DA}_{i_{m-1}}(f,P)) < \cdots < \sigma(\text{DA}_{i_1}(f,P)))$. However, this is impossible under the assumption that $\sum_{a \in A} q_a \geq n$. □

4.2. Domain for TTC to be fair for each weakly acyclic priority profile

Let $m \in \{1, \cdots, |A|\}$. Let $\bar{A} \equiv (\bar{A}_k)_{k=1}^m$ be a partition of A. Let $\bar{\mathcal{A}}$ be the set of all partitions of A. Let $\bar{A} \in \bar{\mathcal{A}}$. For each $a \in A$, define $U(a, \bar{A})$ to be the components of $\bar{A}$ with a smaller subscript than the component that a is in. That is, for some $k \in \{1, \cdots, m\}$, if $a \in \bar{A}_k$, then $U(a, \bar{A}) = \bigcup_{t=1}^{k-1} \bar{A}_t$.

Now, for each $\bar{A} \equiv (\bar{A}_k)_{k=1}^m \in \bar{\mathcal{A}}$, let $\mathcal{P}_0(\bar{A}) \subseteq \Pi$ be the set of preferences such that each school in $\bar{A}_1$ is preferred to each school in $\bar{A}_2$, each school in $\bar{A}_2$ is preferred to each school in $\bar{A}_3$, and so on.[16] For each $k \in \{1, \cdots, m\}$, let $S_k \equiv \{a \in \bar{A}_k : q_a \geq 2\}$ be the set of schools in $\bar{A}_k$ with capacities greater than one. Let $P \in \Pi^N$ be such that (i) for each $i \in N$, $P_i \in \mathcal{P}_0(\bar{A})$ and (ii) for each $k \in \{1, \cdots, m\}$ such that $|S_k| \geq 1$ and each $i, j \in N$, $P_i|_{S_k} = P_j|_{S_k}$. Let $\Pi^N_{\bar{A}}$ be the set of all such preference profiles.

$\bar{\mathcal{A}}$-partitioned preference profiles, $\Pi^N_{\bar{\mathcal{A}}}$: there is $\bar{A} \in \bar{\mathcal{A}}$ such that $P \in \Pi^N_{\bar{A}}$.

Example 4. A preference profile in $\Pi^N_{\bar{A}}$
Let $N \equiv \{1,2,3,4,5\}$, $A \equiv \{a,b,c,d,e\}$, and $q = (1,1,2,2,2)$,

$$\bar{A} \equiv (\bar{A}_1, \bar{A}_2, \bar{A}_3) = (\{c\}, \{a,b\}, \{d,e\})$$

[16] Note that if $m = 1$, then there is no restriction placed on preference profile, and if $m = |A|$, then it is the identical preference profile.

P_1	P_2	P_3	P_4	P_5
c	c	c	c	c
a	b	b	a	b
b	a	a	b	a
d	d	d	d	d
e	e	e	e	e

Theorem 4. If $P \in \Pi^N_{\text{iden}} \cup \Pi^N_{\text{iden-nc}} \cup \Pi^N_{\text{nc}} \cup \Pi^N_{\bar{\mathcal{A}}}$, then

$$\text{for each } f \in \mathcal{F}^{\text{weak } q\text{-acy}},\ \text{TTC}(f,P) \in \text{Fair}(f,P).$$

Proof. If $P \in \Pi^N_{\text{iden}} \cup \Pi^N_{\text{iden-nc}} \cup \Pi^N_{\text{nc}}$, then by Theorem 1, for each $f \in \mathcal{F}^A$, $\text{TTC}(f,P) \in \text{Fair}(f,P)$. Let $P \in \Pi^N_{\bar{\mathcal{A}}}$. We show that for each $f \in \mathcal{F}^{\text{weak } q\text{-acy}}$, there is no school that violates f at ($\text{TTC}(f,P)$,P). Let $x \equiv \text{TTC}(f,P)$. Suppose, by contradiction, that there is a school, say a, violating f at (x,P). Then, there is $t \in \{1,\cdots,m\}$ such that $a \in \bar{A}_t$. Then, there is a pair $i,j \in N$ such that $x_i\ P_j\ x_j$ and $f_{x_i}(j) < f_{x_i}(i)$.

First, we show that $f_{x_i}(j) > 1$. Otherwise, $f_{x_i}(j) = 1$ and student j is temporarily assigned to x_i. According to the TTC rule, student j ends up being assigned to a school that he prefers to x_i, a contradiction.

Second, we show that there is $k \in N$ such that $f_{x_i}(k) < f_{x_i}(j)$ and is included in a top-trading cycle together with student i. If there is no such student k, then according to the TTC rule, student i is never assigned to school x_i while student j is not assigned to the same school. That is, there is a top-trading cycle in which student i points to x_i, school x_i points to student k, and student k points to x_k. Thus, $x_k\ P_k\ x_i$ and $x_i\ P_i\ x_k$. By the definition of $\Pi^N_{\bar{\mathcal{A}}}$, x_i and x_k belong to one component of partition $\bar{A}$ and $q_{x_i} = q_{x_k} = 1$. Without loss of generality, let $q_{x_k} = 1$.

Third, we show that $f_{x_i}(j) \geq q_{x_i} + 1$. Otherwise, again, student j is temporarily assigned to x_i and according to the TTC rule, student j ends up being assigned to a school that he prefers to x_i, a contradiction.

$$\begin{array}{ll}
P_j: & \cdots\ P_j\ x_i\ P_j\ \cdots\ P_j\ x_j\ P_j\ \cdots \\[1em]
f_{x_i}: & \cdots < f_{x_i}(k) < \cdots < f_{x_i}(j) < \cdots < f_{x_i}(i) \\[1em]
f_{x_k}: & \cdots < f_{x_k}(i) < \begin{array}{l} \cdots < f_{x_k}(j) < \cdots \\ \cdots < f_{x_k}(k) < \cdots \end{array}
\end{array}$$

From the fact that $q_{x_i} + 1 \leq f_{x_i}(j) < f_{x_i}(i)$, we have $q_{x_i} + 2 = q_{x_i} + q_{x_k} + 1 \leq f_{x_i}(i)$. By weak q-acyclicity, $|f_{x_k}(i) - f_{x_i}(i)| \leq 1$. Then, $f_{x_k}(k) \geq q_{x_i} + q_{x_k} + 1$ and $f_{x_k}(k) - f_{x_i}(k) \geq 2$, a violation of the weak q-acyclicity of f. □

Remark 3. There is no inclusion relation between the set of preference profiles on which the DA rule is *efficient* and the set of preference profiles on which the TTC rule is *fair* for any weakly q-acyclic priority profile. There are $P \in \Pi^N_{\Sigma\text{-seq}} \cup \Pi^N_{\Sigma\text{-lin}}$ and $f \in \mathcal{F}^A$ such that $\mathrm{DA}(f,P) \notin \mathrm{Eff}(P)$. There are $P \in \Pi^N_{\mathcal{A}}$ and $f \in \mathcal{F}^{\text{weak } q\text{-acy}}$, $\mathrm{TTC}(f,P) \notin \mathrm{Fair}(f,P)$.

Appendix 1. Characterization of Strong q-acyclicity

Kesten (2006) adapts the TTC rule to economies with variable populations and variable schools and studies *consistency* as discussed in Section 3.1. He introduces an even smaller domain of priority profiles than the one in Theorem K.

We say that $\boldsymbol{f}$ **contains a weak q-cycle** if

(1) there are $i,j,k \in N$ and $a,b \in A$ such that $\begin{cases} f_a(i) < f_a(j) < f_a(k) \\ f_b(k) < f_b(i), f_b(j), \text{ and} \end{cases}$

(2) there are $N_a \subseteq N \setminus \{i,j,k\}$: $\begin{cases} N_a \subseteq U^f_a(k) \\ |N_a| = q_a - 1 \end{cases}$.

We say that f is **strongly q-acyclic** if f does not contain a weak q-cycle. Let $\boldsymbol{\mathcal{F}}^{\textbf{strong } q\textbf{-acy}}$ be the set of strongly q-acyclic priority profiles.

On the domain of strongly q-acyclic priority profiles, the TTC rule adapted to variable populations is consistent. Conversely, if a priority profile is not strongly q-acyclic, there is a preference profile at which the (adapted) TTC assignment is not efficient. We provide a characterization of such priority profiles.

Proposition 2. Strong q-acyclicity of a priority profile f is equivalent to the following:

$$\begin{cases} \text{Conditions } (a) \text{ and } (b) \text{ hold and} & \\ \text{Condition } (c)\text{:} & \begin{array}{l} \text{for each pair } a,b \in A \text{ with } f_a(i) \geq q_a + 2, \\ \text{and each } i \in N, \\ f_a(i) - f_b(i) \leq 1. \end{array} \end{cases}$$

Proof. ($\Rightarrow$) Suppose that f is strongly q-acyclic. We show that conditions (a), (b), and (c) hold. If f is strongly q-acyclic, then it is q-acyclic and weakly q-acyclic (Kesten, 2006). Thus, conditions (a) and (b) hold. Suppose that condition (c) does not hold. Then, there are $a,b \in A$ and $i \in N$ such that $f_a(i) \geq q_a + 2$ and $f_a(i) - f_b(i) \geq 2$. Then, $|U_a(i) \setminus U_b(i)| \geq 2$ and there is $N' \subseteq U_a(i) \setminus L_b(i)$ such that $|N'| = q_a - 1$. Then, for each pair $j,k \in U_a(i) \setminus U_b(i)$, f contains a q-cycle involving j,k,i, and $N' \cup \{j,k\}$.

($\Leftarrow$) Suppose that conditions (a), (b) and (c) hold. We show that f is strongly q-acyclic. Suppose, by contradiction, that f contains a weak q-cycle: there are $i,j,k \in N$, $a,b \in A$, and $N' \subseteq U_a(k) \setminus \{i, j\}$ such that $f_a(i) < f_a(j) < f_a(k)$, $f_b(k) < f_b(i), f_b(j)$, and $|N'| = q_a - 1$. There are two cases:

Case 1: $f_b(k) < f_b(i) < f_b(j)$. From condition (c) and the fact that $f_a(k) \geq q_a + 2$, we obtain $f_b(k) \geq f_a(k) - 1$. Since $f_b(i) > f_b(k)$, we have $f_b(i) \geq f_a(k)$. Since $f_a(i) < f_a(j) < f_a(k)$, we have $f_a(i) \leq f_a(k) - 2$. Thus, $f_b(i) - f_a(i) \geq 2$. Let $s \equiv f_a(i)$ and $t \equiv f_b(i)$. Then, $U_a(i) = s - 1$ and $U_b(i) = t - 1$. Since $t \geq s + 2$, we have $|U_a(i) \setminus U_b(i)| \geq t - s$ and $|U_a(i) \cap U_b(i)| \leq s - 1$. Note that $j \notin N'$, and $f_a(j) < f_a(k) \leq t$. Thus, there is $l \in U_a(i) \setminus U_b(i)$ such that $f_a(l) \geq t + 1$. Since $l \in U_b(i)$, $f_b(l) \leq t - 1$. Thus $f_a(l) - f_b(l) \geq 2$, a violation of condition (c).

Case 2: $f_b(k) < f_b(j) < f_b(i)$. From condition (c) and the fact that $f_a(k) \geq q_a + 2$, we obtain $f_b(k) \geq f_a(k) - 1$. Since $f_b(k) < f_b(j)$, we have $f_b(j) \geq f_a(k) + 1$. Since $f_a(j) < f_a(k)$, we have $f_a(j) \leq f_a(k) - 1$. Thus, $f_b(j) - f_a(j) \geq 1$. Let $s \equiv f_a(j)$ and $t \equiv f_b(j)$. Since $i \in U_a(j) \setminus U_b(j)$, $|U_a(j) \cap U_b(j)| \leq s - 2$. Thus, $|U_b(j) \setminus U_a(j)| \geq t - s + 1$. Thus, there is $l \in U_b(j) \setminus U_a(j)$ such that $f_a(l) \geq t + 1$. Since $l \in U_b(j)$, $f_b(l) \leq t - 1$. Thus, $f_a(l) - f_b(l) \geq 2$, a violation of condition (c). □

Appendix 2. Asymptotic results on fair TTC

We have established the existence of a maximal domain of preference profiles on which the TTC rule is *fair*. In each finite economy, the likelihood that we happen to have such a preference profile is always positive. However, we could ask ourselves how much it is likely for us to have such a profile as the size of the economy varies. We construct a sequence of economies of increasing size. At each step, we draw each student's preference from the uniform distribution over all possible preferences over schools. We then calculate the probability that the profile is in our maximal domain. We calculate the probability that this be the case. For each finite economy, the probability that the preference profile is in the maximal domain in Theorem 1 is positive.

We model economies of increasing size. Let $(e^k)_{k \in \mathbb{N}_+} \equiv (A^k, N^k, f^k, P^k, q^k)_{k \in \mathbb{N}_+}$. Let Π_0^k be the set of all strict preferences over A^k. Let U^k be the uniform distribution over Π_0^k.[17] Let $\mathcal{F}^k$ be the set of all priority profiles over N^k. Let $\Pi(k)^{\text{iden}}$, $\Pi(k)^{\text{iden-nc}}$, $\Pi(k)^{\text{nc}} \subseteq \Pi^k$ be the sets of identical preference profiles, no-conflict profiles, and composites of identical profiles and no-conflict profiles defined at e^k, respectively. We introduce two types of sequences of economies.

[17] Kojima and Pathak (2009) also study a sequence of expanding economies, constructed in a similar way to ours. However, they have a "null object", the option to be assigned to no school and in each economy, the number of schools preferred to the null object is bounded above by a constant. See also Immorlica and Mahdian (2005).

There are two different and plausible ways of defining a sequence of "expanding" economies. First, we let the number of students and the number of schools increase, the capacity of each school being kept fixed. This fixed number can be interpreted as a physical constraint such as a floor space: as the number of students increases, some new schools are built. In contrast, if the authority wants to keep the total number of schools fixed, so as to minimize administrative costs, for example, and if each school is able to augment its facilities to accommodate an increasing population, a sequence of economies with an increasing population can be defined as follows: the number of students and the capacity of each school increase, the number of schools begin kept fixed. Our conclusion is that in the limit, for either type of sequences of economies, the probability that the preference profile belongs to our maximal domain converges to zero. Although it is unpleasant news, this asymptotic impossibility result is in the spirit of earlier incompatibility result of fairness and efficiency (Balinski and Sömnez, 1999).

Appendix 2.1. A sequence of economies with increasing number of schools and constant capacities

We introduce a sequence of economies of increasing number of students and schools, each school's capacity being kept fixed. We say that $(e^k)_{k\in\mathbb{N}_+}$ is a **type-1 sequence of economies** if there is a constant $\bar{q} \in \mathbb{N}_+$ such that for each $k \in \mathbb{N}_+$, e^k satisfies conditions (1) to (5):

(1) $|N^k| = \bar{q}k$,
(2) $|A^k| = k$,
(3) for each $i \in N^k$, $P_i \in \Pi_0^k$ is drawn independently from U^k.
(4) for each $a \in A^k$, $q_a^k = \bar{q}$.
(5) for each $a \in A^k$, f_a is arbitrarily chosen from $\mathcal{F}^k$.

Condition (1) says that the number of students in e^k grows $\bar{q}$-times faster than the number of schools. Condition (2) says that the number of schools is exactly k. Condition (3) says that the event that each student has each preference in Π_0^k is equally likely, and there is no correlation in preferences between students. Condition (4) says that the capacities of schools is a constant. Last, condition (5) says that no restriction is placed on the priority profile.

Our next result is that as the number of students increases in type-1 sequences of economies, it becomes eventually impossible for the TTC rule to be *fair*.

Proposition 3. Let $(e^k)_{k\in\mathbb{N}_+} \equiv (A^k, N^k, q^k, f^k, P^k, q_k)_{k\in\mathbb{N}_+}$ be a type-1 sequence of economies. As k increases to infinity, the probability that $P^k \in \Pi(k)^{\text{iden}} \cup \Pi(k)^{\text{iden-nc}} \cup \Pi(k)^{\text{nc}}$ goes to zero.

Proof. Let $k \in \mathbb{N}_+$. To avoid triviality, let $k \geq 2$. For each pair $n, n' \in \mathbb{N}_+$ with $n \geq n'$, let $\left(\begin{array}{c} n \\ n' \end{array} \right) \equiv \frac{n!}{(n-n')!n'!}$. Then,

$$\begin{aligned}
\text{prob } (P^k \in \Pi(k)^{\text{iden}}) &= \tfrac{((k-1)!)^{\bar{q}k}}{(k!)^{\bar{q}k}} = \tfrac{1}{(k)^{\bar{q}k}} \\
\text{prob } (P^k \in \Pi(k)^{\text{nc}}) &= \tfrac{1}{(k!)^{\bar{q}k}} \left| \begin{array}{c} \text{the lists of students'} \\ \text{most preferred schools} \\ \text{that have no conflict at } q^k \end{array} \right| ((k-1)!)^{\bar{q}k} \\
&= \tfrac{1}{(k!)^{\bar{q}k}} \left(\begin{array}{c} \bar{q}k \\ \bar{q} \end{array} \right) \left(\begin{array}{c} \bar{q}k - \bar{q} \\ \bar{q} \end{array} \right) \left(\begin{array}{c} \bar{q}k - \bar{q}(k-1) \\ \bar{q} \end{array} \right) ((k-1)!)^{\bar{q}k} \\
&= \tfrac{1}{(k)^{\bar{q}k}} \tfrac{(\bar{q}k)!}{(\bar{q}!)^k}.
\end{aligned}$$

Last, consider $P^k \in \Pi(k)^{\text{iden-nc}}$. Then, there is $t \in \{2, \cdots, k-1\}$ such that each student has the same preferences over his most preferred school down to his $(t-1)$-th most preferred school, but students' t-th most preferred schools are not the same across students. We show that if k is large enough, then $\Pi(k)^{\text{iden-nc}} = \emptyset$. Note that for each $k \in \mathbb{N}_+$, the list of students' t-th most preferred schools at P^k have conflict at q^k: $|(A^k)^t(P)| \leq k - t + 1$, for each $a \in (A^k)^t(P)$, $q^k_a = \bar{q}$, and there are $k\bar{q}$ students. Then, there is $a \in (A^k)^t(P)$ such that $|N(a, P : (A^k)^t(P))| \geq q_a + 1$. For such a school a, $q_a \geq k\bar{q} - (t-1)\bar{q} - 1$, equivalently, $\bar{q} \geq k\bar{q} - (t-1)\bar{q} - 1$, which is impossible for a sufficiently large k. Altogether,

$$\text{prob } (P^k \in \Pi(k)^{\text{iden}} \cup \Pi(k)^{\text{nc}} \cup \Pi(k)^{\text{iden-nc}}) = \frac{1}{(k)^{\bar{q}k}} + \frac{1}{(k)^{\bar{q}k}} \frac{(\bar{q}k)!}{(\bar{q}!)^k}.$$

As k increases to infinity, this probability converges to zero. □

Appendix 2.2. A sequence of economies with increasing capacities and constant number of schools

We construct a different sequence of economies of increasing number of students and each school's capacity, the number of schools being kept fixed. We say that $(e^k)_{k \in \mathbb{N}_+}$ is a **type-2 sequence of economies** if there is a constant $\alpha \in \mathbb{N}_+$ such that for each $k \in \mathbb{N}_+$, e^k satisfies conditions $(1)'$ to $(5)'$:[18]

$(1)'$ $|N^k| = \alpha k$,
$(2)'$ $|A^k| = \alpha$,

[18]To avoid triviality, let $\alpha \geq 3$.

(3)′ for each $i \in N^k$, $P_i \in \Pi_0^k$ is drawn independently from U^k.
(4)′ for each $a \in A^k$, $q_a^k = k$.
(5)′ for each $a \in A^k$, f_a is arbitrarily chosen from $\mathcal{F}^k$.

Conditions (1)′, (3)′, and (5)′ are the same as (1) (3) and (3) of the previous section. Condition (2)′ says that the number of schools is a constant. Condition (4)′ says that the capacity of each school grows as the number of students.

The following result is that as the number of students increases in type-2 sequences of economies, it becomes eventually impossible for the TTC rule to be *fair*.

Proposition 4. Let $(e^k)_{k\in\mathbb{N}_+} \equiv (A^k, N^k, q^k, f^k, P^k, q^k)_{k\in\mathbb{N}_+}$ be a type-2 sequence of economies. As k increases to infinity, the probability that $P^k \in \Pi(k)^{\text{iden}} \cup \Pi(k)^{\text{iden-nc}} \cup \Pi(k)^{\text{nc}}$ goes to zero.

Proof. Let $k \in \mathbb{N}_+$. Then,

$$\begin{aligned}
\text{prob } (P^k \in \Pi(k)^{\text{iden}}) &= \tfrac{((\alpha-1)!)^{\alpha k}}{(\alpha!)^{\alpha k}} = \tfrac{1}{(\alpha)^{\alpha k}} \\
\text{prob } (P^k \in \Pi(k)^{\text{nc}}) &= \tfrac{1}{(\alpha!)^{\alpha k}} \left| \begin{array}{c} \text{the lists of students'} \\ \text{most preferred schools} \\ \text{that have no conflict } q^k \end{array} \right| ((\alpha-1)!)^{\alpha k} \\
&= \tfrac{1}{(\alpha!)^{\alpha k}} \begin{pmatrix} \alpha k \\ k \end{pmatrix} \begin{pmatrix} \alpha k - k \\ k \end{pmatrix} \begin{pmatrix} \alpha k - (\alpha-1)k \\ k \end{pmatrix} ((\alpha-1)!)^{\alpha k} \\
&= \tfrac{1}{(\alpha)^{\alpha k}} \tfrac{(\alpha k)!}{(k!)^{\alpha}}.
\end{aligned}$$

Last, consider $P^k \in \Pi(k)^{\text{iden-nc}}$. Then, there is $t \in \{2, \cdots, \alpha-1\}$ such that each student has the same preferences over his most preferred school down to his $(t-1)$-th most preferred school, but students' t-th most preferred schools are not the same across students. We show that $\Pi(k)^{\text{iden-nc}} = \emptyset$. Note that for each $k \in \mathbb{N}_+$, the list of students' t-th most preferred schools at P^k cannot have no conflict at q^k: $|(A^k)^t(P)| \leq \alpha - t + 1$, for each $a \in (A^k)^t(P)$, $q_a^k = k$, and there are αk students. Then, there is $a \in A^t(P)$ such that $|N(a, P : A^t(P))| \geq q_a + 1$. For such school a, $q_a \geq \alpha k - (t-1)k - 1$, equivalently, $k \geq \alpha k - (t-1)k - 1$, a contradiction. Altogether,

$$\text{prob } (P^k \in \Pi(k)^{\text{iden}} \cup \Pi(k)^{\text{nc}} \cup \Pi(k)^{\text{iden-nc}}) = \frac{1}{(\alpha)^{\alpha k}} + \frac{1}{(\alpha)^{\alpha k}} \frac{(\alpha k)!}{(k!)^{\alpha}}.$$

As k increases to infinity, this probability converges to zero. □

References

[1] Abdulkadiroğlu, A., P. A. Pathak, A. E. Roth, and T. Sönmez, 2006. "Changing the Boston School Choice Mechanism," mimeo.

[2] Abdulkadiroğlu, A., P. A. Pathak, and A. E. Roth, 2009."Strategy-Proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match," American Economic Review, vol. 99(5), pages 1954-78, December.

[3] Abdulkadiroğlu, A. and T. Sönmez, 2003. "School Choice: A Mechanism Design Approach," American Economic Review, American Economic Association, vol. 93(3), pages 729-747, June.

[4] Alcalde, J. and S. Barbera, 1994. "Top Dominance and the Possibility of Strategy-proof Stable Solutions to Matching Problems," Economic Theory, vol. 4, pages 417-435.

[5] Balinski, M. and T. Sönmez, 1999. "A Tale of Two Mechanisms: Student Placement," Journal of Economic Theory, Elsevier, vol. 84(1), pages 73-94, January.

[6] Barbera, S., H., Sonnenschein, and L. Zhou, 1991, "Voting by committees", Econometrica vol. 59, pages 595-609.

[7] Berga, D. and S. Serizawa, 2000. "Maximal domain for strategy-proof rules with one public good," Journal of Economic Theory, vol. 90, pages 39-61.

[8] Ching, S. and S. Serizawa, 1998. "Maximal domain for the existence of strategy-proof rules," Journal of Economic Theory vol. 78, pages 157-166.

[9] Ehlers, L, and A. Erdil, 2010. "Efficient assignment respecting priorities," Journal of Economic Theory, Elsevier, vol. 145(3), pages 1269-1282, May.

[10] Erdil. A. and H., Ergin, 2008. "What's the Matter with Tie-Breaking? Improving Efficiency in School Choice," American Economic Review, vol. 98(3), pages 669-89, June.

[11] Ergin, H. I., 2002. "Efficient Resource Allocation on the Basis of Priorities," Econometrica, Econometric Society, vol. 70(6), pages 2489-2497, November.

[12] Gale, D. and L.S. Shapley, 1962. "College admissions and the stability of marriage," American Mathematical Monthly, vol. 69, pages 9-15.

[13] Immorlica, N, and M., Mahdian, 2005. "Marriage, Honesty, and Stability," In *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, 53-62. Philadelphia: Society for Industrial and Applied Mathematics.

[14] Kesten, O., 2006. "On two competing mechanisms for priority-based assignment problems," Journal of Economic Theory, Elsevier, vol. 127(1), pages 155-171, March.

[15] Kojima, F. 2007. "When Can Manipulations be Avoided in Two-Sided Matching Markets? – Maximal Domain Results" The B.E. Journal of Theoretical Economics, vol. 7(1), article 32.

[16] Kojima, F. and P. A. Pathak, 2009. "Incentives and Stability in Large Two-Sided Matching Markets," American Economic Review, American Economic Association, vol. 99(3), pages 608-27, June.

[17] Massó, J. and A. Neme, 2001. "Maximal domain of preferences in the division problem," Games and Economic Behavior, vol. 37, pages 367-381.

[18] Thomson, W., 2007. "Consistent Allocation Rules," mimeo.

[19] Thomson, W., 2010, "Strategy-proof Allocation Rules," mimeo.