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A viability theory approach to a two-stage optimal control problem of technology adoption

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Abstract

A new technology adoption problem can be modelled as a two-stage control problem, in which model parameters ("technology") might be altered at some time. An optimal solution to utility maximisation for this class of problems needs to contain information on the time, at which the change will take place (0, finite or never), along with the optimal control strategies before and after the change. For the change, or switch, to occur the "new technology" value function needs to dominate the "old technology" value function, after the switch. We charaterise the value function using the fact that its hypograph is a viability kernel of an auxiliary problem and we study when the graphs can intersect. If they do not, the switch cannot occur at a positive time. Using this characterisation we analyse a technology adoption problem and show how to recognise the models, for which the switch will occur at time zero or never.

Keywords: technology adoption, value function, viability kernel, viscosity solutions.

JEL Classification: C6, C61, C69

MSC: 34H05, 34K35, 49J15, 49L25, 91B02, 91B62, 93C15

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Glossary

D, E, K	closed sets in $\mathbb{R}^N$; D will be typically used for *viability domain*
$f(\cdot,\cdot), \psi(\cdot,\cdot) : \mathbb{R}^N \times U \to \mathbb{R}^N$	system's dynamics, continuous in either argument (later assumed affine in first argument and indexed by technology)
$\phi(\cdot,\cdot,\cdot,\cdot) : \mathbb{R}^{N+3} \to \mathbb{R}^{N+3}$	auxiliary system's dynamics
$\Phi(\cdot), \Psi(\cdot)$	system's dynamics; set valued maps
$L : [0,\, T] \times \mathbb{R}^N \times U \to \mathbb{R}$	instantaneous utility (utility integrand); bounded function
$\mathcal{NP}_D(x)$	set of proximal normals to D at $x \in D$
$U \subseteq \mathbb{R}^M$	set of control values
$u(\cdot) : [0,\, T] \to U$	control; measurable function
$\mathcal{U}_{[t,T]}$	set of measurable controls on $[t, T]$ with values in U
$V^T(\cdot,\cdot) : [0,\, T] \times \mathbb{R}^N \to \mathbb{R}$	value function for T-horizon optimal control
$\underline{w}(\cdot,\cdot), \overline{w}(\cdot,\cdot)$	sub- and supersolution to the Hamilton-Jacobi-Bellman equation
$x(\cdot) : [0,\, T] \to \mathbb{R}^N$	state variable
$\mathcal{V}\mathfrak{iab}_\Psi(K,\, E)$	*viability kernel* in K with target E, for dynamics Ψ
$\mathcal{E}\mathfrak{pi}(w), \mathcal{H}\mathfrak{ypo}(w)$	*epigraph, hypograph* of the function w

1 Introduction

The aim of this paper[1] is two-fold. First, we want to demonstrate economic applicability of recent results in viability theory concerning some equivalence between the value function and the viability kernel. In particular, we will examine a collection of continuous-time optimal-control problems with affine dynamics to decide that their value function graphs cannot intersect. Second, we want to use the established result to prove existence, or the lack of it, of a switching time in a two-stage optimal control problem of technology adoption.

A two-stage control problem is one, in which model parameters might be modified at some time. For example, a system's dynamics, which describes accumulation of pollution, may be altered through installation of new filters. More generally, two-stage (or multi-stage) control problems are concerned with switching between alternative and consecutive regimes,

[1] This paper draws from and extends [21].

where the switching times between regimes are endogenously determined. Such problems have been studied by [35], [1], [36], [23] and [11] and [12] in the context of maximum principle or *multiprocess* maximum principle [4]. In particular [11] and [12] applied this technique to investigate *technology adoption*. We use a model discussed in the latter and study switching-time existence using viability theory.

An optimal solution to utility maximisation in a two-stage control problem needs to contain information on the time, at which the regime switch will take place (0, finite or never) as well as the optimal control strategies before and after the change. For the change, or switch, to occur the "new filter" value function needs to dominate the "old filter" value function, after the switch. Intuitively, if the stock of an investment capital exceeds certain threshold, installing a new technology will be justified.

We will index two-stage optimal control problems by a parameter responsible for the system's dynamics and characterise the corresponding value functions. If their graphs, each corresponding to a different parameter, do not intersect then the switch [2] will not occur and the "new" technology will not be adopted. If the graphs intersect, then it is optimal to replace the old technology by the new one, should the the state variable take the system to the cross-over point. We will charaterise the value functions using the fact that their hypographs are viability kernels of some auxiliary viability problems and study when the hypographs are contained in each other.

What follows is a brief outline of what this paper contains. In Section 2, we describe a basic optimal control model, which we use in Section 6 to study a technology switching problem.[3] Basic results concerning viability theory and viscosity solutions are presented in Section 3. In Section 4, a relationship between the viability kernel of an auxiliary problem and the basic finite-horizon optimal control problem are established. A proposition concerning a relationship between value functions and Hamiltonians is formulated in Section 5. This result is then used in Section 6 to study existence of a switching time in a technology adoption problem. The concluding remarks summarise our findings.

[2] Presumably, in a two-stage optimal control problem one switch at most can occur.

[3] According to our knowledge this will be the first application of viability theory to microeconomics. For macroeconomic applications refer to publications [20], [9] and working papers [10], [19], [22], [27] [26]. For viability theory applications to environmental economics see [8], [24], [16] and [25]; for applications to financial analysis see [28] and the references provided there.

2 Formulation of an optimal control problem

We consider a control system whose dynamics is given by:

$$\dot{x}(s) = f(x(s), u(s)) \tag{1}$$

where the state variable x belongs to $\mathbb{R}^N$, the control $u(\cdot) : [0, \infty) \to U \subset \mathbb{R}^M$ is a measurable function and $f : \mathbb{R}^N \times U \to \mathbb{R}^N$.

The optimal control problem is

$$\max_{u(\cdot)} \int_t^T L\left(s, x(s; t, x, u(\cdot)), u(s)\right) ds \tag{2}$$

where $x(\cdot; t, x, u(\cdot))$ denotes absolutely continuous solutions to (1), with $0 < T < \infty$ and $L : [0,\, T] \times \mathbb{R}^N \times U \to \mathbb{R}$ is a given bounded function. We adopt the convention that $x(\cdot; t, x, u(\cdot))$ denotes the solution to (1) starting from $(t, x) \in [0, T] \times \mathbb{R}^N$.

If we denote the set of measurable controls on $[t, T]$ with values in U by $\mathcal{U}_{[t,T]}$ then the value function corresponding to the optimal control problem (1) and (2) is given by:

$$V^T(t, x) = \sup_{u \in \mathcal{U}_{[t,T]}} \int_t^T L\left(s, x(s; t, x, u(\cdot)), u(s)\right) ds \tag{3}$$

Our goal is to establish conditions allowing us to compare value functions that correspond to different system's dynamics $f(\cdot, \cdot)$, perhaps "indexed" by technologies. To do this, we will characterise the value function (3) through a viability kernel and also as a solution to an equation of Hamilton-Jacobi-Bellman type.

At this stage we hint on a result about viability characterisation obtained in [14]. The result establishes that the epigraph of the minimal time to reach a target set is a viability kernel of an auxiliary control process. Later, in Section 4, we will prove an analogous result for a more general optimisation problem (2), (1).

We will also use some available and well known results (see *e.g.*, [5], [6], [17]) regarding a Lipschitzian value function. In particular, under continuity assumptions on system's dynamics $f(\cdot, \cdot)$ and utility integrand $L(\cdot, \cdot, \cdot)$ (see Section 3.1) the value function (3) is the unique Lipschitz viscosity solution[4]

[4]A viscosity solution of a partial differential equation is a continuous function that satisfies the equation and whose derivatives are considered in a generalised sense. See Section 3.3 for precise definitions.

of the following equation:

$$\begin{cases} \dfrac{\partial V^T}{\partial t}(t,x) + H\Big(t, x, \dfrac{\partial V^T}{\partial x}(t,x)\Big) = 0 \\ (t,x) \in [0,T) \times \mathbb{R}^N; \\ \text{with the final condition } V^T(T,x) = 0, \text{ for all } x \in \mathbb{R}^N \end{cases} \tag{4}$$

where the Hamiltonian [5] $H : \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is:

$$H(t,x,p) = \max_{u \in U} (\langle p, f(x,u) \rangle + L(t,x,u)) \,. \tag{5}$$

Our main result will enable us to compare value functions associated with different technologies. However, rather than obtaining $V^T(t,x)$ as a solution to the Hamilton-Jacobi equation (4), we will characterise the value function through a viability kernel of an auxiliary problem related to the original optimal control problem.

The approach to value function characterisation by viability kernels was dealt with in mathematical publications [3], [13], [18], [29], [31], [32], [33]. Our use of this approach to economic problems' solution is novel. Furthermore, we are unaware of any "applied" problem whose solution would be based on the links between the viscosity supersolution, the value function's hypograph and the viability kernel of an auxiliary problem, all to be defined below.

3 Preliminaries

3.1 Definitions, assumptions and notation

We will assume that the dynamics $f : \mathbb{R}^N \times U \to \mathbb{R}^N$ in equation (1) is a continuous function and that it satisfies:

$$\begin{cases} ||f(x,u)|| \leq c_1(1 + \|x\|) \\ ||f(x,u) - f(y,u)|| \leq c_1 \|x - y\| \end{cases} \quad \forall x, y \in \mathbb{R}^N, u \in U \tag{6}$$

where $c_1 > 0$ is constant; the control set U is a compact subset of $\mathbb{R}^N$. We can therefore describe the system's velocities at x as $f(x,U)$ where

$$f(x,U) = \{f(x,u), u \in U\} \quad \text{is a convex set} \quad \forall x \in \mathbb{R}^N. \tag{7}$$

[5] Notice that we depart from the traditional definition according to which the Hamiltonian will be the contents of brackets $(\cdot)$ in (5) (*i.e.*,"maximand") rather than the result of the maximisation, as we have define it, following the literature on viscosity solutions, see *e.g.* [5], [6], [17].

It is well known that under (6), for every $(t,x) \in [0,\infty)\times\mathbb{R}^N$, the Cauchy Problem (CP):

$$\begin{cases} \dot{x}(s) = f(x(s), u(s)) \text{ for almost every } s \in [t,\infty) \; ; \\ x(t) = x \end{cases} \tag{CP}$$

has a unique absolutely continuous solution, denoted by $x(\cdot; t, x, u(\cdot))$.

We will also assume that $L : [0,\, T] \times \mathbb{R}^N \times U \to \mathbb{R}^N$ is continuous and satisfies :

$$\begin{cases} ||L(t,x,u)|| \le c_2(1+\|x\|) \\ ||L(t,x,u) - L(t,y,u)|| \le c_2 \|x-y\| \end{cases} \quad \forall x, y \in \mathbb{R}^N, u \in U,\, t \in [0, \in T] \tag{8}$$

where $c_2 > 0$ is constant and

$$\forall x \in \mathbb{R}^N,\, t \in [0,\, T] \;\; L(t,x,U) = \{L(t,x,u), u \in U\} \text{ is convex.} \tag{9}$$

Later we will study an example where the function $f : \mathbb{R} \times U \to \mathbb{R}$ is linear in either variable and has the following form:

$$f(x,u) = \theta u - \mu x \tag{10}$$

with $\theta, \mu \in \mathbb{R}$ that can be associated with some technology and $L : t \in [0,\, T] \times \mathbb{R} \times U \to \mathbb{R}_+$

$$L(t,x,u) = e^{-\rho t} g(u,x) \tag{11}$$

where $g(u,x)$ is bounded, continuous and concave in each argument, decreasing in x; $\rho \in \mathbb{R}$.

3.2 Viability theory

Here we will present the notion of *viability-domain-with-a-target* introduced in [31] (for existence and characterisation of feedback controls assuring viability see [37]). We will characterise this set (*i.e.*, viability domain) using the Viability Theorem provided in [13] (Theorem 2.3):

Proposition 1 *We assume that D and E are closed sets. Let us suppose that $\psi : \mathbb{R}^N \times U \to \mathbb{R}^N$ is a continuous function, Lipschitz in the first variable; furthermore, for every x we define set valued map $\psi(x,U) = \{\psi(x,u); u \in U\}$ which is supposed to be Lipschitz continuous with convex, compact, nonempty values.*

Then the two following assertions are equivalent[6]:

[6]Here $\mathcal{NP}_D(x)$ denotes the set of *proximal normals* to D at x *i.e.*, the set of $p \in \mathbb{R}^N$ such that the distance of $x+p$ to D is equal to $||p||$.

i.

$$\forall x \in D\backslash E, \quad \forall p \in \mathcal{NP}_D(x), \quad \min_u \langle \psi(x,u), p\rangle \le 0 \tag{12}$$

(respectively, $\max_u \langle \psi(x,u), p\rangle \le 0$*);*

ii. there exists $u \in \mathcal{U}_{[t,T]}$ *such that*

(respectively, for all $u \in \mathcal{U}_{[t,T]}$*)*

$$\text{the solution of} \quad \begin{cases} \dot{x}(s) = \psi(x(s), u(s)) \text{ for almost every } s \\ x(t) = x \end{cases} \tag{13}$$

remains in D *as long as it does not reach* E*.*

Notice that the inequality $\min_u \langle \psi(x,u), p\rangle \le 0$ in (12) means that there *exists* a control for which the system's velocity $\dot{x}$ "points inside" the set $D\backslash E$. Respectively, $\max_u \langle \psi(x,u), p\rangle \le 0$ means that the system's velocity $\dot{x}$ "points inside" the set $D\backslash E$ for *all* controls from U.

When i) (or ii)) holds we say that D is a *viability domain* with target E (or, respectively, D is an *invariance domain* with target E) for the dynamics ψ. When $E = \emptyset$, then the proposition concerns the classical notion of viability (respectively, invariance) domain [3].

Definition 2 *Let* K *be a closed set. We call* viability kernel *in* K *with target* E*, for a dynamics* Ψ *denoted:*

$$\mathcal{V}\mathfrak{iab}_\Psi(K,\ E)$$

the largest closed subset of K*, which is a viability domain with target* E *for* Ψ*.*

It was proved (see for instance [2] and [31]) that $\mathcal{V}\mathfrak{iab}_\Psi(K, \emptyset)$ is also the set of x such that there exists $x(\cdot)$, a solution of

$$\dot{x}(s) \in \Psi(x(s)) \tag{14}$$

starting from x, which is defined on $[0, \infty)$ and $x(s) \in K$ for all $s \ge 0$. Respectively, $\mathcal{V}\mathfrak{iab}_\Psi(K, E)$ (*i.e.*, when $E \neq \emptyset$) is also the set of x such that there exists $x(\cdot)$, a solution of

$$\dot{x}(s) \in \Psi(x(s))$$

starting from x, which is defined on $[0, \tau)$ and $x(s) \in K$ for all $s \in [0, \tau)$ and if τ is finite then we have $x(\tau) \in E$.

Our conclusions regarding value functions will be based on the fact that the definition of a solution to a PDE of the type (4) gives some invariance properties of sets related to the value function (see Propositions 1 and 4). More precisely, the hypograph[7] of a supersolution to (4) is a viability domain in $[0,T]\times\mathbb{R}^{N+2}$ with some target for the auxiliary system's dynamics ϕ:

$$(t,x,z,r)\to\phi(t,x,z,r)=(1,f(x,U);L(t,x,U),0) \tag{15}$$

and the epigraph of a subsolution is an invariance domain in $[0,T]\times\mathbb{R}^{N+2}$ with some target for the auxiliary system's dynamics $-\phi$:

$$(t,x,z,r)\to-\phi(t,x,z,r)=-(1,f(x,U);L(t,x,U),0). \tag{16}$$

In particular, we will exploit the fact that the *largest* closed viability domain (kernel) in $[0,T]\times\mathbb{R}^{N+2}$ for dynamics ϕ (with a target) is the hypograph of the *biggest* subsolution[8] (value function) to the Hamilton-Jacobi-Belleman equation (4).

We will use $\mathcal{E}\mathfrak{pi}$ for the epigraph and $\mathcal{H}\mathrm{y}\mathfrak{po}$ for the hypograph.

3.3 Viscosity Solutions

Let us define a viscosity solution to the first order Hamilton-Jacobi-Bellman equation (cf. [5] for instance):

Definition 3 *A* viscosity supersolution *of (4) is a lower semicontinuous (l.s.c.) function* $\overline{w}:[0,T)\times\mathbb{R}^N\to\mathbb{R}$ *if and only if*

$$\text{for any } \varphi\in C^1 \text{ and when } (t,x) \text{ is a local minimum of } (\overline{w}-\varphi),\quad \frac{\partial\varphi}{\partial t}(t,x)+H\big(t,x,\frac{\partial\varphi}{\partial x}(t,x)\big)\le 0\,.$$

A viscosity subsolution *of (4) is an upper semicontinuous (u.s.c.) function* $\underline{w}:(0,T]\times\mathbb{R}^N\to\mathbb{R}$ *if and only if*

$$\text{for any } \varphi\in C^1 \text{ and when } (t,x) \text{ is a local maximum of } (\underline{w}-\varphi),\quad \frac{\partial\varphi}{\partial t}(t,x)+H\big(t,x,\frac{\partial\varphi}{\partial x}(t,x)\big)\ge 0.$$

A viscosity solution *of (4) is a function which is both subsolution and supersolution (so, in particular, it is continuous).*

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[7]For $w:[0,T]\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ we have:
$\mathcal{E}\mathfrak{pi}(w):=\{(t,x,r)\in[0,T]\times\mathbb{R}^N\times\mathbb{R}\mid w(t,x)\le r\}$;
$\mathcal{H}\mathrm{y}\mathfrak{po}(w):=\{(t,x,r)\in[0,T]\times\mathbb{R}^N\times\mathbb{R}\mid w(t,x)\ge r\}$.

[8] Biggest with respect to canonical order in the class of functions.

There are several different definitions of discontinuous viscosity solutions. In particular, Ishii's solutions (cf. [5]) are based on semicontinuous envelopes of functions; there are also Barron-Jensen-Frankowska's semicontinuous solutions ([5], [7]) for convex Hamiltonians) and Subbotin's minimax solution [34] (called bilateral solutions in [6]) see also [30]. We think that the definition that we use in this paper is perhaps the most appropriate for the study of our problem. In particular, we find that it enables us to adequately compare solutions to the Hamilton-Jacobi-Bellman equations.

To establish a link between the viscosity solutions and viability we will provide an equivalent definition of super- and subsolutions to (4) in terms of *proximal normals*[9]. The proof of the equivalence between the two definitions and a result formulated as the following proposition can be found in [29].

Proposition 4 *A* viscosity supersolution *to (4) is a l.s.c. function* $\overline{w} : [0,T) \times \mathbb{R}^N \to \mathbb{R}$ *such that:*

$$\text{for any } (p_t, p_x, p_r) \in \mathcal{NP}_{\mathcal{E}\mathsf{pi}(\overline{w})}\big(t, x, \overline{w}(t,x)\big),$$
$$p_t + H(t, x, p_x) \le 0$$

A viscosity subsolution to *(4) is an u.s.c. function* $\underline{w} : [0,T) \times \mathbb{R}^N \to \mathbb{R}$ *such that:*

$$\text{for any } (p_t, p_x, p_r) \in \mathcal{NP}_{\mathcal{H}\mathsf{ypo}(\underline{w})}\big(t, x, \underline{w}(t,x)\big),$$
$$p_t + H(t, x, p_x) \ge 0.$$

4 The Optimal Control Problem with Finite Horizon

This section is dedicated to the characterisation of the value function of a finite-horizon optimal control problem through the Hamilton-Jacobi-Bellman equation (4).

4.1 The associated Mayer problem

Consider the Bolza optimal control problem with the following value function:

$$V^T(t,x) = \sup_{u \in \mathcal{U}_{[t,T]}} \left\{ g\left(x(T; t, x, u(\cdot))\right) + \int_t^T L\left(s, x(s; t, x, u(\cdot)), u(s)\right) ds \right\}. \tag{17}$$

[9] Refer to footnote 6 and Proposition 1.

Function $g(\cdot)$ is a "scrap value" function at the final time T, which satisfies

$$\left\{ \begin{array}{c} |g(x)| \le c_2 \text{ for all } x \in \mathbb{R}^N, \\ g \text{ is upper-semicontinuous in } \mathbb{R}^N \end{array} \right. ; \tag{18}$$

$L : \mathbb{R} \times \mathbb{R}^N \times U \to \mathbb{R}$ satisfies (8), (9). If g is discontinuous then so is, in general, the value function $V^T(t, x)$.

We will consider the modified system's dynamics:

$$\dot{y}(t) = (f(x(t), u(t)); L(t, x(t), u(t)). \tag{19}$$

Here

$$y(\cdot; t, y, u(\cdot)) := (x(\cdot; t, x, u(\cdot)); z(\cdot; t, z, u(\cdot)))$$

is the solution of (19) starting at $(t, x, z) := (t, y) \in [0, T] \times \mathbb{R}^{N+1}$. The *set* of solutions starting at (t, y) will be denoted $S(t, y) := \{y(\cdot; t, y, u(\cdot)); u \in \mathcal{U}_{[t,T]}\}$.

Let $h : \mathbb{R}^{N+1} \to \mathbb{R}$, given by

$$h(y) := g(x) + z \quad \text{ with } \quad y := (x, z)\,.$$

We define an associated Mayer problem as follows:

$$\text{maximise } h\left(y(T; t, y, u(\cdot))\right) \tag{20}$$

over all absolutely continuous solutions of (19).

With the above notations, we define the following value function corresponding to problem (20) subject to (19):

$$W^T(t, y) = \sup_{u \in \mathcal{U}_{[t,T]}} h\big(y\big(T; t, y, u(\cdot)\big)\big). \tag{21}$$

Note that the following relation is true[10]:

$$W^T(t, y) = V^T(t, x) + z\,.$$

We will study the properties of W^T; as a consequence, a characterisation for V^T will be obtained.

Before giving the main result of this section, we cite some classical results and recall the well known results when the function h is Lipschitz (for details, see [5], [6], [17]).

[10] Notice that $z(s) = z + \int_t^s L dt$ where z is some initial condition and

$$\begin{aligned} V^T(t, x) &= \sup_u [g(x(T)) + \int_t^T L dt] = \sup_u [g(x(T)) + z(T) - z] \\ &= \sup_u [h((x(T)), z(T)) - z] = \sup_u [h(y(T)) - z] = -z + \sup_u [h(y(T))] \end{aligned} \tag{22}$$

4.2 Regularity and the Principle of Optimality

We first recall some results concerning the regularity of W^T.

Lemma 5 *Suppose that (6), (7), (8), (9), hold true. Assume that h is upper semicontinuous. Then we have:*

i. (Existence of optimal control) There exists an optimal trajectory starting from each point $(t,y) \in [0,T] \times \mathbb{R}^{N+1}$ i.e., there exists $\bar{y}(\cdot) \in S(t,y)$ such that

$$W^T(t,y) = h(\bar{y}(T; t, y, \bar{u}(\cdot))) \text{ for all } (t,y) \in [0,T] \times \mathbb{R}^{N+1}$$

ii. W^T is upper semicontinuous.

Next we recall the Bellman Principle of Optimality, from which the Hamilton-Jacobi-Bellman PDE is derived, satisfied by the value function.

Proposition 6 *(Principle of Optimality) Let $g : \mathbb{R}^N \to \mathbb{R}$ be a bounded function and suppose that (6), (7), (8), (9) hold true. Then for all $(t,y) \in [0,T] \times \mathbb{R}^{N+1}$ and $\alpha > 0$ such that $t + \alpha \leq T$:*

$$W^T(t,y) = \sup_{u \in \mathcal{U}_{[t,T]}} W^T(t+\alpha, y(t+\alpha)). \tag{23}$$

4.3 The Hamilton-Jacobi Partial Differential Equation for the Mayer problem

Using the Principle of Optimality for the optimal control problem with a finite horizon (20), (19) we can prove that when the value function W^T is regular enough (*e.g.*, u.s.c.) then this function is the viscosity solution in the sense of Definition 3 of the following PDE:

$$\begin{cases} \dfrac{\partial W}{\partial t}(t,y) + \bar{H}\big(t, y, \dfrac{\partial W}{\partial y}(t,y)\big) = 0 \\ (t,y) \in [0,T) \times \mathbb{R}^{N+1}; \quad W(T,\cdot) = h(\cdot) \end{cases} \tag{24}$$

where the Hamiltonian $\bar{H} : [0,T) \times \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \to \bar{\mathbb{R}}$ is:

$$\bar{H}(t,y,q) = \max_{u \in U} \langle q, (f(x,u), L(t,x,u)) \rangle \,. \tag{25}$$

Proposition 7 *If h is a Lipschitz function then W^T is the unique Lipschitz viscosity solution to (24) with the final condition $W^T(T,\cdot) = h(\cdot)$.*

This result, based on the Principle of Optimality, is classical (see [5], [6], [17]). Also, it is easy to check that the value function is Lipschitzian when h is Lipschitzian.

Remark 8 *Using (22), we can verify that* $\frac{\partial W^T}{\partial z} = 1$ *for almost all* $(t, y) \in [0, T] \times \mathbb{R}^{N+1}$ *and as a consequence* V^T *is the unique Lipschitz viscosity solution to (4) with the final condition* $V^T(T, \cdot) = g(\cdot)$*.*

4.4 The upper semicontinuous case for the Mayer problem

In this section we assume that the function h is upper semicontinuous (u.s.c.). If so, the value function W^T is also upper semicontinuous, as we have already said it in Lemma 5.

Using results from *e.g.*, [29], [32] or [33] we obtain the following theorem, which says that the value function is the biggest[11] u.s.c. subsolution of (24).

Theorem 9 *If (6), (7) (8), (9) hold true then* $\mathcal{H}\mathrm{ypo}(W^T)$ *is viability kernel in* $[0, T] \times \mathbb{R}^{N+2}$ *with target* $\{T\} \times \mathcal{H}\mathrm{ypo}(h)$ *for the dynamics* $(t, x, z, r) \to \phi(t, x, z, r) = (1, f(x, U), L(x, U), 0)$ *:*

$$\mathcal{H}\mathrm{ypo}(W^T) = \mathcal{V}\mathrm{iab}_\phi\big([0, T] \times \mathbb{R}^{N+2}, \{T\} \times \mathcal{H}\mathrm{ypo}(h)\big)$$

As a consequence, the value function is the biggest upper semicontinuous subsolution so, it is solution to (24); furthermore, it verifies the final condition $W^T(T, \cdot) = h(\cdot)$*.*

Also notice that $\mathcal{H}\mathrm{ypo}(W^T)$ is a closed set because of the assumption on function h's upper semicontinuity. This helps us to characterize the hypographs as viability domains.

4.5 An example of a viability kernel

We use a simple numerical example to illustrate the relationship between *value function* of an optimal control problem and *viability kernel* of the corresponding auxiliary control system. Consider the following control system:

$$\left\{ \begin{array}{lll} \text{(i)} & \dot{x}(s) = f(x(s), u(s)), & s \geq t \\ \text{(ii)} & x(t) = x\,. & \end{array} \right. \tag{26}$$

Here we choose f to be a map from $\mathbb{R} \times [-1, 0]$ into $\mathbb{R}$ given by

$$f(x, u) := u$$

[11] See footnote 8.

and the controls $u(\cdot) : [0, T] \to [-1, 0]$ are measurable functions.

With any solution $x(\cdot; t, x, u(\cdot))$ to (26) starting from $(t, x) \in [0, T] \times \mathbb{R}$, we associate the following value function:

$$V^T(t, x) = \sup_{u(\cdot)} h\left(x(T; t, x, u(\cdot))\right) . \tag{27}$$

Recall that we have

$$V^T(T, x) = h\left(x\right) . \tag{28}$$

Consequently, a possible target for trajectory $x(\cdot; t, x, u(\cdot))$ that solves (26) will be $\{T\} \times \mathcal{H}\mathrm{ypo}(h)$ because we ignore the system's behaviour beyond T.

In this example, we choose $h : \mathbb{R} \to \mathbb{R}$ as

$$h(x) = x$$

and $T > 0$ is fixed (say, $T = 1$). We notice that looking for $V^T(t, x)$ in (27) is equivalent to maximise $x(T)$.

So, we have that

$$V^T(t, x) = \sup_{u(\cdot)} h\left(x(T; t, x, u(\cdot)\right) = \sup_{u(\cdot)} \left\{ x + \int_t^T u(s) ds \right\} = x \tag{29}$$

and, clearly, the optimal control is zero.

Then, we want to compute the hypograph of the value function. It is

$$\begin{aligned} \mathcal{H}\mathrm{ypo}(V^T) &= \left\{(t, x, r) \in [0, T] \times \mathbb{R}^2; V^T(t, x) \geq r\right\} \\ &= \left\{(t, x, r) \in [0, T] \times \mathbb{R}^2; x \geq r\right\} . \end{aligned}$$

We can see $\mathcal{H}\mathrm{ypo}(V^T)$ in Figure 1. This is the set which is the intersection of the three halfspaces: $r \geq x$, $t \geq 0$ and $t \leq T$.

We will now show that this set (*i.e.*, hypograph of value function V^T) is the viability kernel of the auxiliary viability problem with target $\{T\} \times \mathcal{H}\mathrm{ypo}(h)$ (*i.e.*, the back "wall"), defined as follows:

$$\left\{ \begin{array}{rcl} \dot{t} &=& 1 \\ \dot{x} &=& u \\ \dot{r} &=& 0 \end{array} \right. . \tag{30}$$

As before, the controls $u(\cdot) : [0, T] \to [-1, 0]$ are measurable functions. We can represent the viability kernel for this three-dimensional system with target $\{T\} \times \mathcal{H}\mathrm{ypo}(h)$ in the same Figure 1.

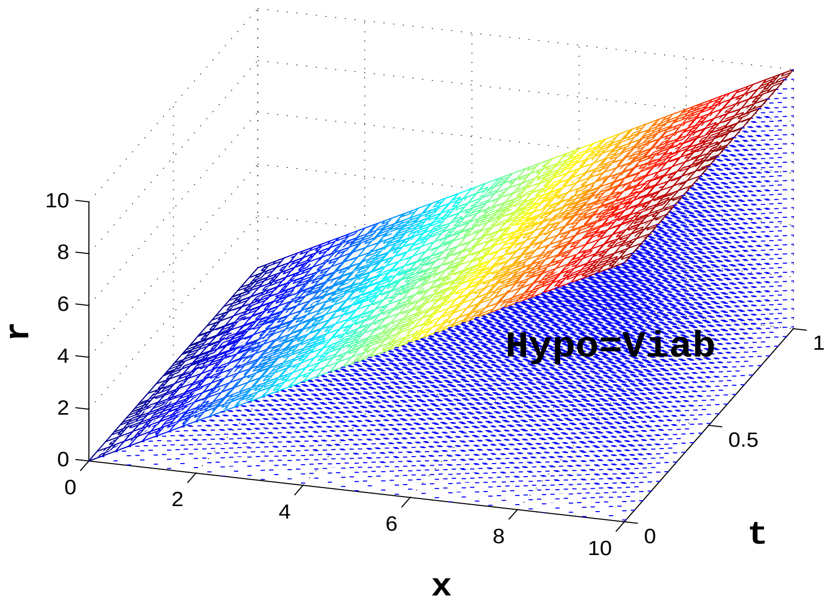

Figure 1: Viability kernel.

The trajectories of this system run from the front "wall" ($t = 0$) in the direction of increasing $t > 0$. They remain at the starting level $x(0)$ if $u(t) = 0 \; \forall \; t \in [0, \, 1]$ or "decrease" in value $x(\tau) < x(t)$ if $u(\tau) < 0$ for some $\tau \in [0, \, T]$. Notice that $r = x$ on the upper "wall" of the hypograph. This coincides with the graph of the value function (29).

In the following claim we will *prove* that the above hypograph of $V^T(t, x)$ is the viability kernel for (30) with target $\{T\} \times \mathcal{H}\mathrm{y}\mathfrak{po}(h)$. *I.e.*, we will show that a trajectory of the dynamic system (30) that starts anywhere inside the hypograph, remains in the hypograph and reaches the "terminal" wall at $t = 1$. Here, we formulate the claim.

Claim 10

$$\begin{aligned} \mathcal{H}\mathrm{y}\mathfrak{po}(V^T) &= \mathcal{V}\mathfrak{iab}_{(1,f,0)} \left[[0, T] \times \mathbb{R}^2 ; \{T\} \times \mathcal{H}\mathrm{y}\mathfrak{po}(h) \right] \\ &= \left\{ (t, x, r) \in [0, T] \times \mathbb{R}^2 ; x \geq r \right\}. \end{aligned}$$

Proof 11 *If we consider* $(t, x, r) \in \mathcal{H}\mathrm{ypo}(V^T)$ *then the solution* $(t + s, x + (t + s) \cdot 0, r)$ *of* (30) *starting from* (t, x, r) *stays in* $\mathcal{H}\mathrm{ypo}(V^T)$*, so* $\mathcal{H}\mathrm{ypo}(V^T)$ *is a viability domain with target* $\{T\} \times \mathcal{H}\mathrm{ypo}(h)$*. So, we have that*

$$\mathcal{H}\mathrm{ypo}(V^T) \subset \mathcal{V}\mathrm{iab}_{(1,f,0)}\left[[0, T] \times \mathbb{R}^2; \{T\} \times \mathcal{H}\mathrm{ypo}(h)\right], \tag{31}$$

because of the definition of the viability kernel and of the equality $V^T(T, x) = h(x)$

Conversely, suppose that D *is a viability domain in* $[0, T] \times \mathbb{R}^2$ *for (30) with target* $\{T\} \times \mathcal{H}\mathrm{ypo}(h)$ *and let* $(t, x, r) \in D$*. If so, then there exists* $u(\cdot)$ *such that* $\left(t + s, x + \int_t^{t+s} u(\tau)d\tau, r\right) \in D$*. Hence, for* $s = T - t$ *we have* $\left(T, x + \int_t^T u(\tau)d\tau, r\right) \in \{T\} \times \mathcal{H}\mathrm{ypo}(h)$*, which means that the trajectory meets the target hence* $x + \int_t^T u(\tau)d\tau = x(T) \geq r$*. Consequently,*

$$r \leq x + \int_t^T u(\tau)d\tau \ \leq x = V^T(t, x) \tag{32}$$

because $u(\cdot)$ *has negative values. We observe that* $(t, x, r) \in \mathcal{H}\mathrm{ypo}(V^T)$ *and conclude*

$$D \subset \mathcal{H}\mathrm{ypo}(V^T). \tag{33}$$

This finishes the proof because we have (31) and (33).

The above claim demonstrates analytically the fact that we have observed in Figure 1: the hypograph of $V^T(t, x)$ is identical with the viability kernel for (30) with target $\{T\} \times \mathcal{H}\mathrm{ypo}(h)$.

In the rest of this paper, similar equivalences will help us inferring about dominance (or non-dominance) of graphs of value functions.

5 Comparisons of value functions

Now, we will formulate the main result of this paper on a relationship between value functions' implied by the relationship between the corresponding Hamiltonians. The result will be proved for two optimal control problems indexed by $i \in \{1, 2\}$, satisfying the same hypotheses as in the previous sections.

Proposition 12 *If* $\bar{H}_1 \leq \bar{H}_2$ *then* $W_2^T \leq W_1^T$*. Similarly if* $\bar{H}_1 \geq \bar{H}_2$ *then* $W_2^T \geq W_1^T$*.*

Proof 13 *We will give the proof for the first part of the proposition; the second part can be proved in a similar manner. Also, our proof will finish when we have shown that* $\bar{H}_1 \leq \bar{H}_2$ *implies* $\mathcal{H}\mathfrak{ypo}(W_2^T) \subset \mathcal{H}\mathfrak{ypo}(W_1^T)$ *because the inclusion is trivially equivalent to* $W_2^T \leq W_1^T$*.*

We know from Proposition 7 and Remark 8 that the value functions W_i^T *are viscosity solution of the following PDE:*

$$\begin{cases} \dfrac{\partial W^T}{\partial t}(t,y) + \bar{H}\big(t,y,\dfrac{\partial W^T}{\partial y}(t,y))\big) = 0 \\ \\ (t,y) \in [0,T) \times \mathbb{R}^{N+1}, \quad W_i^T(T,\cdot) = h_i(\cdot). \end{cases} \tag{34}$$

where the Hamiltonians $\bar{H}_i : \mathbb{R}^{N+3} \times \mathbb{R}^{N+3} \to \bar{\mathbb{R}}$ *are given by:*

$$\bar{H}_i(t,y,r,q) = \max_{u_i \in U_i} \langle q, (f_i(x,u), L_i(t,x,u)) \rangle .$$

Denoting

$$(t,x,z,r) \to \phi_i(t,x,z,r) = (1, f_i(x,U_i); L_i(x,U_i), 0) .$$

we have from Theorem 9 that

$$\mathcal{H}\mathfrak{ypo}(W_i^T) = \mathcal{V}\mathfrak{iab}_{\phi_i}\big([0,T] \times \mathbb{R}^{N+2}, \{T\} \times \mathcal{H}\mathfrak{ypo}(h_i)\big) . \tag{35}$$

Because $h_i(\cdot)$ *is u.s.c., value function* W^T *is also u.s.c. This means that the value function is the biggest upper semicontinuous subsolution so, it is solution to the Bellman equation (34). Consequently, using results of [18] and Proposition 4, the property (35) is equivalent to:*

$$\begin{cases} p_t + \bar{H}_i(t,y,p_y) &= & 0 \\ \textit{for all } (t,y) \in \ (0,T) & \times & \mathbb{R}^{N+2} \\ \textit{and } (p_t,p_y,p_r) & \in & \mathcal{NP}_{\mathcal{H}\mathfrak{ypo}(W_i^T)}\big(t,y,W_i^T(t,y)\big) \end{cases} \tag{36}$$

with the limit conditions at 0 and T (see [18] for details).

Consequently, for the case of $i = 1,2$*, if* $\bar{H}_1 \leq \bar{H}_2$ *and (36) is satisfied for* $i = 2$ *then we have that*

$$\begin{cases} p_t + \bar{H}_1(t,y,p_y)) \leq p_t + \bar{H}_2(t,y,p_y)) = 0 \\ \textit{for all } (p_t,p_y,p_r) \in \mathcal{NP}_{\mathcal{H}\mathfrak{ypo}(W_2^T)}\big(t,y,W_2^T(t,y)\big) \end{cases} \tag{37}$$

hence $\mathcal{H}\mathfrak{ypo}(W_2^T)$ *is a viability domain for* ϕ_1*. So, we have that*

$$\mathcal{V}\mathfrak{iab}_{\phi_2}\Big([0,T]\times\mathbb{R}^{N+2}, \{T\}\times\mathcal{H}\mathfrak{ypo}(h_2)\Big) \subset \mathcal{V}\mathfrak{iab}_{\phi_1}\Big([0,T]\times\mathbb{R}^{N+2}, \{T\}\times\mathcal{H}\mathfrak{ypo}(h_1)\Big)$$

because $\mathcal{H}\mathfrak{ypo}(W_1^T)$ *is viability kernel for* ϕ_1*. Consequently,* $\mathcal{H}\mathfrak{ypo}(W_2^T) \subset \mathcal{H}\mathfrak{ypo}(W_1^T)$ *and* $W_2^T \leq W_1^T$*, which finishes the proof.*

6 Technology switching problem

6.1 Related optimal control problems

We consider control systems indexed by $i \in \{1, 2, \ldots, n\}$, n is finite, whose dynamics are given by:

$$\dot{x}_i(s) = f_i(x_i(s), u_i(s)) \tag{38}$$

where the state variable x_i belongs to $\mathbb{R}^N$, the control $u_i(\cdot) : [0, \infty) \to U$ is a measurable function and $f_i : \mathbb{R}^N \times U \to \mathbb{R}^N$.

The control problem consists of

$$\text{Maximise} \int_t^T L\left(x_i(s; t, x, u(\cdot)), u(s)\right) ds \tag{39}$$

over all absolutely continuous solutions of (38), where $x_i(\cdot; t, x, u(\cdot))$ denotes the solution of (38) starting from $(t, x) \in [0, \infty) \times \mathbb{R}^N$.

Here $L : \mathbb{R}^N \times U \to \mathbb{R}$ is a given bounded function. If we denote by $\mathcal{U}_{[t,T]}$ the set of measurable controls on $[t, T]$ with values in U, then the value function corresponding to the optimal control problem (38) and (39), which is similar to problem (1)-(2), is given by:

$$V_i^T(t, x) = \sup_{u \in \mathcal{U}_i(t)} \int_t^T L\left(x_i(s; t, x, u(\cdot)), u(s)\right) ds \tag{40}$$

$$\begin{aligned} W_i^T(t, y) &= \sup_{u \in \mathcal{U}_i(t)} \left(h_i(y_i(T; t, x, u(\cdot)))\right) \qquad (41) \\ &= \sup_{u \in \mathcal{U}_i(t)} \left(z + \int_t^T L\left(x_i(s; t, x, u(\cdot)), u(s)\right) ds\right). \end{aligned}$$

We note that by (22)

$$W_i^T(t, y) = V_i^T(t, x) + z \text{ for all } (t, y) = (t, x, z) \in [0, T] \times \mathbb{R}^{N+1}$$

and that here $h_i(x, z) = z$ because there is no scrap value function. We conclude that comparing W_i^T (for different i) is equivalent to comparing V_i^T.

Below we provide an example where the result obtained in Proposition 12 enables us to compare the value functions of two related optimal control problems without solving them explicitly.

6.2 A motivating example

Example 14 *Consider an optimal control problem with linear dynamics indexed by "technology"* $i = 1, 2$

$$\dot{x} = f_i(x, u), \quad f_i(x, u) := \theta_i u - \mu_i x, \quad \theta_i > 0,\, \mu_i \geq 0 \tag{42}$$

where u *is control and* x *is state, and with the following concave utility function*

$$L(t, x, u) = e^{-\rho t}(\ln u - \beta x) \quad \rho > 0,\, \beta > 0\,. \tag{43}$$

Assess if a positive switching time between the technologies usage exists.

The above model is a version of the macroeconomic model considered in [12], which we give a microeconomic interpretation in this paper. Here is a situation that may lead to the above model.

An industry's output $Y(t)$ is produced proportionally to input $I(t)$ (*e.g.*, $I(t)$ could be fuel or water). If so, $Y(t) = A_i I(t)$ where $A_i > 1$ is the marginal productivity[12] under technology i.

The flow of output $Y(t)$ produced in technology i causes emissions $E_i(t) = \alpha_i Y(t),\ \alpha_i > 0$ that accumulate and contribute to pollution stock $x(t)$ as in the following state equation

$$\dot{x}(t) = \alpha_i Y(t) - \mu_j x(t)\,. \tag{44}$$

Here, $\mu_j \geq 0$ is the self-cleaning coefficient, which may be decomposed to a natural decay coefficient $\mu_0 \geq 0$ and a term that depends on cleaning technology j; the latter might be related to the production technology i but not necessarily. If those two are unrelated and *technology adoption* concerns the production technology only, then the self-cleaning dynamics is $-\mu_0 x(t)$. In this case, for simplicity, one can omit the subindex $_0$ and write $-\mu x(t)$. If *technology adoption* concerned both production and cleaning technologies jointly, then we would write the right hand side of (44) as $\alpha_i Y(t) - \mu_i x(t)$; if the adoption was just about the cleaning technology, we could write it as $\alpha Y(t) - \mu_i x(t)$. In the remainder of this paper we will deal with the first of the above cases and write the systems dynamics as $\alpha_i Y(t) - \mu_i x(t)$.

Output $Y(t)$ is used for input $I(t)$ and consumption (or wages) $u(t)$, so

$$Y(t) = A_i I(t) = I(t) + u(t)$$

[12] The authors of [12] call their model an AK-type and refer to A_i as to the marginal productivity of capital.

from where

$$I(t) = \frac{u(t)}{A_i - 1} \, . \tag{45}$$

Combing (45) with (44) yields systems dynamics (42) where

$$\theta_i := \frac{\alpha_i A_i}{A_i - 1} \tag{46}$$

is the technology indicator that aggregates the information on the technology productivity and emission propensity.

The utility function (11) captures the industry manager's combined preferences for consumption u and aversion to *pollution stock* x. The latter might be the case if the industry is using a resource (like water) that becomes polluted by the production process (*e.g.*, consider a paper pulp mill whose water inlet is below its outlet); or because the government is measuring $x(t)$ and taxing the industry $\beta x(t)$.

We will suppose that the optimisation horizon is here finite and sufficiently long for us to assume that the *scrap value* impact on control is negligible.

In continuous time and given a technology ($i = 1$, say), the manager is using an optimal strategy $u(x(t))$ (which is elementary[13] for this model). However, given the availability of a new technology $i = 2$ with lower emissions $\alpha_2 < \alpha_1$, the manager will consider the new technology adoption. The adoption would happen, in discrete time, if the new technology value function dominated the old one. Given that the new technology productivity might be lower (*i.e.*, $A_2 < A_1$) and that the "current" pollution level can (still) be relatively low, the decision about the switching time is not obvious. Our study will show if such a time exists.[14]

6.3 Hamiltonian dominance

Here the auxiliary process dynamics is $\phi_i(t, x, z, r) = (1, \theta_i U - \mu_i x, e^{-\rho t}(\ln U - \beta x), 0)$. We aim to examine the viability kernels for the auxiliary dynamics associated with each technology (compare (15)). In other words, we want to see if a hypograph of the value function of one technology is included in the hypograph of the value function of the other technology. If so, we

[13]It is also elementary to prove that value function is linear in this case.

[14]We assume zero implementation cost of the new technology. However, if our study of the Hamiltonians (see Proposition 12) indicated that the viability kernels inclusion were not proper we would say that the switch between the technologies would *not* occur because of a lack of incentive for a change.

will conclude that there is no positive switching time between the use of the technologies.

Because of Result 12 we can rely on the relationship between the Hamiltonians. Let us write the Hamiltonian (25) for technology i:

$$\begin{aligned}\bar{H}_i(t,y,r,q) &= \max_{u\in U}\langle q,(\theta_i u-\mu_i x, e^{-\rho t}(\ln u-\beta x))\rangle \\ &= \langle q,(\theta_i u_i-\mu_i x, e^{-\rho t}(\ln u_i-\beta x))\rangle\,.\end{aligned}$$

Here u_i is the maximiser, y,r,q are fixed, of dimensions $2,1,2$ respectively.

We will assume that *technology adoption* concerns here the production and cleaning technologies jointly and that the new technology is "better" *i.e.*, $\mu_2 \geq \mu_1$.[15]

To be attractive, the new technology will certainly have lower emissions $\alpha_2 < \alpha_1$. It turns out (below) that if $A_2 > A_1$ then the technology indicator θ_2, which relates the emission accumulation to consumption, decreases (improves) for $\alpha_2 \leq \alpha_1$ (*i.e.*, θ_2 improves even if $\alpha_2 = \alpha_1$). Hence the "interesting" case is when $\alpha_2 \leq \alpha_1$ and the new technology is less productive than the old one, $A_2 < A_1$.

We will assume that the coefficients α_i, A_i, $i = 1,2$ are such that $\theta_2 > \theta_1$ and prove that $\bar{H}_1 \leq \bar{H}_2$. Hence we will obtain a sufficient condition for the case where there is *no* switch at $t > 0$.

Indeed we have that

$$\begin{aligned}\bar{H}_1(t,y,r,q) &= \max_{u\in U}\langle q,(\theta_1 u-\mu_1 x, e^{-\rho t}(\ln u-\beta x))\rangle \\ &= \langle q,(\theta_1 u_1-\mu_1 x, e^{-\rho t}(\ln u_1-\beta x))\rangle\end{aligned}$$

It is sufficient to find a u such that

$$\begin{aligned}\bar{H}_1(t,y,r,q) &= \max_{u\in U}\langle q,(\theta_1 u-\mu_1 x, e^{-\rho t}(\ln u-\beta x))\rangle \\ &= \langle q,(\theta_1 u_1-\mu_1 x, e^{-\rho t}(\ln u_1-\beta x))\rangle \\ &\leq \langle q,(\theta_2 u-\mu_2 x, e^{-\rho t}(\ln u-\beta x))\rangle \\ &\leq \max_{u\in U}\langle q,(\theta_2 u-\mu_2 x, e^{-\rho t}(\ln u-\beta x))\rangle \\ &= \bar{H}_2(t,y,r,q)\,.\end{aligned}$$

If we denote $q := (q_x, q_z)$ and

$$\Gamma(u) := \langle q,(\theta_2 u-\mu_2 x, e^{-\rho t}(\ln u-\beta x))\rangle - \langle q,(\theta_1 u_1-\mu_1 x, e^{-\rho t}(\ln u_1-\beta x))\rangle$$

[15] Obviously, the proof below remains correct if technology adoption concerns the production process only and the self-cleaning dynamics is $-\mu x(t)$.

then

$$\begin{aligned}\Gamma(u) &:= \langle q, (\theta_2 u - \mu_2 x, e^{-\rho t}(\ln u - \beta x))\rangle - \langle q, (\theta_1 u_1 - \mu_1 x, e^{-\rho t}(\ln u_1 - \beta x))\rangle \\ &:= q_x(\theta_2 u - \theta_1 u_1 + (\mu_1 - \mu_2)x) + q_z e^{-\rho t}(\ln u - \ln u_1)\end{aligned}$$

from where we see that

$$\lim_{u\to\infty} \Gamma(u) := \infty \text{ if } q_x > 0\,.$$

Alternatively, if $q_x \leq 0$

$$\Gamma(u_1) \geq 0, \text{because } \theta_1 < \theta_2 \text{ and } \mu_2 \geq \mu_1$$

as $\theta_i, \mu_i, \beta, \rho, u, x$ are positive real numbers.

It is easy to prove that if $\theta_2 < \theta_1$ then the $\bar{H}_1 > \bar{H}_2$ and the new technology should be adopted immediately ($t = 0$).

We can generalise the result obtained for the technology adoption example in the following remark.

Remark 15 *For optimal control problems with linear dynamics indexed through* θ_i *as in (42) and utility functions*

$$L(t, x, u) = e^{-\rho t} \left(l_1(u) - l_2(x) \right), \quad \rho > 0 \tag{47}$$

where $l_1(\cdot)$ *is strictly increasing,* $\lim_{u \to \infty} \frac{u}{l_1(u)} = \infty$ *and* $l_2(\cdot)$ *is a positive function, if* $\theta_2 > \theta_1$ *then Hamiltonian* H_2 *dominates Hamiltonian* H_1 *i.e.,* $H_2 \geq H_1$*. Consequently, the hypograph of value function* W_2 *is included in the hypograph of value function* W_1 *i.e.,* $\mathcal{H}\mathrm{ypo}(W_2^T) \subset \mathcal{H}\mathrm{ypo}(W_1^T)$ *.*

With respect to the technology adoption problem we can say that for the class of utility functions specified in Remark 15, which characterise a trade-off between satisfaction from consumption $l_1(\cdot)$ and disutility due to pollution $l_2(\cdot)$, the adoption cannot occur at $t > 0$ (compare results of [12]).

7 Concluding remarks

We have presented an approach suitable for the determination whether a "new" technology will replace the "old" technology. All the regulator needs to do is to compare the Hamiltonians of the optimal control problems formulated for each technology. We have seen that a linear dynamics and a "trade-off" type utility function exclude such a possibility.

More generally, the approach presented in this paper helps solve a two-stage optimal control problem by indicating when the problem will have no second stage.

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