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Constrained infinite group relaxations of MIPs

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Abstract

Recently minimal and extreme inequalities for continuous group relaxations of general mixed integer sets have been characterized. In this paper, we consider a stronger relaxation of general mixed integer sets by allowing constraints, such as bounds, on the free integer variables in the continuous group relaxation. We generalize a number of results for the continuous infinite group relaxation to this stronger relaxation and characterize the extreme inequalities when there are two integer variables.

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1 Introduction

Following on from the pioneering work of Gomory and Johnson [15, 16] and Johnson [18] in the 1970's there has been renewed interest in the last few years in studying infinite group relaxations of mixed integer programs (MIPs). Various efforts to strengthen inequalities obtained from group relaxations have also been made, such as adding bounds on the variables or taking into account the integrality of some of the variables. To describe these directions of work more precisely and also to see where this paper fits in, we first describe a generic relaxation of a simplex tableau where the basic variables are constrained to be integral.

Definition 1.1 ($R(f, S, W, G)$) *Let $R(f, S, W, G)$ be the set of points $x \in \mathbb{Z}^m$ together with functions $y : W \to \mathbb{R}_+$ and $z : G \to \mathbb{Z}_+$ which satisfy*

$$\begin{aligned} x &= f + \sum_{w \in W} w y(w) + \sum_{u \in G} u z(u), \\ x &\in S := \{v \in \mathbb{Z}^m \mid Av \leq b\}, \\ 0 &\leq y(w) \leq T(w) \, \forall w \in W, \\ 0 &\leq z(u) \leq U(u), \, z(u) \in \mathbb{Z}_+, \, \forall u \in G, \\ & \quad y, z \text{ have finite support}, \end{aligned}$$

where $W \subseteq \mathbb{R}^m$, $G \subseteq \mathbb{R}^m$, $A \in \mathbb{Q}^{q \times m}$ and $b \in \mathbb{Q}^{q \times 1}$, $f \in (\mathbb{Q}^m \backslash \mathbb{Z}^m) \cap conv(S)$, $T(w) \in \mathbb{R}_+ \cup \{+\infty\}$ $\forall w \in W$, and $U(u) \in \mathbb{R}_+ \cup \{+\infty\}$ $\forall u \in G$.

The basic variables, the continuous and integer nonbasic variables, and the right-hand-side of the simplex tableau are modeled by x, y and z, and f in the relaxation $R(f, S, W, G)$ respectively.

When $S = \mathbb{Z}^m$ and the bounds T and U are set to $+\infty$, $R(f, S, W, G)$ is the mixed integer group relaxation. When in addition $G = \emptyset$, it is the continuous group relaxation that has been recently addressed in Andersen et al. [2], Borozan and Cornuéjols [9], Cornuéjols and Margot [12], and Zambelli [23]. A lattice-free set is a set $K \subseteq \mathbb{R}^m$ such that $\text{int}(K) \cap \mathbb{Z}^m = \emptyset$. It has been shown that for the continuous infinite group relaxation ($W = \mathbb{R}^m$) there is a close relationship between maximal lattice-free convex sets (convex sets that are lattice-free and maximal wrt to this property) and minimal valid inequalities (undominated inequalities, see Section 2). In two dimensions, maximal lattice-free polyhedra generating extreme inequalities for continuous group relaxations are well understood; see Andersen et al. [2] and Cornuéjols and Margot [12]. Very recently Andersen et al. [1] have considered the effect of adding bounds on the continuous nonbasic variables y and

Dey and Wolsey [13] have examined how inequalities can be strengthened when some of the nonbasic variables are integer.

Observe that constraints on the basic variables x that define S can be rewritten as constraints on the nonbasic variables y and z. Burdet and Johnson [11] present an algorithmic approach for solving an optimization problem over the set $R(f, \mathbb{Z}^m, \emptyset, G)$ with additional constraints on the nonbasic variables. Johnson [19] added constraints on the set of basic variables. In [19], he considered the case where S is a finite set and derived properties of facet-defining inequalities for $R(f, S, W, \emptyset)$. Here we pursue a closely related case. More explicitly, we restrict the basic variables x to a set $S \subseteq \mathbb{Z}^m$ where $S = P \cap \mathbb{Z}^m$ may be finite or infinite, P is a polyhedron and $\text{conv}(S)$ is full-dimensional. One reason for allowing $S \subsetneq \mathbb{Z}^m$ and not necessarily finite is to allow for the fact that the basic integer variables are typically non-negative but are not necessarily bounded.

In this paper, we address whether and how the recent results linking maximal lattice-free convex sets and extreme inequalities for the continuous group relaxation extend to the case where $S \subsetneq \mathbb{Z}^m$. Natural questions that arise are:

1. Given a function $\pi : \mathbb{R}^m \to \mathbb{R}$ associated with a minimal valid inequality of the form $\sum_{w \in \mathbb{R}^m} \pi(w)y(w) \geq 1$ for $R(f, S, \mathbb{R}^m, \emptyset)$, the set $P(\pi) = \{w \in \mathbb{R}^m \mid \pi(w - f) \leq 1\}$ is a maximal S-free convex set (i.e. a convex set with no point of S in its interior and maximal wrt to this property; see Section 2 for formal definitions). This is proven by Basu et al. [7]. Thus maximal S-free convex sets form a natural extension of the maximal lattice-free convex sets. Are maximal S-free convex sets polyhedra, like maximal lattice-free convex sets? Under some technical conditions we show that maximal S-free convex sets are polyhedra with one point of S in the relative interior of each facet (See Appendix 1).

2. Maximal lattice-free polyhedra in m-dimensions have at most 2^m facets. Can we bound the number of facets of maximal S-free convex sets? In Section 3 we show that the maximum number of facets of a maximal S-free convex set K is $2^m - t$ where t is the 'order' of K wrt to a formulation of the set S. If $S = \mathbb{Z}^m$, the order of maximal lattice-free convex sets is 0 and therefore this result generalizes the result for maximal lattice-free polyhedra.

3. Given a maximal lattice-free polyhedron with f in its interior, there exists a unique minimal inequality for the continuous infinite group relax-

ation corresponding to it. Does there exist such a relationship between maximal S-free convex sets and minimal inequalities for $R(f, S, \mathbb{R}^m, \emptyset)$? In Section 4 we address this question. Given a maximal S-free polyhedron K, we construct a minimal inequality π such that $K = P(\pi)$ and show that there exists no other inequality π' such that $\pi' \neq \pi$ and $P(\pi') = K$.

4. For the case of two integer variables, what shapes do 'interesting' maximal S-free polyhedra take or which sets lead to extreme valid inequalities? This question is addressed in Section 5. We show that there are two families of maximal S-free polyhedra (which are not lattice-free) that lead to extreme inequalities for $R(f, S, \mathbb{R}^m, \emptyset)$. We note here that Johnson [19] proves that these inequalities are extreme when W and S are finite. The result in this section establishes the converse, i.e. it is sufficient to consider maximal S-free polyhedra of these two families to yield all extreme inequalities for $R(f, S, W, \emptyset)$.

We illustrate the points made above with an example.

Example 1.1 *Let* $f = (0.5, 0.5)$ *and let* $S = \{(x_1, x_2) \in \mathbb{Z}^2 \,|\, 0 \leq x_1 \leq 1,\, x_2 \geq 0\}$. *It can be verified (Section 4) that the function*

$$\pi(w) = \begin{cases} 1.5w_1 + 0.5w_2 & \text{if } w_2 \geq 0,\ 3w_1 \geq w_2 \\ -4.5w_1 + 2.5w_2 & \text{if } w_2 \geq 2w_1,\ w_2 \geq 3w_1 \\ 1.5w_1 - 0.5w_2 & \text{if } w_2 \leq 0, 2w_1 \geq w_2 \end{cases} \tag{1}$$

yields a valid inequality for $R(f, S, \mathbb{R}^2, \emptyset)$ *of the form* $\sum_{w \in \mathbb{R}^2} \pi(w)y(w) \geq 1$. *This can be used to generate a valid inequality for a mixed integer set such as:*

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} y_1 + \begin{pmatrix} -1 \\ -2 \end{pmatrix} y_2 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} y_3 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} y_4 \\ x_1 \in \{0, 1\},\ x_2 \in \mathbb{Z}_+ \\ y_1, y_2, y_3, y_4 \geq 0. \end{aligned} \tag{2}$$

Let r^i *denote the column corresponding to* y_i *above. Then* $\sum_{i=1}^4 \pi(r^i)y_i \geq 1$ *is the inequality*

$$3y_1 - 0.5y_2 + 0.5y_3 + 1.5y_4 \geq 1,$$

which is a facet-defining for (2) and cuts off the solution $y = \bar{0}$ *and* $x = (0.5, 0.5)^T$. *Observe that*

- *'Group inequalities', i.e., inequalities obtained when $R(f, S, W, G)$ is the group relaxation, have only non-negative coefficients. On the other hand, $\pi(w)$ takes negative values for some w.*

- *The set $P(\pi) := \{w \in \mathbb{R}^m \,|\, \pi(w - f) \leq 1\} = \{(w_1, w_2) \in \mathbb{R}^2 | -9w_1 + 5w_2 \leq 0, 3w_1 + w_2 \leq 4, 3w_1 - w_2 \leq 3\}$ is illustrated in Figure 1. Notice that it is an S-free convex set, i.e., it does not contain any point of S in its interior. It can be verified that it is a maximal S-free convex set since it is a polyhedron and contains one point of S in the relative interior of each facet.*

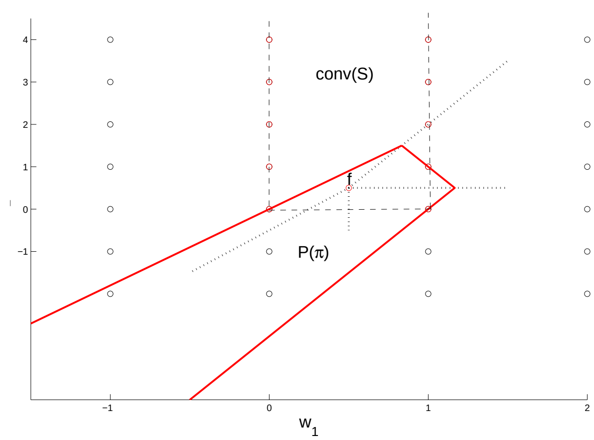

Figure 1: Example of $P(\pi)$.

- *In Section 4 it will be shown that given a maximal S-free polyhedron K with f in its interior, there exists a unique function $\pi^K : \mathbb{R}^m \to \mathbb{R}$ such that $P(\pi^K) = K$ and π^K is an minimal inequality for $R(f, S, \mathbb{R}^m, \emptyset)$. Thus, π described by (1) is a minimal inequality for $R(f, S, \mathbb{R}^2, \emptyset)$.*

- *$P(\pi)$ has three facets. It will be shown that when $m = 2$, all 'interesting' maximal S-free polyhedra (that are not lattice-free) have at most three facets.*

- *The constraints on the x variables are $0 \leq x_1 \leq 1$ and $x_2 \geq 0$. It can be verified that if we relax S to be the set defined by the constraint $x_2 \geq 0$, i.e., we let $S' : \{(x_1, x_2) \in \mathbb{Z}^2 \,|\, x_2 \geq 0\}$, then π is still a valid inequality for $R(f, S', \mathbb{R}^2, \emptyset)$. Informally, the maximum number of inequalities* *defining S that are 'critical' in the validity of π will be defined as the order of the S-free convex set. In Section 3 it will be shown that the number of facets of maximal S-free polyhedron is bounded from above by 2^m less the order of the inequality wrt a formulation of S. Moreover, in Section 5.2 it will be shown that almost all 'interesting' facets in the case of two rows have an order of 1, i.e., their validity depends on at most one constraint defining S.*

2 Valid Inequalities and Maximal S-free Convex Sets

We consider the relaxation $R(f, S, W, G)$ where $G = \emptyset$, $W \subseteq \mathbb{R}^m$, $S \subseteq \mathbb{Z}^m$, and $T(w) = +\infty\ \forall w \in W$. Since $G = \emptyset$, we use the symbol $R(f, S, W)$ to represent $R(f, S, W, G)$. In the case where $W = \mathbb{R}^m$ we use the symbol $R(f, S)$.

Any valid inequality for $R(f, S, W)$ that cuts off the fractional point $x = f$ and $y = \bar{0}$, can be scaled and rewritten as $\sum_{w \in W} \pi(w)y(w) \geq 1$. Instead of considering all valid inequalities for $R(f, S, W)$, we focus our attention on this sub-class of valid inequalities. We next formally define valid inequalities, a hierarchy of 'strong' valid inequalities (similar to those studied for the group relaxation) and a set $P(\pi) \subseteq \mathbb{R}^m$ corresponding to any valid inequality π that plays an important role in analyzing the strength of the inequality. The definition of inequalities and their hierarchy were introduced for the group problem in Gomory and Johnson [17] and that of $P(\pi)$ for the continuous group problem was defined in Borozan and Cornuéjols [9].

Definition 2.1 (Valid, Minimal, and Extreme Inequalities)

1. *The function $\pi : W \to \mathbb{R}$ is a valid inequality for $R(f, S, W)$ if*

$$\sum_{w \in W} \pi(w)y(w) \geq 1 \quad \forall y \in R(f, S, W).$$

The terms 'valid inequality' and 'valid function' are used interchangeably.

2. *Let $P(\pi) \subseteq \mathbb{R}^m$ be the set $P(\pi) = \{w \in \mathbb{R}^m \,|\, \pi(w - f) \leq 1\}$.*

3. *A valid function π is minimal if there exists no valid function π' for $R(f, S, W)$ such that $\pi' \neq \pi$ and $\pi' \leq \pi$.*

4. *A valid function π is extreme if there do not exist two valid functions π^1 and π^2 for $R(f, S, W)$ such that $\pi^1 \neq \pi^2$ and $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$.*

Proposition 2.1 below presents some properties of minimal inequalities for $R(f, S)$. The proof of Proposition 2.1 is similar to that in Borozan and Cornuéjols [9] (also see Johnson [19]). We first present a definition.

Definition 2.2 (Maximal S-free Convex Set) *A convex set $K \subset \mathbb{R}^m$ is a maximal S-free convex set if int$(K) \cap S = \emptyset$ and there exists no convex set K' such that int$(K') \cap S = \emptyset$ and $K' \supsetneq K$.*

Note here that when $S = \mathbb{Z}^m$, a maximal S-free convex set is called a maximal lattice-free convex set (Lovász [20]).

Proposition 2.1 (Minimality $\Rightarrow$) *If $\pi : \mathbb{R}^m \to \mathbb{R}$ is a minimal function for $R(f, S)$, then*

1. *$\pi(\bar{0}) = 0$, π is positively homogenous, subadditive, and convex,*
2. *$P(\pi)$ is a S-free convex set,*
3. *$f \in P(\pi)$.*

In fact the following stronger result can be proven: if $\pi : \mathbb{R}^m \to \mathbb{R}$ is a minimal function for $R(f, S)$, then $P(\pi)$ is a maximal S-free convex set. A very elegant proof of this fact is given by Basu et al. [7].

3 Properties of Maximal S-free Convex Sets

In Appendix 1, it is shown that under some technical condition a maximal S-free convex set is a polyhedron. More precisely the following statement is verified in Appendix 1: Let K be an S-free convex set with the following properties: (1) $K \cap \text{conv}(S)$ is full-dimensional (2) if rec.cone$(K \cap \text{conv}(S)) \supsetneq \{0\}$, then there exists $d^1, ..., d^t \in \mathbb{Z}^m$ such that $d^1, ..., d^t \in$ rec.cone$(K \cap \text{conv}(S))$ and $\text{lin}\{d^1, ..., d^t\} = \text{lin}(\text{rec.cone}(K \cap \text{conv}(S)))$. Then K is a maximal S-free convex set if and only if it is a polyhedron that contains at least one point of S in the relative interior of each facet. Basu et al. [7] recently showed that the above mentioned technical conditions are not necessary (see also Basu et al. [6]).

Figure 2 illustrates some maximal S-free convex sets in two dimensions ($S = \{(x_1, x_2) \in \mathbb{Z}^2 \,|\, x_2 \leq -3\}$).

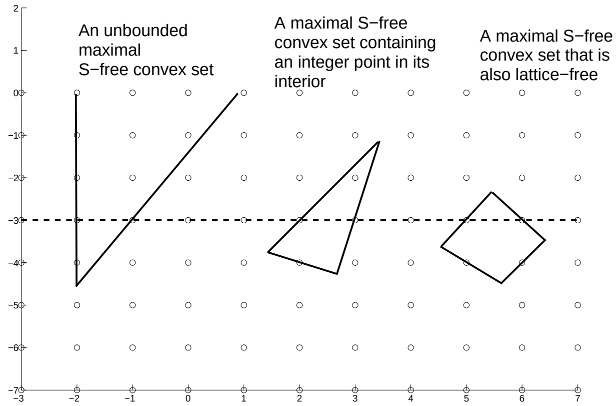

Figure 2: Some maximal S-free convex sets.

The maximum number of inequalities required to define an m-dimensional maximal lattice-free convex set is 2^m (Doignon [14], Bell [8], Scarf [22]). Next we present a bound on the number of facets of a maximal S-free polyhedron in m-dimensions, when it contains one or more integer points in its interior. Informally, the bound is obtained as follows: Add linear inequalities to the description of K sequentially, until it becomes maximal lattice-free. Then the bound on the number of facets of K is 2^m less the number of inequalities added to the description of K. The linear inequalities we add to the description of K are parallel to the linear inequalities used to describe S.

Before presenting this result, we introduce some notation.

Definition 3.1 (Formulation, Critical Subset, and Order) *Let K be an S-free polyhedron. A polyhedral set $P \subseteq \mathbb{R}^m$ is called a 'formulation' for S if $P \cap \mathbb{Z}^m = S$ where $P = \cap_{1 \leq j \leq c} P^j$ and $P^j = \{x \in \mathbb{R}^m \,|\, (a^j)^T x \leq b^j\}$ ($a^j \in \mathbb{Z}^{m \times 1}$ and $b^j \in \mathbb{Z}$). For any subset $\mathcal{J}$ of $\{1, ..., c\}$, denote $S^{P,\mathcal{J}} = (\cap_{j \in \mathcal{J}} P^j) \cap \mathbb{Z}^m$. A subset $\mathcal{J} \subseteq \{1, ..., c\}$ is critical if*

1. *K is $S^{P,\mathcal{J}}$-free, and*
2. *For each $j \in \mathcal{J}$ (if $\mathcal{J}$ is nonempty), $\exists\ p \in int(K) \cap \mathbb{Z}^m$ such that $(a^j)^T p > b^j$ and $(a^k)^T p \leq b^k\ \forall k \in \mathcal{J} \setminus \{j\}$.*

If K is an S-free convex set, P is a formulation of S, and t is the cardinality of the largest critical subset of $\{1, ..., c\}$, then K is of order t with respect to P.

Condition (1.) in Definition 3.1 implies that if we remove all the linear inequalities describing S except those in the set $\mathcal{J}$, K still remains S-free (since $S \subseteq S^{P,\mathcal{J}}$). Condition (2.) in Definition 3.1 implies that each linear inequality $(a^j)^T x \leq b^j$ in the set $\mathcal{J}$ is necessary to maintain the $S^{P,\mathcal{J}}$-free status of K, since there exist integer points in the interior of K that are infeasible for $S^{P,\mathcal{J}}$ where the infeasibility is solely due to the j^{th} inequality in the set $\mathcal{J}$. Together conditions (1.) and (2.) imply that if we remove all the inequalities in $\{1, ..., c\} \setminus \mathcal{J}$ the set K remains $S^{P,\mathcal{J}}$-free, but removing additional inequalities does not preserve this property.

Example 3.1 *1. Consider P, S, and K as defined below.*

$$\begin{aligned} P &:= \{(x_1, x_2) \in \mathbb{R}^2 | x_2 \geq 2, x_1 + x_2 \leq 4\}, \quad S = P \cap \mathbb{Z}^2 \\ K &:= \{(x_1, x_2) \in \mathbb{R}^2 | 2.25x_1 + x_2 \geq 4.25, 2.75x_1 + x_2 \leq 5.75\} \end{aligned}$$

(See Figure 3) Then K is a maximal S-free convex set of order 2 wrt to the formulation P.

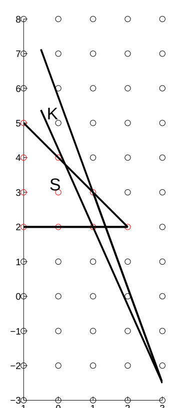

Figure 3: Order of K is 2 wrt P.

2. It is possible to have two subsets $\mathcal{J}^1, \mathcal{J}^2 \subseteq \{1, ..., c\}$ such that both $\mathcal{J}^1$

and $\mathcal{J}^2$ are critical. Let

$$
\begin{array}{rcl}
P & := & \left\{ (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{rcl} x_1 & \leq & 0.5 \\ x_1 + x_2 & \leq & 1.5 \\ x_1 - x_2 & \leq & 0.5 \end{array} \right\} \\
S & := & P \cap \mathbb{Z}^2 \\
K & := & \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 0\}
\end{array}
$$

Then observe that the subsets $\mathcal{J}^1 = \{1\}$ and $\mathcal{J}^2 = \{2, 3\}$ of $\{1, 2, 3\}$ are critical.

Observation 3.1 *The order of an S-free convex set with respect to a formulation P is well-defined.*

Proof: Since the number of subsets of $\{1, ..., c\}$ is finite, it is sufficient to verify that there exists a critical subset of $\{1, ..., c\}$. Let $\mathbb{J} = \{\mathcal{J} \subseteq \{1, ..., c\} \,|\, K \text{ is } S^{P,\mathcal{J}} - \text{free}\}$. Clearly $\mathbb{J}$ is non-empty since K is S-free. Let $J^1 \in \mathbb{J}$ be such that if $J^2 \in \mathbb{J}$, then $J^2 \subsetneq J^1$. By construction J^1 satisfies condition (1.) of Definition 3.1. If J^1 is the empty set, then the proof is complete. Otherwise, as no proper subset of J^1 belongs to $\mathbb{J}$, on removing any one constraint $(a^j)^T x \leq b^j$ from those in J^1, K is not $S^{P,J\setminus\{j\}}$-free. Therefore there exists $p^k \in \text{int}(K) \cap \mathbb{Z}^m$ such that $(a^j)^T p^k > b^j$ and $(a^k)^T p^k \leq b^k$ $\forall k \in J^1$. Therefore J^1 satisfies condition (2.) of Definition 3.1. $\square$

Next we present a lemma that illustrates that we can obtain maximal lattice-free convex sets by sequentially adding linear inequalities to the description of K.

Lemma 3.1 *Let K be a maximal S-free polyhedron of order t wrt formulation P where $t \geq 1$. Let c inequalities describe P and let $\mathcal{J} = \{1, ..., j\}$ be a critical subset of $\{1, ..., c\}$ of maximal cardinality. Let $Q^1 = \{q \in int(K) \cap \mathbb{Z}^m \,|\, (a^1)^T q > b^1, (a^k)^T q \leq b^k \,\forall k \in \mathcal{J} \setminus \{1\}\}$ and $\hat{b}^1 = min\{(a^1)^T x \,|\, x \in Q^1\}$. Set $K^1 := K \cap \{x \in \mathbb{R}^m \,|\, (a^1)^T x \leq \hat{b}^1\}$ and $\bar{S}^1 := S^{P,\mathcal{J}\setminus\{1\}}$. Then,*

1. *K^1 is a maximal m-dimensional $\bar{S}^1$-free convex set of order at least $t - 1$ wrt the polyhedron defined by the inequalities in the set $\mathcal{J} \setminus \{1\}$*

2. *The number of facets of K^1 is one more than the number of facets of K.*

3. *If $p \in S \cap bnd(K)$, then $p \in bnd(K^1)$.*

Proof: For simplicity take $S := S^{P,\mathcal{J}}$ and take P to be the polyhedron described by the inequalities in the set $\mathcal{J}$. Note that $Q^1 \neq \emptyset$ by definition of $\mathcal{J}$. Also note that by the well-ordering of the integers $\hat{b}^1 = \min\{(a^1)^T x \,|\, x \in Q^1\}$ exists and $\hat{b}^1 > b^1$.

1. Claim 1: K^1 is an $\bar{S}^1$-free convex set. Note that the set of integer points in $\mathrm{int}(K^1)$ is a subset of $(\mathrm{int}(K) \cap \mathbb{Z}^m) \setminus Q^1$. By definition of Q^1, if $q \in (\mathrm{int}(K) \cap \mathbb{Z}^m) \setminus Q^1$, there exists $j \neq 1$ such that $(a^j)^T q > b^j$. This proves that K^1 is a $\bar{S}^1$-free convex set.
   Claim 2: K^1 is a maximal $\bar{S}^1$-free convex set: By Proposition 6.3, to verify that K^1 is a maximal $\bar{S}^1$-free convex set, we verify the following:

   (a) The set
   $$K^1 \cap \{x \in \mathbb{R}^m \,|\, (a^1)^T x = \hat{b}^1\}$$
   is a facet of K^1 and there exists a $q \in \bar{S}^1$ that lies in the relative interior of this facet: Let K be the set $(g^i)^T x \leq h^i$ for $1 \leq i \leq z$. Notice from the definition of $\hat{b}^1$ that there exists at least one integer point p belonging to Q^1 that satisfies the inequality $(a^1)^T x \leq \hat{b}^1$ at equality. Since by definition of Q^1, $p \in \mathrm{int}(K)$, we obtain that
   $$\begin{aligned}(g^i)^T p < h^i \,\forall 1 \leq i \leq z\\ (a^1)^T p = \hat{b}^1.\end{aligned}$$
   Thus p is a point on the boundary of K^1 and there is exactly one inequality that defines K^1 that is satisfied at equality by p. Therefore this inequality $(a^1)^T x \leq \hat{b}^1$ is facet-defining and p lies in the relative interior of this facet.

   (b) All the other facets of K^1 are the facets of K that contain integer points belonging to $\bar{S}^1$ in their relative interior: Note that $\bar{S}^1 \supseteq S$. Let K be the set $(g^i)^T x \leq h^i$ for $1 \leq i \leq z$. Consider the u^{th} facet of K, $(g^u)^T x \leq d^u$. Since K is a maximal S-free polyhedron, every facet of K has a point $p \in S$ in its relative interior, i.e.,
   $$\begin{aligned}(g^u)^T p = h^u\\ (g^v)^T p < h^v, v \neq u.\end{aligned}$$
   Since $p \in S$, we have $(a^1)^T p \leq b^1$. However, since by definition $\hat{b}^1 > b^1$, one has,
   $$(a^1)^T p < \hat{b}^1.$$

Thus p belongs to the boundary of K^1 and satisfies exactly one inequality that defines K^1 at equality. Therefore the inequality $(g^u)^T x \leq h^u$ represents a facet of K^1 and $p \in \bar{S}^1$ belongs to the relative interior of this facet of K^1.

Claim 3: K^1 is a maximal $\bar{S}^1$-free convex set of order at least $t-1$: Notice that addition of the inequality $(a^1)^T x \leq \hat{b}^1$ to the description of K does not cut off any of the integer points belonging to $Q^j = \{q \in \text{int}(K) \,|\, (a^j)^T q > b^j, (a^k)^T q \leq b^k \,\forall k \in \mathcal{J} \setminus \{j\}\}$ from the interior of K^1. These sets are non-empty by the definition of $\mathcal{J}$. This proves the result.

2. This follows from the proof of part (1.).

3. This follows from the proof of part (1.). □

Proposition 3.1 *If K is a maximal m-dimensional S-free polyhedron of order t wrt to any formulation P, then K has at most $2^m - t$ facets.*

Proof: By repeating the procedure described in Lemma 3.1 t times we obtain a maximal lattice-free convex set. Since maximal lattice-free convex sets have at most 2^m facets we obtain the result. □

Corollary 3.1 *Every full-dimensional maximal S-free polyhedron in m dimensions has a maximum order of 2^m wrt any formulation of S.*

Corollary 3.2 *If K is a full-dimensional maximal S-free polyhedron in m dimensions with at least one integer point in its interior, then it has at most $2^m - 1$ facets.*

4 Construction of Minimal and Extreme Inequalities for $R(f, S)$

We begin by discussing the construction of minimal inequalities using maximal S-free polyhedra. We use $K - f$ to denote the set $\{x - f \,|\, x \in K\}$.

Construction 4.1 (Construction of Valid inequality: π^K)

1. *Let $K \subset \mathbb{R}^m$ be a maximal S-free polyhedron such that $K \cap conv(S)$ is full-dimensional and K contains f in its interior.*

2. *Let* $K^* = \{w \,|\, w^T x \leq 1 \,\forall x \in K - f\}$ *be the polar of* $K - f$. *Let* $K^o = \{w \in K^* \,|\, \exists x \in K - f \; s.t. \, w^T x = 1\} \subseteq K^*$.

3. *Define the function* $\pi^K : \mathbb{R}^m \to \mathbb{R}$ *as* $\pi^K(u) = sup_{g \in K^o}\{(g)^T u\}$. *Since* $K - f$ *can be written as a set* $\{x \,|\, (g^j)^T x \leq 1, j \in \{1, ..., l\}\}$ *(as* $\bar{0}$ *belongs to the interior of* $K - f$ *and* K *is polyhedral),* π^K *can be written as*

$$\pi^K(u) = max_{1 \leq j \leq l}\{(g^j)^T u\}.$$

The natural interpretation of the function π^K is that it is the inequality obtained after applying the disjunction $\vee_{1 \leq j \leq l}(R^L(f,S) \wedge \{(x,y) \,|\, (g^j)^T x \geq 1\})$, where $R^L(f,S)$ is the linear programming relaxation of $R(f,S)$ (Balas [4]). We next prove that π^K is a minimal inequality for $R(f,S)$, and that given a maximal S-free polyhedron K containing f in its interior, this is the unique way to construct a minimal valid inequality using K. We note here that Johnson [19] gave the construction of π^K starting directly from conv(K^o).

To compare these inequalities with respect to group inequalities, observe that the set K^o is a subset of the polar of K. π^K being the support function of K^o has a value at most as large as the gauge function of K (since the gauge function of K is the support function of the polar of K). In particular, if int(rec.cone(K)) is empty (which happens for example when $S = \mathbb{Z}^m$ and $R(f,S)$ is the group relaxation), the gauge function of K is equal to π^K. Then the function π^K is the intersection cut presented in Balas [3] (See also Burdet [10]). On the other hand if int(rec.cone(K)) $\neq \emptyset$, then π^K is strictly stronger than the gauge function of K and takes negative values for points in the interior of the recession cone of K.

Proposition 4.1 (π^K is Minimal) *Let* $K \subset \mathbb{R}^m$ *be a maximal S-free polyhedral set which contains* f *in its interior. Then*

1. π^K *is a valid function for* $R(f,S)$,

2. $P(\pi^K) := \{x \,|\, \pi^K(x - f) \leq 1\} = K$,

3. π^K *is a minimal valid function for* $R(f,S)$,

4. *If* π' *is any minimal valid function such that* $P(\pi') = K$, *then* $\pi' = \pi^K$.

Proof:

1. Proof of validity of π^K: Note first that π^K being a support function is positively homogenous and subadditive (see Rockafellar [21]). Now the

validity is proven by showing that if $w^o \in S$, then $\pi^K(w^o - f) \geq 1$: As $w^o - f \notin \text{int}(K - f)$, by the separation theorem there exists a half-space $\alpha^T x \leq \beta$ such that $\alpha^T(w^o - f) = \beta$ and $\alpha^T x \leq \beta\ \forall x \in K - f$. Let $\beta^0 = \max\{\alpha^T x | x \in K - f\}$. As $\bar{0} \in \text{int}(K - f)$, $\beta^0 > 0$. Also $\beta^0 \leq \beta$. Then setting $\widehat{\alpha} = \frac{\alpha}{\beta^0}$ we obtain that $\widehat{\alpha} \in K^o$ and $\widehat{\alpha}^T(w^o - f) \geq 1$. Therefore, $\pi^K(w^o - f) \geq \widehat{\alpha}^T(w^0 - f) \geq 1$.

2. $P(\pi^K) = K$:

   - $P(\pi^K) \supseteq K$: We need to show that if $w \in K$, then $\pi^K(w - f) \leq 1$. By definition of K^o, if $u \in K^o$ then $u^T(w - f) \leq 1$. Thus, $\pi^K(w - f) = \max_{g^j \in K^o}\{(g^j)^T(w - f)\} \leq 1$.
   - $P(\pi^K) \subseteq K$: We prove that if $w \notin K$, then $\pi^K(w - f) > 1$. By the separation theorem there exists a half-space $\alpha^T x \leq \beta$ such that $\alpha^T(w - f) = \beta$ and $\alpha^T x < \beta\ \forall x \in K - f$. Let $\beta^0 = \max\{\alpha^T x | x \in K - f\}$. As $\bar{0} \in \text{int}(K - f)$, $\beta^0 > 0$. Also $\beta^0 < \beta$. Then setting $\widehat{\alpha} = \frac{\alpha}{\beta^0}$, we obtain that $\widehat{\alpha} \in K^o$ and $\widehat{\alpha}^T(w - f) > 1$.

3. π^K is minimal: Assume by contradiction that π^K is not minimal. Let π'' be a valid function such that $\pi'' < \pi^K$. Now it is possible to construct a function $\pi' \leq \pi''$ such that π' is valid, π' is subadditive and positively homogenous and $P(\pi')$ is an S-free convex set; see one proof in Basu et al. [6]. Since both π' and π^K (by construction) are positively homogenous, we obtain that $P(\pi') \supseteq P(\pi^K) = K$. Since $P(\pi')$ is an S-free convex set and K is a maximal S-free convex set, we obtain that $P(\pi') = K$.
   Claim 1: If $w \in (\mathbb{R}^m \setminus \text{rec.cone}(K - f))$, then $\pi'(w) = \pi^K(w) = \lambda$ where $f + \frac{w}{\lambda} \in \text{bnd}(K)$. (This is proven in Borozan and Cournuéjols [9]. We present this part for completeness): Let $w \in \mathbb{R}^m \setminus \text{rec.cone}(K - f)$. Since both π' and π^K are positively homogenous, it is sufficient to compare these functions on the boundary of $K - f$. WLOG assume that $(g^1)^T w = 1$ and $(g^j)^T w \leq 1\ \forall j \neq 1$. Then $\pi^K(w) = \max_j\{(g^j)^T w)\} = 1$. If $\pi'(w) < 1$, then there exists $\lambda > 1$ such that $\pi'(\lambda w) = 1$. Therefore $\lambda w \in K - f$. Since $P(\pi') = K$, this is a contradiction as $(g^1)^T(\lambda w) > 1$.
   Claim 2: If u belongs to a face of the recession cone of $K - f$, then $\pi'(u) = 0 = \pi^K(u)$. Since $u \in \text{rec.cone}(K - f)$, $(g^j)^T u \leq 0\ \forall 1 \leq j \leq l$. WLOG assume that $(g^1)^T u = 0$. Then $\pi^K(u) = \max_j\{(g^j)^T u)\} = 0$. By Claim 1, $\pi'(w) = \pi^K(w)$ for all $w \in \text{bnd}(K - f)$. For $\epsilon > 0$, and w' on the face $\{w \,|\, (g^1)^T w = 1\} \cap (K - f)$, we have $w' + \epsilon u$

also belongs to the face $\{w \,|\, (g^1)^T w = 1\}$ (since $(g^1)^T(w' + \epsilon u) = (g^1)^T w' = 1$). Since π' is subadditive and positively homogenous, $\pi'(w') + \epsilon\pi'(u) \geq \pi'(w' + \epsilon u)$ or $\pi'(u) \geq \frac{1}{\epsilon}(\pi'(w' + \epsilon u) - \pi'(w')) = \frac{1}{\epsilon}(\pi^K(w' + \epsilon u) - \pi^K(w')) = \frac{1}{\epsilon}(1 - 1) = 0$. Thus $\pi'(u) = 0$.
Claim 3: If $v \in \text{int}(\text{rec.cone}(K - f))$, then $\pi'(v) = \pi^K(v)$. WLOG assume that $\pi^K(v) = (g^1)^T v < 0$ and $(g^j)^T v \leq (g^1)^T v \; \forall j \neq 1$. Let u be a vector on the face $\{x \,|\, (g^1)^T x = 1\} \cap (K - f)$ or $(g^1)^T u = 1$ and $(g^j)^T u \leq 1 \; \forall j \neq 1$. Let $\gamma = \frac{(g^1)^T u}{-(g^1)^T v}$. Then $g^1(u + \gamma v) = 0$ and $g^j(u + \gamma v) \leq 0 \; \forall j \neq 1$. Thus $u + \gamma v \in \text{bnd}(\text{rec.cone}(K - f))$. By the previous claims, $\pi'(v + \gamma v) = 0$ and $\pi'(u) = 1$. Since π' is subadditive and positively homogeneous $\pi'(u) + \gamma\pi'(v) \geq \pi'(u + \gamma v)$, we obtain $\pi'(v) \geq -\frac{1}{\gamma} = \pi^K(v)$.

4. Let π' be a minimal valid inequality such that $P(\pi') = K$. Then π' is subadditive and positively homogeneous. Using these conditions, observe that the proof of (3.) establishes that $\pi' = \pi^K$.

$\square$

Next we consider the question of extremality of the inequalities π^K. In Proposition 4.4 below we construct a finite set $\mathcal{T} \subset \mathbb{R}^m$ and show that π^K is extreme for $R(f, S)$ if and only if π^K restricted to $\mathcal{T}$ is extreme for $R(f, S, \mathcal{T})$. Cornuéjols and Margot [12] present a similar result for the case where K is a maximal lattice-free convex set in $\mathbb{R}^2$.

We first observe that extreme inequalities must be minimal. The proof relies on the non-negativity of the y variables. See Gomory and Johnson [15] for a proof.

Proposition 4.2 (Extreme $\Rightarrow$ Minimal) *If π is an extreme function for $R(f, S)$, then it is minimal for $R(f, S)$.*

Thus it is sufficient to consider only minimal functions to obtain extreme inequalities for $R(f, S)$. Next we present a preliminary result that is used in the proof of Proposition 4.4. See Gomory and Johnson [15] for a proof.

Proposition 4.3 *If π is a minimal function for $R(f, S)$ and there exist valid functions $\pi_1, \pi_2 : \mathbb{R}^m \to \mathbb{R}$ such that $\pi_1 \neq \pi_2$ and $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$, then π_1 and π_2 are minimal.*

Given a function $\pi : \mathbb{R}^m \to \mathbb{R}$, we denote the restriction of π to the domain $\mathcal{T} \subseteq \mathbb{R}^m$ by $\pi|_{\mathcal{T}}$. The next definition presents the notation used for the construction of the finite subset $\mathcal{T}$ of $\mathbb{R}^m$.

Definition 4.1 *Given $K - f = \{x \in \mathbb{R}^m \mid (g^j)^T x \leq 1, j \in \{1, ..., l\}\}$, define the cone $C^j = \{x \in \mathbb{R}^m \mid (g^j - g^k)^T x \geq 0 \, \forall k \neq j\}$ (If K is a half-space, then set $C^1 := \mathbb{R}^m$). Let V^j be a finite set of generators for C^j.*

Observe that $\pi^K|_{C^j}(u) = (g^j)^T u$.

Proposition 4.4 (Finite $\Leftrightarrow$ Infinite) *Let K be a full-dimensional maximal S-free polyhedron and let $\mathcal{T} := \cup_{1 \leq j \leq l} V^j$, where V^j is a finite set of generators for C^j. Then $\pi^K : \mathbb{R}^m \to \mathbb{R}$ is an extreme inequality for $R(f, S)$ if and only if $\pi^K|_{\mathcal{T}} : \mathcal{T} \to \mathbb{R}$ is an extreme inequality for $R(f, S, \mathcal{T})$.*

Proof: $\Rightarrow$ Let $\pi^K : \mathbb{R}^m \to \mathbb{R}$ be an extreme inequality for $R(f, S)$. Assume by contradiction that $\pi^K|_{\mathcal{T}} : \mathcal{T} \to \mathbb{R}$ is not extreme. So there exist valid inequalities $\tilde{\pi}_1, \tilde{\pi}_2 : \mathcal{T} \to \mathbb{R}$ for $R(f, S, \mathcal{T})$ such that $\tilde{\pi}_1 \neq \tilde{\pi}_2$ and $\pi^K|_{\mathcal{T}} \geq \frac{1}{2}\tilde{\pi}_1 + \frac{1}{2}\tilde{\pi}_2$. Construct the function $\pi_1 : \mathbb{R}^m \to \mathbb{R}$ as follows

$$\pi_1(u) = \min_{C^j \ni u} \left\{ \begin{array}{cl} \min & \sum_{w^i \in V^j} \lambda_i \tilde{\pi}_1(w^i) \\ \text{s.t.} & \sum_{w^i \in V^j} \lambda_i w^i = u \\ & \lambda_i \geq 0 \, \forall i. \end{array} \right.$$

Observe that π_1 (and π_2 defined similarly using $\tilde{\pi}_2$) is well-defined for every $u \in \mathbb{R}^m$ as given any $u \in \mathbb{R}^m$, $u \in C^{j*}$ where $j^* \in \text{argmax}_j\{(g^j)^T u\}$.

Claim: $\pi_1 : \mathbb{R}^m \to \mathbb{R}$ is a valid inequality for $R(f, S)$. For any $w \in \mathbb{R}^m$, let $\lambda^w \in \mathbb{R}^{|\mathcal{T}|}_+$ be such that $\pi_1(w) = \sum_{w^i \in \mathcal{T}} \lambda^w_i \tilde{\pi}_1(w^i)$ and $\sum_{w^i \in \mathcal{T}} \lambda^w_i w^i = w$. Let $\bar{y} \in R(f, S)$. Therefore $\sum_{w \in \mathbb{R}^m} w\bar{y}(w) + f \in S$, or equivalently $\sum_{w \in \mathbb{R}^m} (\sum_{w^i \in \mathcal{T}} \lambda^w_i w^i)\bar{y}(w) + f \in S$. Thus setting $\tilde{\lambda} = \sum_{w \in \mathbb{R}^m} \bar{y}(w)\lambda^w$, we obtain $\sum_{w^i \in \mathcal{T}} \tilde{\lambda}_i w^i + f \in S$. Hence $\tilde{\lambda} \in R(f, S, \mathcal{T})$. Since by definition $\tilde{\pi}_1$ is a valid inequality for $R(f, S, \mathcal{T})$, we obtain $\sum_{w^i \in \mathcal{T}} \tilde{\pi}_1(w^i)\tilde{\lambda}_i \geq 1$ or equivalently $\sum_{w \in \mathbb{R}^m} \pi_1(w)\bar{y}(w) \geq 1$.

Now we verify that $\pi^K \geq \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$ to obtain a contradiction to the fact that π^K is extreme: Choose any $u \in \mathbb{R}^m$ and let $\lambda \in \mathbb{R}^{|V_j|}_+$ be such that $u = \sum_{w^i \in V^j} \lambda_i w^i$. Note that since the function π^K is linear in the cone C^j, we obtain that $\pi^K(u) = (g^j)^T u = \sum_{w^i \in V^j} \lambda_i((g^j)^T w^i) = \sum_{w^i \in V^j} \lambda_i(\pi^K|_{\mathcal{T}}(w^i))$. Also observe that $\pi_1(u) \leq \sum_{w^i \in V^j} \lambda_i \tilde{\pi}_1(w^i)$. Thus we obtain,

$$\begin{aligned} \frac{1}{2}\pi_1(u) + \frac{1}{2}\pi_2(u) &\leq \sum_{w^i \in V^j} \lambda_i(\frac{1}{2}\tilde{\pi}_1(w^i) + \frac{1}{2}\tilde{\pi}_2(w^i)) \\ &\leq \sum_{w^i \in V^j} \lambda_i(\pi^K|_{\mathcal{T}}(w^i)) \\ &= \pi^K(u). \end{aligned}$$

Finally in order to complete the proof, we need to verify that $\pi_1 \neq \pi^K$: Observe that $\pi_1|_{\mathcal{T}} \leq \tilde{\pi}_1$ and $\pi_2|_{\mathcal{T}} \leq \tilde{\pi}_2$. Since $\pi^K|_{\mathcal{T}} \geq \frac{1}{2}\tilde{\pi}_1 + \frac{1}{2}\tilde{\pi}_2$ and $\tilde{\pi}_1 \neq \tilde{\pi}_2$, there exists $u \in \mathcal{T}$ such that $\tilde{\pi}_1(u) < \pi^K(u)$. Thus $\pi_1(u) < \pi^K(u)$.

$\Leftarrow$ Let $\pi^K|_{\mathcal{T}} : \mathcal{T} \to \mathbb{R}$ be an extreme inequality for $R(f, S, \mathcal{T})$. Assume by contradiction that π^K is not extreme. So there exist valid functions $\pi_1, \pi_2 : \mathbb{R}^m \to \mathbb{R}$ such that $\pi_1 \neq \pi_2$ and $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. Note that $\pi_1|_{\mathcal{T}} = \pi_2|_{\mathcal{T}} = \pi^K|_{\mathcal{T}}$. Since π^K is minimal, π_1 and π_2 are minimal. Therefore π_1 and π_2 are subadditive and positively homogenous. Consider any $u \in \mathbb{R}^m$. The point u can be written as a conic combination of the vectors $w^i \in c^j$ for some j, i.e. let $u = \sum_{w^i \in C^j} \lambda_i w^i$. Thus we obtain that $\pi_1(u) = \pi_1(\sum_{w^i \in C^j} \lambda_i w^i) \leq \sum_{w^i \in C^j} \lambda_i \pi_1(w^i) = \sum_{w^i \in C^j} \lambda_i \pi|_{\mathcal{T}}(w^i) = \pi^K(u)$. This is a contradiction since minimality of π^K implies that $\pi_1 = \pi^K$. $\square$

We end this section with a discussion on the construction of π^K. Notice that we have allowed only the case where $f \in \text{int}(K)$ instead of $f \in K$. It turns out that when we consider $R(f, S, W)$ with W is a finite set, it is possible to construct an S-free polyhedron that contains f in its interior to generate any valid cut. This result is proven for intersection cuts based on two-dimensional maximal lattice-free convex sets in Cornuéjols and Margot [12] and for general lattice-free convex sets in Zambelli [23]. Although the proof is modified here, the key ideas are similar. Notice also that we may need to redefine S appropriately.

Proposition 4.5 *Let $W = \{r^1, r^2, ..., r^l\} \subset \mathbb{Q}^m$. Let $\sum_{i=1}^{l} \alpha_i y_i \geq 1$ be a valid inequality for $R(f, S, W)$ where $\alpha_i \in \mathbb{Q}\ \forall i \in \{1, ..., l\}$. Let S satisfy the condition that $\forall \bar{x} \in S$, $\exists \bar{y} \in \mathbb{R}^W_+$ such that $x = f + W\bar{y}$. Then there exists an S-free convex polyhedron K such that $f \in int(K)$ and $\pi^K(r^i) \leq \alpha_i$.*

Proof: Given a valid inequality $\sum_{i=1}^{l} \alpha_i y_1 \geq 1$ for $R(f, S, W)$, the set $P_\alpha = \{x \in \mathbb{R}^m \,|\, \exists y \geq 0 \text{ s.t. } x = f + \sum_{i=1}^{l} r^i y_i, \sum_{i=1}^{l} \alpha_i y_i \leq 1\}$ is an S-free polyhedral set. Let $P_\alpha = \{x \,|\, (g^j)^T x \leq h^j, 1 \leq j \leq c\}$ where $g^j \in \mathbb{Q}^{m \times 1}$ and $h^j \in \mathbb{Q}$ and each inequality satisfies at least one point of P_α at equality, i.e., there are no redundant inequalities.

Claim 1: If $h^j - (g^j)^T f > 0$, then $\frac{(g^j)^T (r^i)}{h^j - (g^j)^T f} \leq \alpha_i$.

1. $\alpha_i > 0$: Then observe that the point $\bar{x} = f + \frac{r^i}{\alpha_i} \in P_\alpha$. Therefore $(g^j)^T(f + \frac{r^i}{\alpha_i}) \leq h^j$ or $\frac{(g^j)^T (r^i)}{h^j - (g^j)^T f} \leq \alpha_i$.

2. $\alpha_i = 0$: Observe first that if $\alpha_i = 0$, then all points of the form $f + \lambda r^i$, $\lambda \geq 0$ belong to P_α. Therefore $(g^j)^T(f + \lambda r^i) \leq h^j\ \forall \lambda \geq 0$. This implies that $(g^j)^T (r^i) \leq 0$. Therefore $\frac{(g^j)^T (r^i)}{h^j - (g^j)^T f} \leq 0 = \alpha_i$.

3. $\alpha_i < 0$: Similar to the previous part, it can be verified that $(g^j)^T(r^i) \leq 0$ whenever $\alpha_i \leq 0$. Now observe that there must exist some $e \in \{1,...,l\}$ such that $(g^j)^T(r^e) > 0$. If not, then for all points in P_α we have $(g^j)^T(f + \sum_{k=1}^{l} y_k r^k) \leq (g^j)^T f < h^j$. This will make the inequality $(g^j)^T x \leq h^j$ redundant, a contradiction. Now observe that $\alpha_e > 0$. If not, then $f + \lambda r^e \in P_\alpha \; \forall \lambda \geq 0$. However for suitably large λ, $(g^j)^T(f + \lambda r^e) > h^j$ which contradicts the fact that $(g^j)^T x \leq h^j$ is valid for P_α. Now observe that if $(g^j)^T(f + \sum_{i=1}^{l} r^i \bar{y}_i) = h^j$ for some $\bar{y} \in \mathbb{R}^l_+$ such that $\sum_{k=1}^{l} \alpha_k \bar{y}_k \leq 1$ (by non-redundancy of inequalities, $\bar{y}$ exists), then $\sum_{k=1}^{l} \alpha_k \bar{y}_k = 1$. If not, then a point of the form $f + \sum_{k=1}^{l} r^k \bar{y}_k + \epsilon r^e$ belongs to P_α for a suitably small positive ϵ. However, $(g^j)^T(f + \sum_{k=1}^{l} r^k \bar{y}_k + \epsilon r^e) > h^j$, a contradiction. Let $\mu = 1 - \alpha_i$. Then observe that $\mu \sum_{k=1}^{l} \alpha_k \bar{y}_k + \alpha_i = 1$. Therefore the point $f + \sum_{k=1}^{l} r^k(\mu \bar{y}_k) + r^i \in P_\alpha$. Therefore $(g^j)^T(f + \mu \sum_{k=1}^{l} r^k \bar{y}_k + r^i) \leq h^j$ or $\frac{(g^j)^T(r^i)}{h^j - (g^j)^T f} \leq 1 - \mu = \alpha_i$.

Claim 2: If $(g^j)^T f = h^j$, then $\forall \, \bar{x} \in S$, $\exists k \neq j$ such that $(g^k)^T \bar{x} \geq h^k$. Choose a point belonging to P_α of the form $f + \epsilon r^i \in P_\alpha$ where $\epsilon > 0$ for some $1 \leq i \leq l$. Then $(g^j)^T(f + \epsilon r^i) \leq h^j$ or $(g^j)^T(r^i) \leq 0 \; \forall i$. By assumption if $\bar{x} \in S$, then $\bar{x} \in f + \text{cone}(W)$. Therefore $(g^j)^T(\bar{x}) = (g^j)^T(f + \sum_{i=1}^{l} \lambda_i r^i) \leq (g^j)^T f = h^j$. As P_α is S-free, if $(g^j)^T \bar{x} < h^j$, then there must exist some $k \neq j$ such that $(g^k)^T \bar{x} \geq h^k$. Next we verify that if $(g^j)^T(\bar{x}) = h^j$ then $\exists k \neq j$ such that $(g^k)^T \bar{x} \geq h^k$. Assume by contradiction that there exists a point $\bar{x} \in S$ in the relative interior of the face $P_\alpha \cap \{x \,|\, (g^j)^T x = h^j\}$. Then there is a point of the form $f + \lambda(\bar{x} - f) \in P_\alpha \cap \{x \,|\, (g^j)^T x = h^j\}$ where $\lambda > 1$. However note that since $\bar{x} \in S$, $\min\{\sum_{i=1}^{l} \alpha_i y_i \,|\, f + \sum_{i=1}^{l} r^i y_i = \bar{x}, y_i \geq 0\} \geq 1$. Thus, $\min\{\sum_{i=1}^{l} \alpha_i y_i \,|\, f + \sum_{i=1}^{l} r^i y_i = f + \lambda(\bar{x} - f), y_i \geq 0\} > 1$ or $f + \lambda(\bar{x} - f) \notin P_\alpha$, a contradiction.

By Claim 2 whenever $(g^j)^T f = h^j$, it is possible to drop the constraint $(g^j)^T x \leq h^j$ from the description of P_α and the resulting set remains S-free. Thus let K be the set defined by the inequalities defining P_α only when $(g^j)^T f < h^j$. Now it follows from the definition of π^K that

$$\pi^K(r^i) = \max_{\{j | (g^j)^T f < h^j\}} \left\{ \frac{(g^j)^T(r^i)}{h^j - (g^j)^T f} \right\}.$$

Thus the result follows from Claim 1. $\square$

5 Maximal S-free Convex Sets in $\mathbb{R}^2$ yielding Extreme Inequalities for $R(f, S)$

We now consider the specific case of two rows. There are two main considerations. The first is whether maximum lattice-free convex sets yield extreme inequalities for $R(f, S)$ when $S \neq \mathbb{Z}^2$. This is dealt with in Section 5.1. The second is a classification of maximal S-free convex sets that yield extreme inequalities and contain an integer point in their interior. This is treated in Section 5.2.

5.1 Maximal Lattice-free Convex Sets in $\mathbb{R}^2$

Maximal S-free convex sets in $\mathbb{R}^2$ of order 0 are maximal lattice-free convex sets. See Lovász [20] for a classification of these sets. Cornuéjols and Margot [12] present the subset of maximal lattice-free convex sets that generate extreme inequality for $R(f, S)$ when $S = \mathbb{Z}^2$. The next proposition shows that their result holds for any maximal S-free polytope which is also lattice-free.

Proposition 5.1 *Let K be a full-dimensional maximal S-free polytope which is also lattice-free. Then π^K is extreme for $R(f, S)$ if and only if π^K is extreme for $R(f, \mathbb{Z}^2)$.*

Proof: $\Rightarrow$ Assume by contradiction that π^K is extreme for $R(f, S)$ and π^K is not extreme for $R(f, \mathbb{Z}^2)$. Then there exist two different functions π_1 and π_2 valid for $R(f, \mathbb{Z}^2)$ such that $\pi^K = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. However if π_1 and π_2 are valid for $R(f, \mathbb{Z}^2)$, then π_1 and π_2 are valid for $R(f, S)$, thus contradicting the extremality of π^K for $R(f, S)$.

$\Leftarrow$ Assume now that π^K is extreme for $R(f, \mathbb{Z}^2)$. Then $P(\pi^K)$ is either a maximal lattice-free triangle or a maximal lattice-free quadrilateral satisfying the ratio condition (see Cornuéjols and Margot[12]). In either case let a^1, a^2, ..., a^c ($c \leq 4$) be the vertices of $P(\pi^K)$. Consider the problem $R(f, \mathbb{Z}^2, \mathcal{T})$:

$$x = f + \sum_{1 \leq i \leq c} r^i y_i$$
$$x \in \mathbb{Z}^2, y_i \in \mathbb{R}_+ \, \forall i \in \{1, ..., c\},$$

where $r^i = a^i - f$. Since K is a maximal S-free polytope, there is at least one integer point in the relative interior of each facet of K belonging to S.

Let p^j, $1 \leq j \leq c$ be the integer points (belonging to S) in the relative interior of the facets of K. For both the cases in which $P(\pi^K)$ is a triangle or a quadrilateral satisfying a ratio condition, Cornuéjols and Margot [12] (see proofs of Theorem 3.8, 3.10) show that there exist c points $y^j \in \mathbb{R}^c_+$, such that

1. $\sum_{1 \leq i \leq c} \pi^K(r^i) y^j_i = 1 \; \forall 1 \leq j \leq c$,
2. $p^j = f + \sum_{1 \leq i \leq c} r^i y^j_i \; \forall 1 \leq j \leq c$,
3. The matrix $[y^1, y^2, ..., y^c]$ has a rank c.

Since the points p^j belong to S as well, we have that y^1, ..., y^c are feasible points for the problem $R(f, S, \mathcal{T})$ and satisfy π^K at equality. Thus, from (3.) above, there cannot exist two vectors $\pi_1 \neq \pi_2$ such that $(\pi_1)^T y^j = 1$ $\forall 1 \leq j \leq c$ and $\pi^K|_{\mathcal{T}} = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. Thus, $\pi^K|_{\mathcal{T}}$ is extreme for $R(f, S, \mathcal{T})$. Therefore by Proposition 4.4, π^K is extreme for $R(f, S)$. $\square$

Next consider the case where K is an unbounded maximal lattice-free convex set, in which case K is a split set. In this case we obtain the following result.

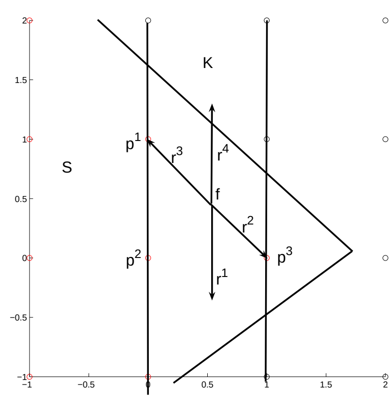

Figure 4: Maximal S-free split set with at least two integer points on one facet generate extreme inequalities.

Proposition 5.2 *Let $K := \{(x_1, x_2) \in \mathbb{R}^2 \,|\, a_0 \leq a_1x_1 + a_2x_2 \leq a_0 + 1\}$ where g.c.d$(a_1, a_2) = 1$, and let K be a maximal S-free convex set. Then π^K is extreme for $R(f, S)$ if and only if at least one of the facets of K contains two points belonging to S in its relative interior.*

Proof: $\Rightarrow$ Let K have at least two integral points on one facet and one integral point on the other facet. Let p^1 and p^2 be two different integer points on one facet and p^3 be the integer point on the second facet of K. (See Figure 4 for an example). Consider the finite problem $R(f, S, \mathcal{T})$

$$x = f + r^1y_1 + r^2y_2 + r^3y_3 + r^4y_4$$
$$x \in S,\ y_i \geq 0\,\forall i \in \{1, ..., 4\},$$

where r^1 is the vector $(a_2, -a_1)$, r^2 is the vector $p^3 - f$, r^3 is the vector $p^1 - f$, and r^4 is the vector $(-a_2, a_1)$. The two cones C^1 and C^2 corresponding to the two facets of K are generated by r^1, r^2, r^4 and r^1, r^3, r^4 respectively. Now note that $\pi^K(r^1) = 0$, $\pi^K(r^2) = 1$, $\pi^K(r^3) = 1$, and $\pi^K(r^4) = 0$. Also note that $p^2 = f + \alpha r^1 + r^3$ where $\alpha > 0$, $p^1 = f + r^3$, $p^3 = f + r^2$, and $p^2 = f + (1 + \alpha)r^1 + r^3 + r^4$. Therefore the following points are satisfied at equality for the inequality $\pi^K(r^1)y_1 + \pi^K(r^2)y_2 + \pi^K(r^3)y_3 + \pi^K(r^4)y_4 \geq 1$:

1. $(y_1, y_2, y_3, y_4) := (\alpha, 0, 1, 0)$
2. $(y_1, y_2, y_3, y_4) := (0, 1, 0, 0)$
3. $(y_1, y_2, y_3, y_4) := (0, 0, 1, 0)$
4. $(y_1, y_2, y_3, y_4) := (1 + \alpha, 0, 1, 1)$.

Note that as $\alpha > 0$, the four points above are linearly independent. Thus the inequality $\pi^K(r^1)y_1 + \pi^K(r^2)y_2 + \pi^K(r^3)y_3 + \pi^K(r^4)y_4 \geq 1$ cannot be written as a convex combination of two different inequalities valid for $R(f, S, \mathcal{T})$. Therefore by Proposition 4.4, π^K is extreme for $R(f, S)$.

$\Leftarrow$ Let there be exactly one integer point on each facet, namely p^1 and p^2. Then $K \cap \text{conv}(S)$ is bounded since otherwise rec.cone(conv(S)) $\cap$ rec.cone$(K) \neq \emptyset$, which implies that at least one of the directions $(a_2, -a_1)$, $(-a_2, a_1)$ belongs to the recession cone of conv(S). Then there must exist at least two integer points on each of the facets of K belonging to S.

Since $K \cap \text{conv}(S)$ is bounded it can be verified that there exists $\epsilon_1 > 0$ such that for all $0 \leq \epsilon < \epsilon_1$ the set $K_1(\epsilon) := \{(x_1, x_2) \in \mathbb{R}^2 \,|\, (a_1 + \epsilon)p^1_1 + (a_2 - \epsilon)p^1_2 \leq (a_1 + \epsilon)x_1 + (a_2 - \epsilon)x_2 \leq (a_1 + \epsilon)p^2_1 + (a_2 - \epsilon)p^2_2\}$ is S-free. Similarly there exists $\epsilon_2 > 0$ such that for all $0 \leq \epsilon < \epsilon_2$ the set $K_2(\epsilon) := \{(x_1, x_2) \in \mathbb{R}^2 \,|\, (a_1 - \epsilon)p^1_1 + (a_2 + \epsilon)p^1_2 \leq (a_1 - \epsilon)x_1 + (a_2 + \epsilon)x_2 \leq (a_1 - \epsilon)p^2_1 + (a_2 + \epsilon)p^2_2\}$ is S-free. Let $\epsilon^0 = \frac{1}{2}\min\{\epsilon_1, \epsilon_2\}$ and set $K_1 := K_1(\epsilon^0)$ and $K_2 := K_2(\epsilon^0)$. Now it can be verified that $\pi^K = \frac{1}{2}\pi^{K_1} + \frac{1}{2}\pi^{K_2}$. $\square$

5.2 Maximal S-free Convex Set with at least One Integer Point in its Interior

We next analyze the case where K is a full-dimensional S-free convex set such that $K \cap \mathrm{conv}(S)$ is bounded[1] with at least integer point belonging to its interior. By Corollary 3.2, K has at most three facets.

We prove the following result in this section.

Theorem 5.1 (Classification) *Let K be a full-dimensional maximal S-free convex set in $\mathbb{R}^2$ with at least one integer point in its interior and let $K \cap conv(S)$ be bounded.*

1. *Order of K:*
   - *(a) If π^K is extreme for $R(f, S)$, then the order of K is at most 2.*
   - *(b) If K is not a half-space and π^K is extreme for $R(f, S)$, then the order of K is at most 1.*
2. *Number of facets of K:*
   - *(a) If K is a half-space, then π^K is extreme for $R(f, S)$ if and only if bnd(K) contains at least two points belonging to S.*
   - *(b) If K has two facets, then π^K is extreme for $R(f, S)$ if and only if one of the facets of K contains at least two points belonging to S.*
   - *(c) If K has three facets, then π^K is extreme for $R(f, S)$.*

We analyze the three cases based on the number of facets of K in the next three subsections, thereby proving Theorem 5.1.

5.2.1 K has one facet

Proposition 5.3 ((2a.) of Theorem 5.1) *If K is an S-free half-space in $\mathbb{R}^2$, then π^K is extreme for $R(f, S)$ if and only if bnd(K) contains at least two points belonging to S.*

Proof: $\Rightarrow$ Let the two integer points belonging to $S \cap \mathrm{bnd}(K)$ be p^1 and p^2. Let $r^1 = p^1 - f$, $r^2 = p^2 - f$ and r^3 be any vector of the form $-\lambda_1 r^1 - \lambda_2 r^2$

[1] This is essentially a method to incorporate assumption (2) of Proposition 6.3. We do not consider the case where this condition is absent.

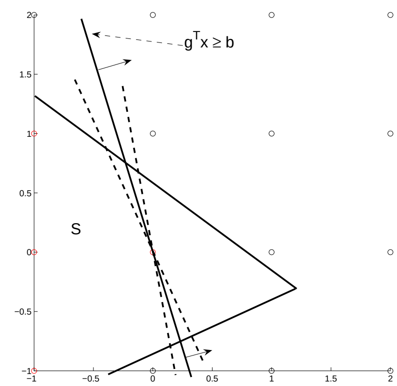

Figure 5: A one facet maximal S-free convex set with one tight integer point on its boundary does not generate an extreme inequality.

where $\lambda_1, \lambda_2 > 0$. Note that since $f \notin \text{bnd}(K)$, we obtain that r^1, r^2, and r^3 span $\mathbb{R}^2$ (This is C^1). Consider the mixed integer set

$$\begin{aligned} & x = f + r^1 y_1 + r^2 y_2 + r^3 y_3 \\ & x \in S,\, y_i \geq 0 \,\forall i \in \{1, 2, 3\}. \end{aligned} \tag{3}$$

By Proposition 4.4, if the inequality

$$\pi^K(r^1)y_1 + \pi^K(r^2)y_2 + \pi^K(r^3)y_3 \geq 1 \tag{4}$$

is extreme for (3), then π^K is extreme for $R(f, S)$. The following points are satisfied at equality for (4):

1. $(y_1, y_2, y_3) := (1, 0, 0)$
2. $(y_1, y_2, y_3) := (0, 1, 0)$
3. $(y_1, y_2, y_3) := (2, 1, \lambda_1 + \lambda_2)$

As the above three points are linearly independent, (4) is an extreme inequality for (3).

$\Leftarrow$ Let K be the set $\{x \in \mathbb{R}^2 \,|\, g^T x \geq b\}$. By assumption there is only one integer point $p = (p_1, p_2)$ belonging to S on the boundary of K. Since by assumption $K \cap \text{conv}(S)$ is bounded, there exists $\epsilon > 0$ such that the set $\{x \in \mathbb{R}^2 \,|\, (g^1)^T x \geq b\}$ and $\{x \in \mathbb{R}^2 \,|\, (g^2)^T x \geq b\}$ are also S-free where $g^1 := g + \epsilon(-p_2, p_1)$, $g^2 := g - \epsilon(-p_2, p_1)$, and $\epsilon > 0$. (See Figure 5 for an example). Now the result follows. $\square$

Proposition 5.4 ((1a.) of Theorem 5.1) *Let K be a full-dimensional S-free convex set in $\mathbb{R}^2$ with at least one integer point in its interior such that $K \cap conv(S)$ is bounded. If π^K is extreme for $R(f,S)$, then the order of K is at most 2.*

Proof: By Lemma 3.1, a maximal S-free polyhedron that contains at least one integer point in its interior has an order of at most 3. If K is of order 3, then it must be a half-space. Using Lemma 3.1, it is possible to add 3 hyperplanes to the description of K to make it into a maximal lattice-free quadrilateral $\tilde{K}$. Also the integer points belonging to S that are tight on the boundary of K remain tight at the boundary of $\tilde{K}$. However, since $\tilde{K}$ is a maximal lattice-free quadrilateral, $\tilde{K}$ has only one integer point on the boundary of each facet (see Lovász [20]). Therefore bnd(K) contains only one integer point belonging to S in the relative interior of each facet. Now by Proposition 5.3, the result follows. □

5.2.2 K has two facets

The proof of the next proposition is similar to that of Proposition 5.2.

Proposition 5.5 ((2b.) of Theorem 5.1) *Let K be a full-dimensional maximal S-free convex set in $\mathbb{R}^2$ with two facets. Then π^K is extreme for $R(f,S)$ if and only if one of the facets of K contains at least two integer points belonging to S.*

Proposition 5.6 ((1b.) of Theorem 5.1) *Let K be a full-dimensional maximal S-free convex set in $\mathbb{R}^2$ with at least one point belonging to S in its interior such that $K \cap conv(S)$ is bounded. If K is not a half-space and π^K is extreme for $R(f,S)$, then the order of K is at most 1.*

Proof: Since K is not a hyperplane, K has two or three facets. It K has three facets, it must be of order 1. Otherwise, by Lemma 3.1 we obtain that if K is order of 2, it is possible to add 2 inequalities to the description of K to make it into a maximal lattice-free quadrilateral $\tilde{K}$. Also the integer points belonging to S that are tight on the boundary of K remain tight at the boundary of $\tilde{K}$. However, since $\tilde{K}$ is a maximal lattice-free quadrilateral, $\tilde{K}$ has only one integer point on the boundary of each facet (see Lovász [20]). Therefore bnd(K) contains only one integer point belonging to S in the relative interior of each facet. Now the result follows from Proposition 5.5. □

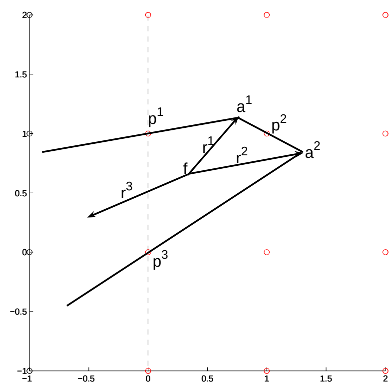

Figure 6: Three facet maximal S-free convex sets generate extreme inequalities.

5.2.3 K has three facets

Proposition 5.7 ((2c.) of Theorem 5.1) *Let K be a three-facet maximal S-free convex set. Then π^K is an extreme function for $R(f, S)$.*

Proof: There are two cases: K is bounded or unbounded. Consider the unbounded case first.

Let $K - f = \{x \in \mathbb{R}^2 \,|\, (g^1)^T x \leq 1, (g^2)^T x \leq 1, (g^3)^T x \leq 1\}$ and let r^1 and r^2 be vertices of $K - f$ satisfying $(g^1)^T r^1 = (g^2)^T r^1 = 1$ and $(g^2)^T r^2 = (g^3)^T r^2 = 1$. Let r^3 be a ray of $K - f$ satisfying $(g^1)^T (r^3) = (g^2)^T (r^3) < 0$. Consider the mixed integer set

$$\begin{aligned} x &= f + r^1 y_1 + r^2 y_2 + r^3 y_3 \\ x &\in S,\ y_i \geq 0\, \forall i\{1, 2, 3\}. \end{aligned} \tag{5}$$

(See Figure 6 for an example). Note that C^1 is generated by r^1, r^3; C^2 is generated by r^1, r^2; and C^3 is generated by r^2, r^3. Let p^k be an integer point in the relative interior of the facet $\{x + f \in \mathbb{R}^2 \,|\, (g^k)^T x = 1\}$ of K. Now it is easily verified that the following points:

1. $(y_1, y_2, y_3) := (\alpha, 1 - \alpha, 0)$ (where $0 < \alpha < 1$ and $p^2 = f + \alpha r^1 + (1 - \alpha) r^2$)

2. $(y_1, y_2, y_3) := (\beta, 0, \gamma)$ (where $\beta, \gamma > 0$ and $p^1 = f + \beta r^1 + \gamma r^2$)

3. $(y_1, y_2, y_3) := (0, \delta, \zeta)$ (where $\delta, \zeta > 0$ and $p^3 = f + \delta r^2 + \zeta r^3$)

satisfy the inequality $\pi(r^1)y_1 + \pi(r^2)y_2 + \pi(r^3)y_3 \geq 1$ at equality. Since

$$\begin{vmatrix} \alpha & 1-\alpha & 0 \\ \beta & 0 & \gamma \\ 0 & \delta & \zeta \end{vmatrix} = -\alpha\gamma\delta - (1-\alpha)\beta\zeta \neq 0 \,(\text{as } 0 < \alpha < 1, \beta, \gamma, \delta, \zeta > 0), \quad (6)$$

we obtain that the three points are linearly independent. Thus $\pi(r^1)y_1 + \pi(r^2)y_2 + \pi(r^3)y_3 \geq 1$ cannot be obtained as a convex combination of two different inequalities that are valid for $R(f, S, \mathcal{T})$ and therefore π^K is extreme for $R(f, S)$ by the application of Proposition 4.4.

A similar proof can be presented for the case where K is bounded. In this case, set r^3 to be the third vertex of $K - f$. □

6 Concluding Remarks

Apart from the questions posed in Section 1, one important question is that of generating valid inequalities for the set $R(f, S, \mathbb{R}^m, G)$ when $G \neq \emptyset$. This is an important case since almost always some nonbasic variables are integral and relaxing them to be continuous variables yields weak coefficients. Let $\pi : \mathbb{R}^m \to \mathbb{R}$ be a valid inequality for $R(f, S, \mathbb{R}^m, \emptyset)$ and let $S = Q \cap \mathbb{Z}^m$ where $Q = \cap_{1 \leq j \leq c} Q^j$ and $Q^j = \{x \in \mathbb{R}^m \,|\, (a^j)^T x \leq b^j\}$. Let $\mathcal{J}$ be a critical subset of $\{1, ..., c\}$ for $P(\pi)$ wrt to Q. Define $\phi : G \to \mathbb{R}$ as

$$\phi(u) = \inf\{\pi(w) \,|\, w = u + x, x \in \mathbb{Z}^m, x \in \text{rec.cone}(\text{conv}(S^{Q,\mathcal{J}}))\}. \quad (7)$$

(Remember $S^{Q,\mathcal{J}} = (\cap_{j \in \mathcal{J}} Q^j) \cap \mathbb{Z}^m$). It can be verified that (π, ϕ) yields a valid inequality for $R(f, S, \mathbb{R}^m, G)$ of the form

$$\sum_{w \in \mathbb{R}^m} \pi(w)y(w) + \sum_{u \in G} \phi(u)z(u) \geq 1. \quad (8)$$

The proof of validity is the following:

1. Since $\mathcal{J}$ is a critical set, $P(\pi)$ is an $S^{Q,\mathcal{J}}$-free convex set. Therefore $\pi : \mathbb{R}^m \to \mathbb{R}$ is a valid inequality for $R(f, S^{Q,\mathcal{J}}, \mathbb{R}^m, \emptyset)$.

2. Since the set $M := \text{rec.cone}(\text{conv}(S^{Q,\mathcal{J}})) \cap \mathbb{Z}^m$ is a monoid, i.e., $0 \in M$ and M is closed under addition, by application of Theorem 1 from Balas and Jeroslow [5], (π, ϕ) yields a valid inequality for $R(f, S^{Q,\mathcal{J}}, \mathbb{R}^m, G)$.

3. Since $S^{Q,\mathcal{J}} \supseteq S$, $R(f, S^{Q,\mathcal{J}}, \mathbb{R}^m, G)$ is a relaxation of $R(f, S, \mathbb{R}^m, G)$. Therefore (π, ϕ) yields a valid inequality for $R(f, S, \mathbb{R}^m, G)$.

However obtaining ϕ via (7) involves solving a MIP which may not always be efficiently solvable. Understanding when ϕ can be obtained efficiently and when ϕ yields strongest possible coefficients (see Dey and Wolsey [13] for some cases when $S = \mathbb{Z}^2$) are interesting directions of research.

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Appendix 1

In Proposition 6.3 we show that full-dimensional maximal S-free convex sets are polyhedral under some technical conditions. The proof presented here essentially uses the same steps as used in Lovász [20] for the case in which $S = \mathbb{Z}^m$.

It is convenient to deal with general lattices in this section. Let $\mathcal{L} \subseteq \mathbb{R}^m$ be the full-dimensional lattice generated by the basis $\{q^1, q^2, ..., q^m\}$ where $q^i \in \mathbb{Z}^m \; \forall \, i \in \{1, ..., m\}$. For the purpose of this section, define $S = \{x \in \mathcal{L} \,|\, (a^j)^T x \leq b^j, j \in \{1, ..., c\}\}$ where $a^j \in \mathbb{Z}^{m \times 1}$ and $b^j \in \mathbb{Z}$. Before presenting the main result in Proposition 6.3, we present two preliminary results in Propositions 6.1 and 6.2.

Proposition 6.1 *Let $K \subset \mathbb{R}^m$ be a full-dimensional S-free closed convex set. If $d \in \mathcal{L}$ is a direction in rec.cone(K) and rec.cone$(conv(S))$, then $\widehat{K} := conv(K + ray(-d))$ is also S-free.*

Proof: Suppose d is a direction both in rec.cone(K) and in rec.cone(conv(S)), i.e., $d^T a^j \leq 0 \; \forall j \in \{1, ..., c\}$. We need to show that if $q' \in \text{int}(\widehat{K}) \setminus K$ then $q' \notin S$. Assume by contradiction that there exists $q \in K$ and $\lambda > 0$ such that $q + \lambda(-d) \in (\text{int}(\widehat{K}) \setminus K) \cap S$.

Claim: $q + \mu d \in \text{int}(K) \; \forall \mu > 0$. Clearly $q + \mu d \in K$, since $q \in K$ and $d \in$ rec.cone(K). If $\forall u \in \mathbb{R}^m$, $\exists \epsilon > 0$ such that $q + \mu d + \epsilon u \in K$, then $q + \mu d \in \text{int}(K)$ (as K is convex). Therefore assume by contradiction that there exists a vector $u \in \mathbb{R}^m$ such that

$$q + \mu d + \epsilon u \notin K \quad \forall \epsilon > 0. \tag{9}$$

Since $q - \lambda d \in \text{int}(\widehat{K})$, there is a $\epsilon_0 > 0$ such that $q - \lambda d + \epsilon_0 u \in \widehat{K}$. If $q - \lambda d + \epsilon_0 u \in K$, then $q + \mu d + \epsilon_0 u \in K$; a contradiction to (9). Therefore by definition of $\widehat{K}$ there exists a point $v \in K$ and scalar $\gamma \in \mathbb{R}_+$ such that $v - \gamma d = q - \lambda d + \epsilon_0 u$. If $\gamma - \lambda \leq \mu$, then $q + \mu d + \epsilon_0 d \in K$ which is a contradiction to (9). If $\gamma - \lambda > \mu$, then the point $(1 - \frac{\mu}{\gamma - \lambda})q + (\frac{\mu}{\gamma - \lambda})v = q + \mu d + \frac{\epsilon_0 \mu}{\gamma - \lambda} u \in K$ which is a contradiction to (9) since $\frac{\epsilon_0 \mu}{\gamma - \lambda} > 0$.

Now consider the point $q + (\lceil \lambda \rceil - \lambda)d$. Since $(\lceil \lambda \rceil - \lambda) \geq 0$, by the previous claim, $q + (\lceil \lambda \rceil - \lambda)d \in \text{int}(K)$. Since by assumption $q - \lambda d \in S$, we obtain that $(q - \lambda d) + (\lceil \lambda \rceil)d \in S$ (as $\lceil \lambda \rceil d \in \mathcal{L}$ and $d \in$ rec.cone(conv(S))). This is a contradiction to the fact that K is S-free. $\square$

Observation 6.1 *Let K be a S-free convex set. Let $d \in \mathcal{L}$ be a recession direction for both K and conv(S). Let $\mathcal{L}^d$, S^d, and K^d be the projections of $\mathcal{L}$,*

S, and K on the linear space orthogonal to the direction d. If $\exists G \subset lin(\mathcal{L}^d)$ such that $G \supseteq K^d$ and G is an S^d-free convex set, then $G + cone\{d, -d\}$ is a S-free convex set and contains K.

Proposition 6.2 *Let $P \subseteq \mathbb{R}^m$ be a rational polyhedron and $K \subset \mathbb{R}^m$ be a closed convex set such that $P \cap K$ is bounded and full-dimensional. Then there exists a polyhedron $Q \subset \mathbb{R}^m$ such that $P \cap Q$ is bounded and $K \subseteq Q$.*

Proof: If P is bounded, the result is obvious. Now consider the case where P is not bounded. Take any point q in the interior of $K \cap P$. Draw rays starting from q in all the directions in rec.cone(P). For any given ray $q + \lambda d$, $\lambda \geq 0$ there exists a point $p = q + \widehat{\lambda} d$ (i.e., p is on the ray) such that $p \in P$ and $p \notin K$ since $P \cap K$ is bounded. By the separation theorem for convex sets (Rockefeller [21]) it is possible to construct a half-space $H^d := \{x \,|\, (g^d)^T x \leq h^d\}$ such that $(g^d)^T p = h^d$ and $K \subseteq H^d$. Let $\mathcal{H}$ be the set of all such half-spaces. Clearly, $K \subset \cap_{H^d \in \mathcal{H}} H^d$ and $(\cap_{H^d \in \mathcal{H}} H^d) \cap P$ is bounded. The proof is complete if it is shown that there exists a finite subset $\tilde{H}$ of $\mathcal{H}$ such that $(\cap_{H^d \in \tilde{H}} H^d) \cap P$ is bounded. Since P is a closed set its recession cone is a closed set. Consider the closed and bounded set $B = \{x \in \mathbb{R}^m \,|\, \|x\| = 1, x \in \text{rec.cone}(P)\}$. Corresponding to a half-plane $H^d := \{x \,|\, (g^d)^T x \leq h^d\}$, consider the open subset O^d of B defined as $B \cap \{x \,|\, (g^d)^T x > 0\}$.

Claim 1: $\cup_{H^d \in \mathcal{H}} O^d$ is an open cover of B. For any $\bar{r} \in B$, we show that some O^d covers $\bar{r}$. By construction there exists a hyperplane $H^d \in \mathcal{H}$ such that $(g^d)^T (q + \lambda \bar{r}) = h^d$, for some $\lambda > 0$, and $K \subseteq H^d$. Since $q \in \text{int}(K)$, we obtain $(g^d)^T q < h^d$ or equivalently $(g^d)^T \bar{r} > 0$. Therefore O^d covers $\bar{r}$.

Claim 2: Let $\tilde{H} \subset \mathcal{H}$. If $\cup_{H^d \in \tilde{H}} O^d$ covers B, then $(\cap_{H^d \in \tilde{H}} H^d) \cap P$ is bounded. Let r be any non-zero vector in the recession cone of P. Then $\bar{r} := \frac{r}{\|r\|}$ belongs to B. Let an element of $H^d \in \tilde{H}$ that covers $\bar{r}$. Thus $(g^d)^T \bar{r} > 0$ or $(g^d)^T r > 0$. Therefore r is not a vector in the recession cone of $H^d \cap P$. Thus no vector of the recession cone of P belongs to the recession cone of $(\cap_{H^d \in \tilde{H}} H^d) \cap P$. Since the recession cone of $(\cap_{H^d \in \tilde{H}} H^d) \cap P$ is a subset of the recession cone of P, this proves the result.

Now since B is closed and bounded, it is a compact set (Heine-Borel theorem). Since $\cup_{H^d \in \mathcal{H}} O^d$ is an open cover of B, there exists a finite subset $\tilde{H}$ of $\mathcal{H}$ that covers B. Now using Claim 2, $(\cap_{H^d \in \tilde{H}} H^d) \cap P$ is bounded, which completes the proof. $\square$

Proposition 6.3 *Let $S = \{x \in \mathcal{L} \,|\, (a^j)^T x \leq b^j, j \in \{1, ..., c\}\}$ where $\mathcal{L}$ is a full-dimensional lattice in $\mathbb{R}^m$. Let K be a full-dimensional S-free* *convex set with the property that (1) $K \cap conv(S)$ is full-dimensional and (2) if $rec.cone(K \cap conv(S)) \supsetneq \{0\}$, then there exists $d^1, ..., d^t \in \mathcal{L}$ such that $d^1, ..., d^t \in rec.cone(K \cap conv(S))$ and $lin\{d^1, ...d^t\} = lin(rec.cone(K \cap conv(S)))$. Then K is a maximal S-free convex set if and only if it is a polyhedron that contains at least one point of S in the relative interior of each facet.*

Proof: $\Rightarrow$ Let K be a full-dimensional maximal S-free convex set satisfying (1) and (2).

Claim 1: K is a polyhedral set. We first construct an S-free polyhedron containing K to prove that K is a polyhedral set. For every $v \in S$, let H^v be a half-space that contains K and contains v on its boundary. (This can be done since K is convex; separation theorem of convex sets). Clearly $K \subseteq \cap_{v \in S} H^v$. Moreover, none of the points in S are contained in the interior of $\cap_{v \in S} H^v$. If S is a finite set, this shows that K is a polyhedron.

If S is not a finite set, we need to show that there is a finite subset F of S such that $K \subseteq \cap_{v \in F} H^v$. If the intersection of rec.cone(K) and rec.cone(conv(S)) is non-empty, then by assumption there is $d^1, ..., d^t \in \mathcal{L}$ such that $d^1, ..., d^t \in$ rec.cone($K \cap$conv(S)) and $\text{lin}\{d^1, ...d^t\} = \text{lin}(\text{rec.cone}(K \cap \text{conv}(S)))$. Then by Proposition 6.1 and by the fact that K is a maximal S-free convex set, $-d^1, ..., -d^t$ belong to rec.cone(K). Let $\mathcal{L}^d, S^d, K^d$ be the projection of $\mathcal{L}, S, K$ respectively on the linear space orthogonal to d^1. Clearly, K^d is an S^d-free convex set and K^d, S^d satisfy assumptions (1) and (2) of the proposition. By repeating this process t times, we obtain a lattice $\tilde{\mathcal{L}}$ and a convex set $\tilde{K}$ that is a maximal $\tilde{S}$-free convex set (by Observation 6.1) such that there exists no direction d belonging to both rec.cone($\tilde{K}$) and rec.cone(conv($\tilde{S}$)). Therefore conv($\tilde{S}$) $\cap \tilde{K}$ is bounded. Now using Proposition 6.2 we obtain that there exists a polyhedron Q such that $Q \cap \text{conv}(\tilde{S})$ is bounded and $\tilde{K} \subset Q$. Since $Q \cap \text{conv}(\tilde{S})$ is bounded, there exists a finite number of points $v \in (Q \cap \tilde{S})$. Therefore $\tilde{K} \subset (\cap_{v \in (Q \cap \tilde{S})} H^v) \cap Q$. Also the set $(\cap_{v \in (Q \cap \tilde{S})} H^v) \cap Q$ is $\tilde{S}-$free. Therefore by maximality $\tilde{K} = (\cap_{v \in (Q \cap \tilde{S})} H^v) \cap Q$ and consequently K is a polyhedron.

Claim 2: Every facet of K contains a point belonging to S in its relative interior. We show that $\tilde{K}$ constructed in the proof of Claim 1 contains a point belonging to $\tilde{S}$ in its relative interior (When S is finite, the proof is the same). Suppose a facet $F := \{x | g^T x \leq h\}$ of $\tilde{K}$ does not contain any point of $\tilde{S}$ in its relative interior. Then there exists a $\epsilon > 0$ such that replacing F by $g^T x \leq h + \epsilon$ in the description of $\tilde{K}$ creates a set that also contains no point of $\tilde{S}$ in its interior (since $\tilde{K} \cap \text{conv}(\tilde{S})$ is bounded). This contradicts the maximality of $\tilde{K}$. Since $\tilde{K}$ is a polyhedron with one point of $\tilde{S}$ in the relative interior of each facet, we obtain that K is a polyhedron that contains at least one point of S in the relative interior of each facet as each of the projection directions belonged to $\mathcal{L}$ and belonged to the recession cone of K and $\text{conv}(S)$.

$\Leftarrow$ Suppose K is a full-dimensional S-free polyhedron $\{x \in \mathbb{R}^m \mid (g^j)^T x \leq b^j, 1 \leq j \leq l\}$ containing a point of S in the relative interior of each facet. Suppose that K is not maximal. Then there exists a convex set K' which strictly contains K and is S-free. However note that K' is completely contained in each of the half-planes $(g^j)^T x \leq b^j$ (since $(g^j)^T x \leq b^j + \epsilon$ contains a point belonging to S in its interior for any $\epsilon > 0$). This implies that $K' = K$. $\square$

Note that assumption (2) is required to prove that if K is a maximal S-free convex set, then K is a polyhedral set. In the proof of the converse, assumption (2) is not required.

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