Laurence Wolsey’s Contributions to Integer Programming

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VERY EARLY YEARS

Who would have believed that

Would become one of the world’s leading researchers in optimization
EARLY YEARS

Laurence’s work on integer programming began at MIT
EARLY YEARS

His thesis was on the group relaxation

\[
\max cx \\
Ax = b \\
x_N \geq 0 \text{ integer} \\
x_B \text{ integer } \leftarrow \text{non-negativity dropped}
\]

Contributions:
1. Extension of the group idea to other relaxations
2. Generalization to mixed integer programming
3. Controlling group size
EARLY YEARS

Laurence spent one year on the faculty of Manchester University. In 1971 he received a CORE fellowship and then accepted a position at UCL at CORE and the Department of Mathematical Engineering.

Why did Laurence decide to accept a PERMANENT position at CORE?
Marguerite Loute Wolsey
MID 1970’s

• Valid Inequalities for Structured Problems
  – Covers and lifted covers from knapsacks
  – Facets for vertex packing problems
  – Lifting techniques
  – Superadditive lifting
Late 1970’s Early 1980’s

• Analysis of Heuristics for Combinatorial Optimization problems
  – Maximizing a submodular set function greedy algorithm gives a bound of \( \frac{e-1}{e} \)
  – Use of LP solutions in bound analysis

• Duality in integer programming
Mid 1980’s

• The lot-sizing work begins
  – Structured inequalities for MIPs
  – Convex hull for uncapacitated lot sizing
    (Barany and Van Roy)
  – Flow covers for MIPs
    (Padberg and Van Roy)
Laurence becomes the leader for structured MIP inequalities
INEQUALITIES

• The (l,S) inequalities for uncapacitated lot-sizing

\[ \sum_{i \in S} y_i + \sum_{i \in S} d_{i\ell} x_i \geq d_{0\ell} \]

\[ x_i \in \{0,1\} \]

for all \( i \leq \ell, \ell \in \{0,\ldots,T\}, S \subseteq \{0,\ldots,\ell\} \]

\[ d_{ij} = \sum_{k=i}^{j} d_k \]
• Flow cover inequalities

\[ N^- \begin{array}{c} \text{Flow in} = \text{flow out} \\
\sum_{i \in C} a_i - b = \lambda > 0 \\
C \text{ is a flow cover} \\
\text{Valid inequality} \\
\sum_{j \in C} \left\{ y_j + (a_j - \lambda)(1 - x_j) \right\} \leq b + \sum_{j \in N^-} \lambda x_j \end{array} \]

\[ N^+ \]
PRODUCTION PLANNING

• 1987 – Yves Pochet’s Ph.D. dissertation
  Lot-Sizing Problems: Reformulations and Cutting Plane Algorithms
  (Laurence’s first student)
  beginning a long list which is now around 20
• back logging
• start-up costs
• multi-item stronger cuts
• constant batches
• Wagner-Whitin costs
• capacities
• culminating in: Production Planning by Mixed Integer Programming
  by
  Pochet and Wolsey
  Springer 2006
LANCHESTER PRIZE 1989

COMPUTATION–I Structured Systems

- MPSARX – 1987 with Van Roy
  Math Programming System – Automatic Reformulation Executor
  A cut-and-branch solver with preprocessing and cuts that recognizes fixed charge and production lot-sizing type constraints. It used a branch-and-bound solver (SCICONIC) to solve LPs with specialized cuts and separation routine. When no further cuts are possible branch-and-bound is involved. Structured MIPs with up to 1000 0-1 variables.

- BC-PROD-2000 with Belvaux
  branch-and-cut code for lot-sizing
COMPUTATION-II General Systems

- bc-opt-1999 with Cordier, Laundy and Marchand
  General branch-and-cut system incorporating model interface routines (classifies and recognizes structure so that separation can be performed for given cut classes.)

Canonical Structures
- Integer knapsack
- Knapsack with a continuous variable
- 0-1 knapsack with GUBs
- Single node flow set
- Fixed charge path set
- Tableaux row
MIXED-INTEGER ROUNDDING I

- Generating all Cuts Recursively – 1990 with Nemhauser
  - extended Chvatal rounding to mixed-integer rows
  - Chvatal-Gomory fractional: MIR – Gomory mixed-integer

\[ x - y \leq \frac{5}{2} \]
\[ x - 2y \leq 2 \]
\[ x - y \leq \text{int.} + f \]
\[ x - \left( \frac{1}{1 - f} \right)y \leq \text{int.} \]
GENERALIZING 0-1 KNAPSACK CUTS

• 0-1 knapsack with one continuous variable – 1999 with Marchand

\[ \sum_j a_j x_j \leq b + s, \quad x_j \in \{0,1\} \]

\[ s \geq 0 \]

(by aggregation of continuous variables this is general for 0-1 MIPs)

The key here is taking advantage of superadditivity and different lifting orders to obtain families of inequalities.
MIXED-INTEGER ROUNDING-II

• Applying MIR to structured inequalities – 2001 with Marchand
  – Many families of strong, valid inequalities are MIR
    – network design inequalities (Magnanti et al)
    – mixed knapsack inequalities
    – weight inequalities (Martin and Weismantel)
  – Aggregation and separation of structured MIR inequalities
• Dynamic knapsack sets – 2003 with Loparic and Marchand

\[ a_1 x_1 + \cdots + a_t x_t \leq d_1 + \cdots + d_t + s, \quad t = 1, \ldots, T \]

• Generalizing simple MIR set – 2003 with Miller
SOME RECENT WORK-I

• Extended Formulations
  Several papers with Conforti, Miller, van Vyve…
  We get integrality in two basic ways
    – cuts
    – higher dimensional representations

  These papers study how to get strong inequalities using higher dimensional representations and projections
Extended formulations

Given

\[ X = \{ x \in \mathbb{R}^n : A'x \leq b', x_i \text{ integer, } i \in I \subseteq N \} \]

Find a dimension \( p \) and a polyhedron

\[ Q = \{ (x, z) \in \mathbb{R}^{n+p} : Ax + Bz \leq b \} \]

such that:

\[ \text{proj}_x (Q) = \text{conv}(X) \]

The linear program \( \max \{ hx + gz : Ax + Bz \leq b \} \) is easy to solve.
Mixed-integer vertex covers

\[ G = (V, E) : \text{bipartite graph}; I \subseteq V : \text{set of integer variables}; \]

\[ \mathcal{X}(G, b, I) \text{ set of mixed-integer vertex covers } x \in \mathbb{R}^V. \]

\[ x_i + x_j \geq b_{ij} \quad ij \in E \]

\[ x_i \geq 0 \quad i \in V \]

\[ x_j \text{ integer} \quad j \in I \]

There is an extended formulation whose size is polynomial in \( |V|, |E| \) and \( k \) (where \( k \) is the smallest number s.t. \( kb \) is integral). [Conforti, Di Summa, Eisenbrand, Wolsey]
SOME RECENT WORK-II

- Two row cuts and lattice free bodies

\[ x_1 = b_1 - \sum_j a_{1j} x_j \]
\[ x_2 = b_2 - \sum_j a_{2j} x_j \]

Strip (split) GMIC  triangle  quadrilateral
Two row cuts and lattice free bodies
  – To get the convex hull of a 2 row system you only need split inequalities and intersection cuts arising from triangles and quadrilaterals in $R^2$ (Anderson, Louveaux, Weismantel and Wolsey)

  – To use these minimal inequalities to obtain strong cuts, it is necessary to lift the integer (nonbasic) variables. Dey and Wolsey do this for lattice free triangles.
THE GRAND CHALLENGE

• Identify an area of integer programming to which Laurence Wolsey has NOT contributed

• Thank you