

2009/44



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and an application to mixing inequalities

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CORE DISCUSSION PAPER
2009/44

**Lifting group inequalities
and an application to mixing inequalities**

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July 2009

Abstract

Given a valid inequality for the mixed integer infinite group relaxation, a lifting based approach is presented that can be used to strengthen this inequality. Bounds on the solution of the corresponding lifting problem and some necessary conditions for the lifted inequality to be minimal for the mixed integer infinite group relaxation are presented. Finally, these results are applied to generate a strengthened version of the mixing inequality that provides a new class of extreme inequalities for the two-row mixed integer infinite group relaxation.

Keywords: mixed integer programs, mixing inequalities, infinite group relaxations.

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This paper presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.

1 Introduction

Given a valid inequality for the mixed integer infinite group relaxation, we consider how such an inequality can be strengthened when it is not minimal. Given an initial inequality that is extreme when restricted to the continuous variables (the continuous infinite group relaxation), two standard approaches are sequential lifting of the integer variables and the use of a fill-in function. As the first is computationally very costly, and the second may not provide very strong inequalities, we study a composite lifting/fill-in approach that may be computationally viable. Applied to the mixing inequalities in two dimensions, this composite approach provides a new class of extreme inequalities for the two row mixed integer infinite group problem. We now introduce the problem and briefly describe related research.

Let I^m be the group defined by the set $\{(u_1, u_2, \dots, u_m) \in \mathbb{R}^m \mid 0 \leq u_1, u_2, \dots, u_m < 1\}$ and let the group operation be defined as addition modulo 1 componentwise. We use the symbols $+$ and $-$ to represent addition and subtraction in both I^m and \mathbb{R}^m . We use the symbol $\bar{0}$ to represent the zero vector in \mathbb{R}^m and I^m . For any $w \in \mathbb{R}^m$, we use the symbol $\mathbb{P}(w)$ to denote the element u in I^m where $u_i = w_i \pmod{1}$.

Given $r \in I^m$, $r \neq \bar{0}$, U a subgroup of I^m and W a subset of \mathbb{R}^m , the infinite group relaxation $MI(U, W, r)$ is the set of pairs (x, y) that satisfy

1. $x : U \rightarrow \mathbb{Z}_+$, $y : W \rightarrow \mathbb{R}_+$, x and y have finite support,
2. $\sum_{u \in U} ux(u) + \mathbb{P}(\sum_{w \in W} wy(w)) = r$.

Gomory and Johnson introduced the infinite group relaxation of mixed integer programs in the 1970's ([12], [13], [15]). A pair of functions (ϕ, π) are called a valid inequality (or valid function) for $MI(U, W, r)$ if $\phi : U \rightarrow \mathbb{R}_+$, $\pi : W \rightarrow \mathbb{R}_+$ and $\sum_{u \in U} \phi(u)x(u) + \sum_{w \in W} \pi(w)y(w) \geq 1$ for all $(x, y) \in MI(U, W, r)$. A valid function (ϕ, π) is said to be minimal for $MI(U, W, r)$ if there does not exist a valid function (ϕ^*, π^*) for $MI(U, W, r)$ different from (ϕ, π) such that $\phi^*(u) \leq \phi(u) \forall u \in U$ and $\pi^*(w) \leq \pi(w) \forall w \in W$. A valid function (ϕ, π) is extreme for $MI(U, W, r)$ if there do not exist valid functions (ϕ_1, π_1) and (ϕ_2, π_2) for $MI(U, W, r)$ such that $(\phi_1, \pi_1) \neq (\phi_2, \pi_2)$ and $(\phi, \pi) = \frac{1}{2}(\phi_1, \pi_1) + \frac{1}{2}(\phi_2, \pi_2)$.

Considerable research in recent years has gone into understanding the minimal and extreme inequalities of the continuous infinite group relaxation $MI(\{\bar{0}\}, \mathbb{R}^m, r)$; see [1],[3],[5],[16]. Given a minimal or extreme valid inequality $\pi : \mathbb{R}^m \rightarrow \mathbb{R}_+$ for $MI(\{\bar{0}\}, \mathbb{R}^m, r)$, [11] and [4] consider the problem of obtaining a function $\phi : I^m \rightarrow \mathbb{R}_+$ such that

$$\sum_{u \in I^m} \phi(u)x(u) + \sum_{w \in \mathbb{R}^m} \pi(w)y(w) \geq 1 \tag{1}$$

is minimal or extreme valid inequality for $MI(I^m, \mathbb{R}^m, r)$. A natural candidate for constructing the function ϕ so that (ϕ, π) is a valid inequality, is to construct the so called trivial fill-in function ([12],[15],[2],[11]). However, the trivial fill-in function is not necessarily a minimal or extreme inequality for $MI(I^m, \mathbb{R}^m, r)$. This motivates us to consider the question of strengthening (1) when it is not minimal.

The approach pursued here for strengthening the inequalities is related to both the lifting approach and the fill-in approach. The basics of this composite fill-in and lifting-based approach and the rationale behind it are discussed in Section 2. In Section 3 we present some bounds on the

solution of the lifting problem. Section 4 presents some necessary conditions for the resulting inequalities to be minimal. Finally in Section 5 we illustrate the use of this technique by strengthening the mixing inequalities.

2 Basics of Lifting

Given a valid inequality (ϕ, π) for $MI(I^m, \mathbb{R}^m, r)$, the goal is to generate a stronger valid inequality for $MI(I^m, \mathbb{R}^m, r)$. We assume that π is an extreme inequality for $MI(\{\bar{0}\}, \mathbb{R}^m, r)$. One traditional approach is to sequentially lift in the integer variables $x(u)$ for all $u \in I^m$ to obtain $\psi : I^m \rightarrow \mathbb{R}_+$ such that (ψ, π) is an extreme inequality for $MI(I^m, \mathbb{R}^m, r)$. Since $x(u)$ is a general integer variable, the calculation of each lifting coefficient $\psi(u)$ involves solving a sequence of mixed integer programs which can be computationally prohibitive. On the other hand, it is possible to construct the fill-in function ([12], [15], [2],[11]) which involves solving one mixed integer program corresponding to each integer variable $x(u)$ for $u \in I^m$, while not necessarily producing a strong inequality. In this paper, we take a middle path between the traditional lifting and the fill-in process.

We begin with a discussion on subadditive valid inequalities and the fill-in approach in Section 2.1. We discuss the approach followed in this paper in Section 2.2.

2.1 Fill-in inequality

We start with a valid inequality

$$\sum_{u \in U} \alpha(u)x(u) + \sum_{w \in W} \beta(w)y(w) \geq 1$$

for $MI(U, W, r)$ where we assume that positive combinations of the columns in W span \mathbb{R}^m .

Now let $\phi_{\alpha, \beta} : I^m \rightarrow \mathbb{R}_+$ be the function such that $\phi_{\alpha, \beta}(\bar{0}) = 0$ and

$$\phi_{\alpha, \beta}(v) = \inf \left\{ \sum_{u \in U} \alpha(u)x(u) + \sum_{w \in W} \beta(w)y(w) \mid (x, y) \in MI(U, W, v) \right\} \quad \forall v \in I^m \setminus \{\bar{0}\}, \quad (2)$$

and let $\pi_{\alpha, \beta} : \mathbb{R}^m \rightarrow \mathbb{R}_+$ be the function

$$\pi_{\alpha, \beta}(v) = \inf \left\{ \sum_{w \in W} \beta(w)y(w) \mid \sum_{w \in \mathbb{R}^m} \beta(w)y(w) = v, \right. \\ \left. y(w) \geq 0, y \text{ has finite support} \right\}. \quad (3)$$

It can be verified that

1. $\phi_{\alpha, \beta}(u) \leq \alpha(u) \quad \forall u \in U$ and $\pi_{\alpha, \beta}(w) \leq \beta(w) \quad \forall w \in W$.
2. $(\phi_{\alpha, \beta}, \pi_{\alpha, \beta})$ is a valid inequality for $MI(I^m, \mathbb{R}^m, r)$.
3. $(\phi_{\alpha, \beta}, \pi_{\alpha, \beta})$ is subadditive, i.e.

$$\phi_{\alpha, \beta}(u^1) + \phi_{\alpha, \beta}(u^2) \geq \phi_{\alpha, \beta}(u^1 + u^2) \quad \forall u^1, u^2 \in I^m, \quad (4)$$

$$\phi_{\alpha, \beta}(u) + \pi_{\alpha, \beta}(w) \geq \phi_{\alpha, \beta}(u^1 + \mathbb{P}(w)) \quad \forall u \in I^m, \forall w \in \mathbb{R}^m, \quad (5)$$

$$\sum_{w \in Q} \pi_{\alpha, \beta}(w)y(w) \geq \pi_{\alpha, \beta}\left(\sum_{w \in Q} wy(w)\right), \quad Q \text{ is a finite subset of } \mathbb{R}^m. \quad (6)$$

$$4. \lim_{h \downarrow 0} \frac{\phi_{\alpha, \beta} \mathbb{P}((hw))}{h} = \pi_{\alpha, \beta}(w) \quad \forall w \in \mathbb{R}^m.$$

See [11] for a proof. We call the functions defined in (2) and (3) the fill-in functions. Essentially given a valid inequality (α, β) , the fill-in function is a subadditive valid inequality that dominates (α, β) .

If $U = \{\bar{0}\}$ and $\alpha(\bar{0}) = 0$, then the inequality $\phi_{\alpha, \beta}$ is called the trivial fill-in function. Moreover if $\beta : W \rightarrow \mathbb{R}_+$ is an extreme inequality for $MI(\{\bar{0}\}, W, r)$, then $\pi_{\alpha, \beta}(w) = \beta(w)$ for $w \in W$.

2.2 Lifting one variable followed by fill-in

Suppose that (ϕ, π) is a subadditive valid inequality for $MI(I^m, \mathbb{R}^m, r)$. The goal is to construct a function $\psi : I^m \rightarrow \mathbb{R}_+$ such that (ψ, π) is a valid inequality and ψ dominates ϕ . In particular, assume that we would like to improve the coefficient of $x(a)$ in the new function.

One approach to obtain a better coefficient for $x(a)$ is to solve the following problem (This is the approach followed in [11]): What is the smallest value of $\hat{\eta}$ such that the inequality

$$\hat{\eta} \tilde{x} + \sum_{w \in \mathbb{R}^m} \pi(w) y(w) \geq 1$$

is valid for all feasible solutions to

$$a \tilde{x} + \mathbb{P} \left(\sum_{w \in \mathbb{R}^m} w y(w) \right) = r$$

where $\tilde{x} \in \mathbb{Z}_+$, $y(w) \in \mathbb{R}_+$ and y has finite support. Specifically the smallest value of $\hat{\eta}$ denoted η can be obtained as $\eta := \sup_{n \in \mathbb{Z}_+, n \geq 1, y(w) \in \mathbb{R}_+} \left(\frac{1}{n} (1 - \sum_{w \in B} \pi(w) y(w)) \mid \mathbb{P}(\sum_{w \in B} w y(w)) = r - na \right)$. However, in this case there is no guarantee that there exists a valid inequality (ψ, π) such that $\psi(a) = \eta$ and $\psi \leq \phi$.

Therefore we solve the following modified lifting problem: What is the smallest value of $\hat{\gamma}$ such that the inequality

$$\hat{\gamma} \tilde{x} + \sum_{u \in I^m} \phi(u) x(u) + \sum_{w \in \mathbb{R}^m} \pi(w) y(w) \geq 1$$

is valid for all feasible solutions to $a \tilde{x} + \sum_{u \in I^m} u x(u) + \mathbb{P}(\sum_{w \in \mathbb{R}^m} w y(w)) = r$ where $\tilde{x} \in \mathbb{Z}_+$, $x(u) \in \mathbb{Z}_+$, $y(w) \in \mathbb{R}_+$, and x and y have finite supports. The smallest value of $\hat{\gamma}$ denoted γ is obtained as

$$\gamma := \sup_{n \in \mathbb{Z}_+, n \geq 1} \left\{ \frac{1 - (\sum_{u \in I^m} \phi(u) x(u) + \sum_{w \in \mathbb{R}^m} \pi(w) y(w))}{n} \mid \sum_{u \in I^m} u x(u) + \mathbb{P} \left(\sum_{w \in \mathbb{R}^m} w y(w) \right) = r - na \right\}.$$

Since (ϕ, π) is a subadditive valid inequality, we obtain

$$\gamma = \sup_{n \in \mathbb{Z}_+, n \geq 1} \left\{ \frac{1 - \phi(r - na)}{n} \right\}. \quad (7)$$

Since \tilde{x} and $x(a)$ are unbounded integer variables, we obtain that the inequality (ϕ', π) is valid for $MI(I^m, \mathbb{R}^m, r)$ where $\phi' : I^m \rightarrow \mathbb{R}_+$ is defined as

$$\phi'(v) = \begin{cases} \phi(v) & \text{if } v \neq a \\ \gamma & \text{if } v = a. \end{cases}$$

Finally, we can construct a subadditive valid inequality that dominates ϕ' using (2). We call this function ϕ^a . Formally, for all $v \in I^m$,

$$\begin{aligned} \phi^a(v) &= \inf \left\{ \sum_{u \in U} \phi'(u)x(u) + \sum_{w \in W} \pi(w)y(w) \mid (x, y) \in MI(U, W, v) \right\} \\ &= \inf \left\{ \sum_{u \in I^m \setminus \{a\}} \phi(u)x(u) + \sum_{w \in \mathbb{R}^m} \pi(w)y(w) + n\gamma \right. \\ &\quad \text{s.t.} \quad \sum_{u \in I^m \setminus \{a\}} ux(u) + \mathbb{P} \left(\sum_{w \in \mathbb{R}^m} wy(w) \right) + na = v, \\ &\quad \left. x(u), n \in \mathbb{Z}_+, y(w) \geq 0, x \text{ and } y \text{ have finite supports.} \right\} \end{aligned} \quad (8)$$

or equivalently

$$\phi^a(v) = \inf_{n \in \mathbb{Z}_+} \{ \phi(v - na) + n\gamma \}. \quad (9)$$

Note here that $\phi^a \leq \phi$ and (ϕ^a, π) is a valid subadditive inequality for $MI(I^m, \mathbb{R}^m, r)$. We note here that this composite approach is very closely related to the approach in [11].

Observe that when we compute γ using (7), we require the value of the function $\phi(r - na)$, which is equivalent to solving the mixed integer program given by (2) for each positive integer n . On the other hand, the computation of $\phi^a(u)$ for $u \neq a$ (in the next section it will be shown that $\phi^a(a) = \gamma$) involves solving exactly one mixed integer program given by (8). Thus the computation of ϕ^a is a computationally cheaper method of constructing a valid inequality than sequentially lifting in each variable which would involve solving a lifting problem (7) corresponding to each $x(u)$, $u \in I^m$.

Based on the above discussion, we are led to the following two questions.

1. How difficult is the computation of γ ? In Section 3, we present an upper bound on the positive integer n that solves the lifting problem (7).
2. What choice of a leads to a “strong” inequality? In Section 4, we present some necessary conditions on the choice of a so that (ϕ^a, π) is a minimal valid inequality for $MI(I^m, \mathbb{R}^m, r)$.

3 Bounds for General Lifting

We make the assumption that $\phi(u) \leq 1 \forall u \in I^m$. This is not a strong assumption as the following result can be easily verified.

Proposition 1. *If (ϕ, π) is a subadditive valid inequality for $MI(I^m, \mathbb{R}^m, r)$, then $(\hat{\phi}, \pi)$ is a subadditive valid inequality for $MI(I^m, \mathbb{R}^m, r)$, where $\hat{\phi}(u) = \min\{\phi(u), 1\} \forall u \in I^m$.*

With this assumption we obtain the following result.

Proposition 2.

$$\gamma = \frac{1 - \phi(r - \tilde{n}a)}{\tilde{n}}$$

for some $\tilde{n} \in \mathbb{Z}_+$, $\tilde{n} \geq 1$.

Proof. Since $\phi(u) \leq 1 \forall u \in I^m$, we obtain that $1 - \phi(r - na) \geq 0 \forall n \in \mathbb{Z}_+$. Now observe that if $\gamma = 0$, then $\frac{1 - \phi(r - a)}{1} = 0$. On the other hand if $\gamma > 0$, then there exists N such that $\frac{1 - \phi(r - na)}{n} < \gamma \forall n \geq N$. \square

From the definition (9) of ϕ^a it is clear that $\phi^a(a) \leq \gamma$ where γ is obtained using (7). The next proposition shows that $\phi^a(a) = \gamma$.

Proposition 3. $\phi^a(a) = \gamma$.

Proof. If $\gamma = 0$, then the proof is complete, since by the validity of ϕ^a , $\phi^a(a) \geq 0$.

Now let $\gamma > 0$. Since $\phi(u) \geq 0 \forall u \in I^m$, we obtain that $n\gamma + \phi(a - na) > \gamma$ for $n > 1$. Therefore $\phi^a(a) = \min\{\phi(a), \gamma\}$. To complete the proof we need to show that $\phi(a) \geq \gamma$. Now observe that by Proposition 2, there exists \tilde{n} such that $\gamma = \frac{1 - \phi(r - \tilde{n}a)}{\tilde{n}}$. By validity of ϕ , we obtain $\tilde{n}\phi(a) + \phi(r - \tilde{n}a) \geq 1$ or equivalently $\phi(a) \geq \frac{1 - \phi(r - \tilde{n}a)}{\tilde{n}} = \gamma$. \square

Let $\mathcal{S}(\phi, a) = \{p \in \mathbb{Z}_+ \setminus \{0\} \mid p\phi^a(a) + \phi(r - pa) = 1\}$ be the set of positive integers that solve the lifting problem (7). This set is well-defined and non-empty due to Proposition 2 and Proposition 3.

Proposition 4. *If $p \in \mathcal{S}(\phi, a)$, then*

1. $\phi^a(ta) + \phi^a(r - ta) = 1 \forall 1 \leq t \leq p, t \in \mathbb{Z}$,
2. $\phi^a(ta) = t\phi^a(a) \forall 1 \leq t \leq p, t \in \mathbb{Z}$.

Proof. For $1 \leq t \leq p, t \in \mathbb{Z}$ observe that

$$\begin{aligned} \phi^a(ta) + \phi^a(r - ta) &= \phi^a(ta) + \inf_{n \in \mathbb{Z}_+} \{n\phi^a(a) + \phi(r - ta - na)\} \\ &\leq t\phi^a(a) + \inf_{n \in \mathbb{Z}_+} \{n\phi^a(a) + \phi(r - ta - na)\} \end{aligned} \quad (10)$$

$$\begin{aligned} &= \inf_{n \in \mathbb{Z}_+} \{(n + t)\phi^a(a) + \phi(r - (t + n)a)\} \\ &\leq p\phi^a(a) + \phi(r - pa) \\ &= 1 \\ &\leq \phi^a(ta) + \phi^a(r - ta). \end{aligned} \quad (11)$$

The first equality follows from the definition of ϕ^a . The first inequality is due to the subadditivity of the function ϕ^a . The second inequality is obtained by setting $n = p - t$. The last equality follows from the definition of $\phi^a(a)$ and the last inequality follows from the subadditivity of the function ϕ^a .

The result follows from the fact that (11) and (10) must be satisfied at equality. \square

The result of Proposition 4 can be used to obtain an upper bound on the lifting coefficient as follows: Suppose that $\phi^a(a) = \frac{1}{p}(1 - \phi(r - pa))$. For any $n \leq p$, we know that $n\phi^a(a) = \phi^a(na) \leq \phi(na)$ or equivalently $\phi^a(a) \leq \frac{\phi(na)}{n}$. This upper bound can be used to improve on the naïve algorithm to determine $\phi^a(a)$ which consists of enumerating $\frac{1 - \phi(r - na)}{n}$ for all n . This is presented in Table 1.

The next corollary is useful to obtain a bound on the integers in $\mathcal{S}(\phi, a)$.

Corollary 5 (Bounds on Lifting). *If $\phi(qa) + \phi(r - qa) = 1$ for some positive integer q , then $\exists p \in \mathcal{S}(\phi, a)$ such that $p \leq q$.*

Table 1: Algorithm to obtain $\phi^a(a)$ within an error of ϵ

-
1. Set $N \leftarrow +\infty$, $UB \leftarrow 1$, $LB \leftarrow 0$, $i \leftarrow 1$.
 2. While $i \leq N$:
 - (a) Compute $\phi(ia)$ and $\phi(r - ia)$. Set $LB \leftarrow \max\{LB, \frac{1-\phi(r-ia)}{i}\}$. Set $UB \leftarrow \min\{UB, \frac{\phi(ia)}{i}\}$.
 - (b) Update N:
 - If $UB \leq LB + \epsilon$, then set $N \leftarrow i$.
 - Else set $N \leftarrow \lceil \frac{1}{LB} \rceil$.
 - (c) $i \leftarrow i + 1$.
 3. Return UB .
-

Proof. Since $\phi^a(u) \leq \phi(u) \forall u \in I^m$, we obtain that $\phi^a(qa) \leq \phi(qa)$ and $\phi^a(r - qa) \leq \phi(r - qa)$. Therefore by subadditivity of ϕ^a , we obtain

$$\phi^a(qa) = \phi(qa). \quad (12)$$

Now suppose that $\exists p' \in \mathcal{S}(\phi, a)$ such that $p' > q$. From Proposition 4, we obtain $\phi^a(qa) = q\phi^a(a)$. Together with (12), this implies that

$$\phi(qa) = q\phi^a(a). \quad (13)$$

Since $\phi(qa) = 1 - \phi(r - qa)$, we obtain that $\phi^a(a) = \frac{1}{q}(1 - \phi(r - qa))$ or equivalently $q \in \mathcal{S}(\phi, a)$. This completes the proof. \square

Similar to Corollary 5, the next result shows that computing the function $\phi^a(u)$ for $u \neq a$ requires an examination of limited number of integers.

Corollary 6 (Bounds on Fill-in). *Let q be the smallest positive integer such that $q\phi^a(a) \geq \phi(qa)$. For any $u \in I^m$, $\exists l \in \mathbb{Z}_+$ such that $l < q$ and $\phi^a(u) = l\phi^a(a) + \phi(u - la)$.*

Proof. It is sufficient to show that if $l \geq q$, then $l\phi^a(a) + \phi(u - la) \geq (l - q)\phi^a(a) + \phi(u - (l - q)a)$. Assume by contradiction that $l\phi^a(a) + \phi(u - la) < (l - q)\phi^a(a) + \phi(u - (l - q)a)$. This implies that $q\phi^a(a) < \phi(u - (l - q)a) - \phi(u - la) \leq \phi(qa)$ where the last inequality follows from the subadditivity of ϕ . Now observe that $q\phi^a(a) < \phi(qa)$ is a contradiction to the definition of q . \square

If $\phi(qa) + \phi(r - qa) = 1$ for some positive integer q , then by (13) we obtain $q\phi^a(a) = \phi(qa)$. So the result of Corollary 6 holds where l is less than the smallest positive integer q satisfying $\phi(qa) + \phi(r - qa) = 1$.

4 Some Necessary Conditions for Lifting and Fill-in to Produce Minimal Inequalities

In this section, we focus on the inequalities for $MI(I^2, \mathbb{R}^2, r)$. However, most of the results generalize to group problems with more rows. We assume that the function ϕ is piecewise linear, continuous, and $\phi(u) = 0$, $u \in I^2$ if and only if $u = \bar{0}$. The notation of continuity of ϕ (whose domain is I^2) is based on the metric topology endowed on I^2 as discussed in Dey et al. [9]. Note that in general, when U and W are finite sets the class of functions generated using (2) are piecewise linear and lower semi-continuous as they are value functions of MIPs.

Definition 7 (Edges of ϕ [8]). *Let ϕ be a continuous function and let ϕ be piecewise linear, i.e. I^2 can be decomposed into finitely many polytopes with non-empty interiors P_1, \dots, P_k , such that ϕ is linear over polytopes P_1, \dots, P_k . Define an edge Q of ϕ to be the one-dimensional intersection of two polytopes such that ϕ has different gradients in these two polytopes.*

A point $u \in I^2$ is called strict local maximum (resp. minimum) point of ϕ if $\exists \epsilon_0 > 0$ such that $\phi(u + \epsilon d) < \phi(u)$ (resp. $\phi(u + \epsilon d) > \phi(u)$) for all directions $d \in \mathbb{R}^2$ where $\|d\| = 1$ and for all $0 < \epsilon < \epsilon_0$.

We prove the following result in this section.

Theorem 8. *Let (ϕ, π) be a subadditive valid inequality for $MI(I^2, \mathbb{R}^2, r)$ that is not minimal for $MI(I^2, \mathbb{R}^2, r)$. Let ϕ be piecewise linear, continuous on I^2 and $\phi(u) = 0$ for $u \in I^2$ if and only if $u = \bar{0}$. If $\phi^a(a) > 0$ and (ϕ^a, π) is a minimal valid inequality for $MI(I^2, \mathbb{R}^2, r)$, then one of the following must hold:*

1. $|\mathcal{S}(\phi, a)| \geq 3$.
2. $\mathcal{S}(\phi, a) = \{p^1, p^2\}$ and either $r - p^1 a$ or $r - p^2 a$ belongs to an edge of ϕ .
3. $\mathcal{S}(\phi, a) = \{p^1\}$ and $r - p^1 a$ is a point of local maximum for ϕ .

Note that if $\phi(a) + \phi(r - a) = 1$, then $\phi^a(a) = \phi(a)$ (this is implied by Corollary 5). The next proposition shows that if $\phi(a) + \phi(r - a) > 1$, then $\phi^a(a) < \phi(a)$.

Proposition 9. *If $\phi(a) + \phi(r - a) > 1$, then $\phi^a(a) < \phi(a)$.*

Proof. Assume by contradiction that $\phi(a) = \phi^a(a)$. Then using Corollary 6, $\phi^a(u) = \phi(u) \forall u \in I^2$. Therefore $\phi^a(u) + \phi^a(r - u) = \phi(a) + \phi(r - a) > 1$, contradicting the result of Proposition 4. \square

Next we present a lemma that is used repeatedly in this section.

Lemma 10. *Let $u \in I^2$ and $\mathcal{K} = \{k \in \mathbb{Z}_+ \mid \phi^a(u) < k\phi^a(a) + \phi(u - ka)\}$. There exists $\epsilon_0 > 0$ such that $\phi^a(u + \epsilon d) < k\phi^a(a) + \phi(u + \epsilon d - ka) \forall 0 \leq \epsilon < \epsilon_0, \forall k \in \mathcal{K}$ and for all directions $d \in \mathbb{R}^2$ with $\|d\| = 1$.*

Proof. Let $\sigma > 0$ be the largest directional derivative of ϕ . By subadditivity of ϕ^a ,

$$\phi^a(u + \epsilon d) \leq \phi^a(u) + \phi^a(\epsilon d) \leq \phi^a(u) + \epsilon \sigma. \quad (14)$$

Since $\phi^a(a) > 0$, $\exists n \in \mathbb{Z}_+$ such that $n\phi^a(a) > 3$. Since $-1 \leq \phi(u - ka) - \phi^a(u) \leq 1$ for any k , we obtain that $\delta := \inf\{k\phi^a(a) + \phi(u - ka) - \phi^a(u) \mid k \in \mathcal{K}\} = \inf\{k\phi^a(a) + \phi(u - ka) - \phi^a(u) \mid k \in \mathbb{Z}_+\}$

$\mathcal{K} \cap \{1, \dots, n-1\} = \min\{k\phi^a(a) + \phi(u - ka) - \phi^a(u) \mid k \in \mathcal{K} \cap \{1, \dots, n-1\}\}$. Therefore δ is the minimum of a finite number of positive values and $\delta > 0$.

Now for any $k \in \mathcal{K}$,

$$\begin{aligned} k\phi^a(a) + \phi(u + \epsilon d - ka) &\geq k\phi^a(a) + \phi(u - ka) - \phi(-\epsilon d) \\ &\geq \phi^a(u) + \delta - \epsilon\sigma \\ &\geq \phi^a(u + \epsilon d) + \delta - 2\epsilon\sigma, \end{aligned}$$

where the first inequality follows from the subadditivity of ϕ , the second inequality follows from the definition of \mathcal{K} , and the last inequality follows from (14). Now choosing $0 \leq \epsilon < \frac{\delta}{4\sigma}$, we obtain that $\phi^a(u + \epsilon d) < k\phi^a(a) + \phi(u + \epsilon d - ka) \forall 0 \leq \epsilon < \frac{\delta}{4\sigma}, \forall k \in \mathcal{K}$ and $\forall d \in \mathbb{R}^2, \|d\| = 1$. \square

Proposition 11. *Let $\phi^a(a) > 0$. If $\phi^a(ta) < \phi(ta) \forall t \in \mathbb{Z}_+$ with $1 \leq t \leq l$, then ta is a strict local minimum point for the function ϕ^a for $1 \leq t \leq l$.*

Proof. The proof for any l involves three cases:

1. Let $k \in \mathbb{Z}$ and $k > l$. Then by subadditivity of ϕ^a and the fact that $\phi^a(a) > 0$ we obtain $\phi^a(la) \leq l\phi^a(a) < k\phi^a(a) + \phi((l-k)a)$. Therefore by Lemma 10, we have that for all $k > l$ there exists $\epsilon_0 > 0$ such that

$$\phi^a(la + \epsilon d) < k\phi^a(a) + \phi((l-k)a + \epsilon d) \quad \forall 0 \leq \epsilon < \epsilon_0, \quad \forall d \in \mathbb{R}^2, \|d\| = 1. \quad (15)$$

2. Let $k = 0$. By assumption $\phi^a(la) < \phi(la)$ or $\phi^a(la) < 0 \cdot \phi^a(a) + \phi(la - 0 \cdot a)$. Again by Lemma 10, we have that there exists $\epsilon_0 > 0$ such that

$$\phi^a(la + \epsilon d) < \phi(la + \epsilon d) \quad \forall 0 \leq \epsilon < \epsilon_0, \quad \forall d \in \mathbb{R}^2, \|d\| = 1. \quad (16)$$

3. Let $k \in \{1, \dots, l-1\}$. Therefore $\forall 0 \leq \epsilon < \epsilon_0, \forall d \in \mathbb{R}^2, \|d\| = 1$, we have

$$\phi^a(la + \epsilon d) \leq k\phi^a(a) + \phi^a((l-k)a + \epsilon d) < k\phi^a(a) + \phi((l-k)a + \epsilon d), \quad (17)$$

where the first inequality is due to subadditivity of ϕ^a and the second inequality follows from (16) for $l := l - k$.

By (15), (16), (17), and the definition of ϕ^a , we have that $\exists \epsilon_0 > 0$ such that

$$\phi^a(la + \epsilon d) = l\phi^a(a) + \phi(\epsilon d) \quad \forall 0 \leq \epsilon < \epsilon_0, \quad \forall d \in \mathbb{R}^2, \|d\| = 1. \quad (18)$$

Since $\phi(\epsilon d) > 0 \forall \epsilon > 0$, the result is proven. \square

Next in Proposition 13 we consider the case where $\mathcal{S}(\phi, a)$ is a singleton. First we need the following result.

Theorem 12 ([15]). *Let $\phi : I^2 \rightarrow \mathbb{R}_+$ and let $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}_+$. For any $r \in I^2 \setminus \{\bar{0}\}$, (ϕ, π) is a minimal valid inequality for $MI(I^2, \mathbb{R}^2, r)$ if and only if*

$$\begin{aligned} \phi(u) + \phi(v) &\geq \phi(u+v) \quad \forall u, v \in I^2 \\ \pi(w) &= \lim_{h \downarrow 0} \frac{\phi(\mathbb{P}(hw))}{h} \quad \forall w \in \mathbb{R}^2 \\ \phi(u) + \phi(r-u) &= 1 \quad \forall u \in I^2. \end{aligned} \quad (19)$$

Proposition 13. *Let $\phi^a(a) > 0$ and let $\mathcal{S}(\phi, a) = \{p\}$. If $r - pa$ is not a strict local maximum point for the function ϕ , then ϕ^a is not minimal.*

Proof. Claim 1: $\phi^a(r-a) = (p-1)\phi^a(a) + \phi(r-pa)$ and $\phi^a(r-a) < k\phi^a(a) + \phi(r-(k+1)a)$ if $k \neq p-1$: By Proposition 4, $\phi^a(r-a) + \phi^a(a) = 1$. Therefore $\phi^a(r-a) = 1 - \phi^a(a) = (p-1)\phi^a(a) + \phi(r-pa)$, where the second equality follows from the definition of $\mathcal{S}(\phi, a)$. Assume by contradiction that $\phi^a(r-a) = k\phi^a(a) + \phi(r-(k+1)a)$ for some $k \neq p-1$. Then,

$$1 = \phi^a(a) + \phi^a(r-a) = (k+1)\phi^a(a) + \phi(r-(k+1)a), \text{ or}$$

$$\phi^a(a) = \frac{1}{k+1}(1 - \phi(r-(k+1)a)). \quad (20)$$

So $k+1 \in \mathcal{S}(\phi, a)$ with $k+1 \neq p$, a contradiction.

Claim 2: $\exists \epsilon_0 > 0$ such that $\phi^a(r-a+\epsilon d) = (p-1)\phi^a(a) + \phi(r-pa+\epsilon d) \forall 0 \leq \epsilon < \epsilon_0$ and for all unit directions d . By Claim 1 and Lemma 10 if $k \neq p-1$, then $\phi^a(r-a+\epsilon d) < k\phi^a(a) + \phi(r-(k+1)a+\epsilon d) \forall 0 \leq \epsilon < \epsilon_0$ and for all unit directions d . Now the claim follows from the definition of ϕ^a .

By Proposition 9, $\phi^a(a) < \phi(a)$. Therefore by Proposition 11 (see (18)), $\phi^a(a+\epsilon d) = \phi^a(a) + \phi(\epsilon d) \forall 0 \leq \epsilon < \epsilon_0$. Also from Claim 2, $\phi^a(r-a+\epsilon d) = (p-1)\phi^a(a) + \phi(r-pa+\epsilon d) \forall 0 \leq \epsilon < \epsilon_0$. Therefore if $\phi(r-pa+\epsilon d) \geq \phi(r-pa)$ for some directions d' and $0 < \epsilon < \epsilon_0$, then $\phi^a(a+\epsilon(-d')) + \phi^a(r-a+\epsilon d') = p\phi^a(a) + \phi(r-pa+\epsilon d') + \phi((- \epsilon d')) \geq p\phi^a(a) + \phi(r-pa) + \phi((- \epsilon d')) = 1 + \phi((- \epsilon d')) > 1$. Now the result follows from Theorem 12. \square

We next analyze the case in which the lifting problem solves at exactly two integers.

Proposition 14. *Let $\phi^a(a) > 0$ and let $\mathcal{S}(\phi, a) = \{p^1, p^2\}$. A necessary condition for ϕ^a to be minimal is that at least one of $r - p^1a$ or $r - p^2a$ belongs to an edge of ϕ .*

Proof. Similar to the proof of Proposition 13, it can be verified that $\phi^a(r-a) < k\phi^a(a) + \phi(r-(k+1)a) \forall k \neq p^1, p^2$. Therefore by Lemma 10 we obtain that $\exists \epsilon_0 > 0$ such that $\forall 0 \leq \epsilon < \epsilon_0$ and $\forall d \in \mathbb{R}^2 \ \|d\| = 1$,

$$\phi^a(r-a+\epsilon d) = \min \left\{ \begin{array}{l} (p^1-1)\phi^a + \phi(r-p^1a+\epsilon d) \\ (p^2-1)\phi^a + \phi(r-p^2a+\epsilon d) \end{array} \right. . \quad (21)$$

By Proposition 9, $\phi^a(a) < \phi(a)$. Therefore by Proposition 11, a is a point of strict local minimum for ϕ^a . Hence using Theorem 12 a necessary condition for ϕ^a to be minimal is that $(r-a)$ is a strict local maximum for the function ϕ^a . If both $r - p^1a$ and $r - p^2a$ do not belong to edges, there must exist a direction \tilde{d} such that $\phi(r-p^1a+\epsilon \tilde{d}) \leq \phi(r-p^1a)$ and $\phi(r-p^2a+\epsilon \tilde{d}) \leq \phi(r-p^2a)$ for all sufficiently small positive ϵ , leading to $(r-a)$ not being a strict local maximum of the function ϕ^a . \square

Propositions 13 and 14 prove Theorem 8.

5 Strengthening Mixing Inequalities

The mixing set, introduced in [14], is a relaxation of several sets arising in classical fixed charge network flow problems such as the constant capacity single item lot sizing problem, the capacitated facility location problem, and the capacitated network design problem.

Definition 15 (Mixing set [14]). $\{(y_0, z) \in \mathbb{R}_+ \times \mathbb{Z}^n \mid y_0 + z_i \geq r_i, \forall i \in \{1, \dots, n\}\}$. We assume that $0 \leq r_1 < r_2 < r_3 < \dots < r_n < 1$.

The convex hull of the feasible points of the mixing set is given by the mixing inequalities.

Definition 16 (Mixing inequalities [14]). *The mixing inequality type 1 with p terms is*

$$y_0 \geq \sum_{k=1}^p (r_{i_k} - r_{i_{k-1}})(1 - z_{i_k}), \quad (22)$$

and the mixing inequality type 2 with p terms is

$$y_0 \geq \sum_{k=1}^p (r_{i_k} - r_{i_{k-1}})(1 - z_{i_k}) - (1 - r_{i_p})z_{i_1}, \quad (23)$$

where $r_{i_0} = 0$ and $i_k > i_{k-1} \forall k$.

Let $r_1, r_2 \in \mathbb{Q}_+$ and $0 < r_1 < r_2 < 1$. Introducing slack variables y_1 and y_2 , the two row mixing set can be rewritten as,

$$\begin{aligned} z_1 + y_0 - y_1 &= r_1 \\ z_2 + y_0 - y_2 &= r_2 \\ y_0, y_1, y_2 &\in \mathbb{R}_+, z_1, z_2 \in \mathbb{Z}. \end{aligned} \quad (24)$$

Let $w^1 = (1, 1)$, $w^2 = (-1, 0)$, and $w^3 = (0, -1)$. Then the mixing set in (24) is equivalent to the set $MI(\{\bar{0}\}, \{w^1, w^2, w^3\}, r)$, and the mixing inequality (22) can be rewritten as

$$\frac{1 - r_2}{D}y_0 + \frac{r_1}{D}y_1 + \frac{r_2 - r_1}{D}y_2 \geq 1 \quad (25)$$

where $D = (r_2 - r_1)(1 - r_2) + r_1(1 - r_1)$. The mixing inequality (23) for the two row mixing set is a MIR inequality. Henceforth we use the notation ϕ^{MIX} and π^{MIX} to denote the functions obtained using (2) and (3) respectively where $\alpha(\bar{0}) = 0$ and β is given by (25), i.e

$$\begin{aligned} \phi^{\text{MIX}}(u_1, u_2) &= \min \quad \beta_0 y_0 + \beta_1 y_1 + \beta_2 y_2 \\ \text{s.t.} \quad z_1 + y_0 - y_1 &= u_1 \\ z_1 + y_0 - y_2 &= u_2 \\ z_1, z_2 &\in \mathbb{Z}, y_0, y_1, y_2 \in \mathbb{R}_+, \end{aligned} \quad (26)$$

where $\beta_0 = \frac{(1-r_2)}{D}$, $\beta_1 = \frac{r_1}{D}$, $\beta_2 = \frac{r_2-r_1}{D}$, and $D = (r_2 - r_1)(1 - r_2) + r_1(1 - r_1)$.

5.1 Strength of π^{MIX}

We show that π^{MIX} is an extreme inequality for $MI(\emptyset, \mathbb{R}^2, r)$. Note that proving this is tantamount to showing that if $(\phi^{\text{MIX}}, \pi^{\text{MIX}}) = \frac{1}{2}(\phi^1, \pi^1) + \frac{1}{2}(\phi^2, \pi^2)$ where (ϕ^j, π^j) are valid inequalities for $MI(I^2, \mathbb{R}^2, r)$, then $\pi^1 = \pi^2 = \pi^{\text{MIX}}$.

The following result is modified from [5].

Theorem 17 ([5]). Let $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ represent a valid inequality for $MI(\emptyset, \mathbb{R}^2, r)$. If the set $P(\pi) := \{\mu \in \mathbb{R}^2 \mid \pi(r - \mu) \leq 1\}$ is a triangle such that each side of $P(\pi)$ contains at least one integer point in its relative interior, then $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ is an extreme inequality for $MI(\emptyset, \mathbb{R}^2, r)$.

The following result can be easily verified, see for example [11] or [7].

Proposition 18 ([11]). $P(\pi^{\text{MIX}})$ is the triangle whose vertices are: $V^0 := (r_1 - \frac{D}{1-r_2}, r_2 - \frac{D}{1-r_2})$, $V^1 := (r_1 + \frac{D}{r_1}, r_2)$, and $V^2 := (r_1, r_2 + \frac{D}{r_2-r_1})$. There is exactly one integer point in the relative interior of each side of the triangle.

Therefore by Theorem 17 and Proposition 18, we obtain that π^{MIX} is extreme for $MI(\emptyset, \mathbb{R}^2, r)$.

5.2 Strength of ϕ^{MIX}

Now we consider the strength of the function ϕ^{MIX} . The following result from [10] indicates that the function $\phi^{\text{MIX}} : I^2 \rightarrow \mathbb{R}_+$ can be strengthened.

Theorem 19 ([10]). Let $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ be a valid inequality for $MI(\emptyset, \mathbb{R}^2, r)$. If the set $P(\pi) := \{\mu \in \mathbb{R}^2 \mid \pi(r - \mu) \leq 1\}$ is a triangle such that each side of $P(\pi)$ contains exactly one integer point on each side, then (ϕ, π) is not a minimal inequality for $MI(I^2, \mathbb{R}^2, r)$, where ϕ is defined as $\phi(u) = \inf_{z \in \mathbb{Z}} \{\pi(u + z)\}$.

By (2) and (3), we obtain that $\phi^{\text{MIX}}(u) = \inf_{z \in \mathbb{Z}^2} \pi^{\text{MIX}}(u + z)$. Hence Theorem 19 and Proposition 18 imply that $(\phi^{\text{MIX}}, \pi^{\text{MIX}})$ is not minimal for $MI(I^2, \mathbb{R}^2, r)$.

5.3 Strengthening ϕ^{MIX}

Throughout this section we assume that $r_1 + r_2 \leq 1$. This is not a serious drawback. Given any two rows of a simplex tableau with right-hand-sides b_1 and b_2 , let $r_j := \mathbb{P}(b_j)$. If $r_1 + r_2 > 1$, then multiplying the two rows with -1 and setting $r_j := \mathbb{P}(b_j)$ (where b_j is the new right-hand-side in the simplex tableau) we obtain $r_1 + r_2 \leq 1$.

Proposition 20. Let $r_1 + r_2 \leq 1$ and let $a := (1 + \frac{r_1}{2} - \frac{r_2}{2}, \frac{r_1+r_2}{2})$. Then $\phi^a(a) = \frac{-r_1+r_2+r_1r_2-r_2^2}{2D}$.

Proof. Claim 1: There exists an optimal solution $(z_1^*, z_2^*, y_0^*, y_1^*, y_2^*)$ of (26), such that $-1 < u_1 - z_1^* < 1$, $-1 < u_2 - z_2^* < 1$. If $u_1 - z_1^* \geq 1$, then observe that $y_0^* \geq 1$. Then setting $z_1^* = z_1^* + 1$, $z_2^* = z_2^* + 1$ and $y_0^* - 1$ yields a better solution. If $u_1 - z_1^* \leq -1$, then $y_1^* \geq 1$. Then setting $z_1^* = z_1^* - 1$ and $y_1^* - 1$ yields a better solution. The rest of the claim can be proven similarly.

Claim 2:

$$\phi^{\text{MIX}}(r - a) = \phi^{\text{MIX}}\left(\frac{r_1 + r_2}{2}, \frac{-r_1 + r_2}{2}\right) = \frac{(1 - r_2)(r_1 + r_2) + 2(r_2 - r_1)r_1}{2D}.$$

Let $(z'_1, z'_2, y'_0, y'_1, y'_2)$ be an optimal solution of (26) when $u = r - a$. Since $0 < \frac{r_1+r_2}{2} \leq 1$ and $0 < \frac{-r_1+r_2}{2} < 1$ by the use of Claim 1, there are four cases:

1. $z'_1 = 0, z'_2 = 0$: In this case $y'_0 = \frac{r_1+r_2}{2}$, $y'_1 = 0$, and $y'_2 = r_1$. Then $\sum_{i=0}^2 \beta_i y'_i = \frac{(1-r_2)(r_1+r_2)+2(r_2-r_1)r_1}{2D}$.
2. $z'_1 = 1, z'_2 = 0$: In this case $y'_0 = \frac{-r_1+r_2}{2}$, $y'_1 = 1 - r_1$, and $y'_2 = 0$. Then $\sum_{i=0}^2 \beta_i y'_i = \frac{(1-r_2)(-r_1+r_2)+2(1-r_1)r_1}{2D} = \frac{(1-r_2)(r_1+r_2)+2(r_2-r_1)r_1}{2D}$.

3. $z'_1 = 0, z'_2 = 1$: In this case $y'_0 = \frac{r_1+r_2}{2}$, $y'_1 = 0$, and $y'_2 = 1 + r_1$. Then $\sum_{i=0}^2 \beta_i y'_i > \frac{(1-r_2)(r_1+r_2)+2(r_2-r_1)r_1}{2D}$.
4. $z'_1 = 1, z'_2 = 1$: In this case $y'_0 = 0$, $y'_1 = 1 - \frac{r_1+r_2}{2}$, and $y'_2 = 1 - \frac{-r_1+r_2}{2}$. Then $\sum_{i=0}^2 \beta_i y'_i = \frac{2r_2+r_1r_2-2r_1^2-r_2^2}{2D} > \frac{(1-r_2)(r_1+r_2)+2(r_2-r_1)r_1}{2D}$.

Claim 3:

$$\begin{aligned}\phi^{\text{MIX}}(r-2a) &= \phi^{\text{MIX}}(r_2, 1-r_1) = \frac{r_1(1-r_2) + (r_2-r_1)r_1}{D}. \\ \phi^{\text{MIX}}(2a) &= \phi^{\text{MIX}}(1+r_1-r_2, r_1+r_2) = \frac{r_1(r_2-r_1) + (r_2-r_1)(1-r_1-r_2)}{D}.\end{aligned}$$

It is possible to verify that $\phi^{\text{MIX}}(r-2a) = \phi^{\text{MIX}}(r_2, 1-r_1) \leq \frac{r_1(1-r_2)+(r_2-r_1)r_1}{D}$ and $\phi^{\text{MIX}}(r-2a) = \phi^{\text{MIX}}(r_2, 1-r_1) \leq \frac{r_1(1-r_2)+(r_2-r_1)r_1}{D}$. This can be done by fixing $z_1 = z_2 = 1$ in (26) and computing the optimal objective function value of resultant linear program. Finally, observe that $\phi^{\text{MIX}}(r-2a) + \phi^{\text{MIX}}(2a) \leq \frac{r_1(1-r_2)+(r_2-r_1)r_1}{D} + \frac{r_1(r_2-r_1)+(r_2-r_1)(1-r_1-r_2)}{D} = 1 \leq \phi^{\text{MIX}}(r-2a) + \phi^{\text{MIX}}(2a)$ where the second inequality follows from the subadditivity of ϕ^a . This completes the proof.

We are now ready to prove the result. Observe first that since $\phi^{\text{MIX}}(r-2a) + \phi^{\text{MIX}}(2a) = 1$, by Corollary 5 there exists an integer $p \leq 2$ such that $\phi^a = \frac{1}{p}(1 - \phi^{\text{MIX}}(r-pa))$, i.e.

$$\phi^a(a) = \max_{n \in \{1,2\}} \frac{1}{n} (1 - \phi^{\text{MIX}}(r-na)).$$

Now by making the necessary computations using Claim 2 and Claim 3, we obtain that

$$1 - \phi^{\text{MIX}}(r-a) = \frac{1}{2}(1 - \phi^{\text{MIX}}(r-2a)) = \frac{-r_1 + r_2 + r_1r_2 - r_2^2}{2D}.$$

□

While we have not used Theorem 8 explicitly, the search for the above mentioned a was guided by it. In particular, note that the proof of Proposition 20 shows that $|\mathcal{S}(\phi^{\text{MIX}}, a)| \geq 2$ for the point a . It can be verified that $r-a$ lies on an edge of ϕ^{MIX} .

Next in Proposition 22 we verify that ϕ^a indeed yields a minimal inequality. In order to prove this we use the following result.

Theorem 21 ([12]). *If $\phi : I^2 \rightarrow \mathbb{R}_+$ is a valid function for $MI(I^2, \emptyset, r)$ and if $\phi(u) + \phi(r-u) \leq 1 \forall u \in I^2$, then ϕ is subadditive.*

Proposition 22. *Let $r_1 + r_2 \leq 1$ and let $a := (1 + \frac{r_1}{2} - \frac{r_2}{2}, \frac{r_1+r_2}{2})$. Then $(\phi^a, \pi^{\text{MIX}})$ is a minimal inequality for $MI(I^2, \mathbb{R}^2, r)$.*

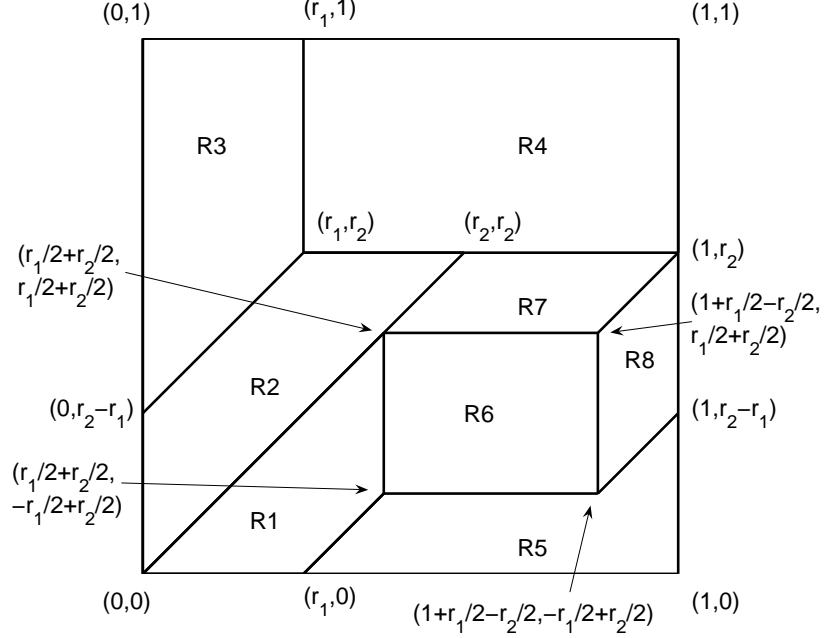


Figure 1: Edges of ϕ^a and $\bar{\phi}$

Proof. Consider the function $\bar{\phi} : I^2 \rightarrow \mathbb{R}_+$ defined as:

$$\bar{\phi}(u_1, u_2) = \begin{cases} \beta_0 u_1 + \beta_2 (u_1 - u_2) & (u_1, u_2) \in R1 \\ \beta_0 u_2 + \beta_1 (u_2 - u_1) & (u_1, u_2) \in R2 \\ \beta_0 u_1 + \beta_2 (u_1 - u_2 + 1) & (u_1, u_2) \in R3 \\ \beta_1 (1 - u_1) + \beta_2 (1 - u_2) & (u_1, u_2) \in R4 \\ \beta_0 u_2 + \beta_1 (u_2 - u_1 + 1) & (u_1, u_2) \in R5 \\ \frac{-r_1 + r_2 + r_1 r_2 - r_2^2}{2D} + \beta_1 \left(1 + \frac{r_1}{2} - \frac{r_2}{2} - u_1\right) + \beta_2 \left(\frac{r_1 + r_2}{2} - u_2\right) & (u_1, u_2) \in R6 \\ \frac{-r_1 + r_2 + r_1 r_2 - r_2^2}{2D} + \beta_0 \left(u_2 - \frac{r_1 + r_2}{2}\right) + \beta_1 (1 - r_2 - u_1 + u_2) & (u_1, u_2) \in R7 \\ \frac{-r_1 + r_2 + r_1 r_2 - r_2^2}{2D} + \beta_0 \left(u_1 - 1 - \frac{r_1}{2} + \frac{r_2}{2}\right) + \beta_2 (r_2 - 1 + u_1 - u_2) & (u_1, u_2) \in R8, \end{cases} \quad (27)$$

where

- $R1$ is the region defined by the line segments $(0, 0) - (r_1, 0) - \left(\frac{r_1 + r_2}{2}, \frac{-r_1 + r_2}{2}\right) - \left(\frac{r_1 + r_2}{2}, \frac{r_1 + r_2}{2}\right) - (0, 0)$,
- $R2$ is the region defined by the line segments $(0, 0) - (r_2, r_2) - (r_1, r_2) - (0, r_2 - r_1) - (0, 0)$,
- $R3$ is the region defined by the line segments $(0, r_2 - r_1) - (r_1, r_2) - (r_1, 1) - (0, 1) - (0, r_2 - r_1)$,
- $R4$ is the region defined by the line segments $(r_1, r_2) - (1, r_2) - (1, 1) - (r_1, 1) - (r_1, r_2)$,
- $R5$ is the region defined by the line segments $(r_1, 0) - (1, 0) - (1, r_2 - r_1) - \left(1 + \frac{r_1 - r_2}{2}, \frac{-r_1 + r_2}{2}\right) - \left(\frac{r_1 + r_2}{2}, \frac{-r_1 + r_2}{2}\right) - (r_1, 0)$,

- $R6$ is the region defined by the line segments $(\frac{r_1+r_2}{2}, \frac{-r_1+r_2}{2}) - (1 + \frac{r_1-r_2}{2}, \frac{-r_1+r_2}{2}) - (1 + \frac{r_1-r_2}{2}, \frac{r_1+r_2}{2}) - (\frac{r_1+r_2}{2}, \frac{r_1+r_2}{2}) - (\frac{r_1+r_2}{2}, \frac{-r_1+r_2}{2})$,
- $R7$ is the region defined by the line segments $(\frac{r_1+r_2}{2}, \frac{r_1+r_2}{2}) - (1 + \frac{r_1-r_2}{2}, \frac{r_1+r_2}{2}) - (1, r_2) - (r_2, r_2) - (\frac{r_1+r_2}{2}, \frac{r_1+r_2}{2})$,
- $R8$ is the region defined by the line segments $(1 + \frac{r_1-r_2}{2}, \frac{-r_1+r_2}{2}) - (1, -r_1 + r_2) - (1, r_2) - (1 + \frac{r_1-r_2}{2}, \frac{r_1+r_2}{2}) - (1 + \frac{r_1-r_2}{2}, \frac{-r_1+r_2}{2})$.

(See Figure 1.)

Claim 1: $\bar{\phi}(u) \geq \phi^a(u) \forall u \in I^2$. Consider the case of $R1$. Since $u_1 \geq u_2$ for $u \in R1$, setting $z_1 = z_2 = 0$, $y_0 = u_1$, $y_1 = 0$, and $y_2 = u_1 - u_2$ yields a feasible solution to (26). Thus $\phi^a(u) \leq \phi^{\text{MIX}}(u) \leq \beta_0 u_1 + \beta_2(u_1 - u_2) = \bar{\phi}(u)$. Similar proofs can be presented for the cases $R2$, $R3$, $R4$, and $R5$.

For the case of $R6$, observe that using Proposition 20 we obtain $a := (1 + \frac{r_1}{2} - \frac{r_2}{2}, \frac{r_1+r_2}{2})$ and $\phi^a(a) = \frac{-r_1+r_2+r_1r_2-r_2^2}{2D}$. Also $\phi^{\text{MIX}}(1 + u - a) \leq \beta_1(1 + \frac{r_1}{2} - \frac{r_2}{2} - u_1) + \beta_2(\frac{r_1+r_2}{2} - u_2)$. Thus $\bar{\phi}(u) = \frac{-r_1+r_2+r_1r_2-r_2^2}{2D} + \beta_1(1 + \frac{r_1}{2} - \frac{r_2}{2} - u_1) + \beta_2(\frac{r_1+r_2}{2} - u_2) \geq \phi^a(a) + \phi^{\text{MIX}}(1 + u - a) \geq \phi^a(a) + \phi^a(1 + u - a) \geq \phi^a(u)$. Similar proofs can be presented for $R7$ and $R8$.

Claim 2: $\bar{\phi}(u) + \bar{\phi}(r - u) = 1 \forall u \in I^2$. There are four cases.

1. $0 \leq u_1 \leq r_1$ and $0 \leq u_2 \leq r_2$. There are two sub cases.
 - $u \in R1$. Then $r - u \in R3$. Thus $\bar{\phi}(u) + \bar{\phi}(r - u) = \beta_0(u_1) + \beta_2(u_1 - u_2) + \beta_0(r_1 - u_1) + \beta_2(r_1 - u_1 - r_2 + u_2 + 1) = \beta_0(r_1) + \beta_2(r_1 - r_2 + 1) = 1$.
 - $u \in R2$. Then $r - u \in R2$. Thus $\bar{\phi}(u) + \bar{\phi}(r - u) = \beta_0(u_2) + \beta_1(u_2 - u_1) + \beta_0(r_2 - u_2) + \beta_1(r_2 - u_2 - r_1 + u_1) = \beta_0 r_2 + \beta_1(r_2 - r_1) = 1$.
2. $0 \leq u_1 \leq r_1$ and $r_2 \leq u_2 \leq 1$. Then $u \in R3$. Thus $\bar{\phi}(u) + \bar{\phi}(r - u) = \beta_0(u_1) + \beta_2(u_1 - u_2 + 1) + \beta_0(r_1 - u_1) + \beta_2(r_1 - u_1 - r_2 + u_2 - 1 + 1) = \beta_0 r_1 + \beta_2(r_1 - r_2 + 1) = 1$.
3. $r_1 \leq u_1 \leq 1$ and $r_2 \leq u_2 \leq 1$. Then $u \in R4$. Thus $\bar{\phi}(u) + \bar{\phi}(r - u) = \beta_1(1 - u_1) + \beta_2(1 - u_2) + \beta_1(u_1 - r_1) + \beta_2(u_2 - r_2) = \beta_1(1 - r_1) + \beta_2(1 - r_2) = 1$.
4. $r_1 \leq u_1 \leq 1$ and $0 \leq u_2 \leq r_2$. There are four cases.
 - If $u \in R1$, then $r - u \in R8$. Thus $\bar{\phi}(u) + \bar{\phi}(r - u) = \beta_0(u_1) + \beta_2(u_1 - u_2) + \phi^a(a) + \beta_0(\frac{r_1}{2} + \frac{r_2}{2} - u_1) + \beta_2(r_1 - u_1 + u_2) = \phi^a(a) + \beta_0(\frac{r_1+r_2}{2}) + \beta_2(r_1) = 1$.
 - If $u \in R6$, then $r - u \in R6$. Thus $\bar{\phi}(u) + \bar{\phi}(r - u) = 2\phi^a(a) + \beta_1(1 + \frac{r_1-r_2}{2} - u_1) + \beta_2(\frac{r_1}{2} + \frac{r_2}{2} - u_2) + \beta_1(\frac{-r_1-r_2}{2} + u_1) + \beta_2(\frac{r_1}{2} - \frac{r_2}{2} + u_2) = 2\phi^a(a) + \beta_1(1 - r_2) + \beta_2(r_1) = 1$.
 - If $u \in R7$, then $r - u \in R5$. Thus $\bar{\phi}(u) + \bar{\phi}(r - u) = \phi^a(a) + \beta_0(u_2 - \frac{r_1}{2} - \frac{r_2}{2}) + \beta_1(1 - r_2 - u_1 + u_2) + \beta_0(r_2 - u_2) + \beta_1(r_2 - u_2 - r_1 + u_1) = \phi^a(a) + \beta_0(-\frac{r_1}{2} + \frac{r_2}{2}) + \beta_1(1 - r_1) = 1$.
 - If $u \in R2$, then $r - u \in R5$. Then $\bar{\phi}(u) + \bar{\phi}(r - u) = \beta_0 u_2 + \beta_1(u_2 - u_1) + \beta_0(r_2 - u_2) + \beta_1(r_2 - u_2 - r_1 + u_1) = 1$.

From Claim 1 we have that $(\bar{\phi}, \pi^{\text{MIX}})$ is a valid inequality for $MI(I^2, \mathbb{R}^2, r)$. Then using Claim 2 and Theorem 21 we obtain $\bar{\phi}(u) + \bar{\phi}(v) \geq \bar{\phi}(u + v) \forall u, v \in I^2$. By computing the value of the function π^{MIX} explicitly, it can be verified that $\pi^{\text{MIX}}(w) = \lim_{h \downarrow 0} \frac{\bar{\phi}(\mathbb{P}(hw))}{h} \forall w \in \mathbb{R}^2$.

Hence by the use of Theorem 12, $(\bar{\phi}, \pi^{\text{MIX}})$ is a minimal inequality for $MI(I^2, \mathbb{R}^2, r)$. However, since $\bar{\phi} \geq \phi^a$ and (ϕ^a, π) is a valid inequality, we obtain that $\phi^a = \bar{\phi}$ and $(\phi^a, \pi^{\text{MIX}})$ is a minimal inequality for $MI(I^2, \mathbb{R}^2, r)$. \square

We note here that it can be verified that $\phi^{\text{MIX}}(u) = \phi^a(u)$ for $u \in R1, R2, R3, R4, R5$. Hence lifting $x(a)$ followed by the construction of the fill-in function ϕ^a improves the coefficients of ϕ^{MIX} in the regions $R6, R7, R8$ and yields a minimal inequality.

Theorem 23. *Let $r_1 + r_2 \leq 1$ and let $a := (1 + \frac{r_1}{2} - \frac{r_2}{2}, \frac{r_1+r_2}{2})$. Then $(\phi^a, \pi^{\text{MIX}})$ is an extreme inequality for $MI(I^2, \mathbb{R}^2, r)$.*

Proof. We use the following result from [11]: Let π is an extreme inequality for $MI(\emptyset, \mathbb{R}^2, r)$. Construct the function $\tilde{\phi}^a$ as follows:

- First compute $\tilde{\gamma} := \sup_{n \in \mathbb{Z}_+, n \geq 1, y(w) \in \mathbb{R}_+} (\frac{1}{n}(1 - \sum_{w \in B} \pi(w)y(w)) \mid \mathbb{P}(\sum_{w \in B} wy(w)) = r - na)$.
- Then compute $\tilde{\phi}^a(u) := \inf_{n \in \mathbb{Z}_+} \{n\tilde{\alpha} + \pi(w) \mid \mathbb{P}(w) = v\} \forall u \in I^2$.

If $(\tilde{\phi}^a, \pi)$ is a minimal inequality for $MI(I^2, \mathbb{R}^2, r)$, then $(\tilde{\phi}^a, \pi)$ is extreme for $MI(I^2, \mathbb{R}^2, r)$.

The result now follows from Proposition 22, Theorem 17 and the the fact that ϕ^a is equivalent to the function $\tilde{\phi}^a$. \square

We end with the observation that we do not have to solve the lifting problem (7) or fill-in problem (8) to obtain the function ϕ^a as the proof of Proposition 22 gives a closed form expression for ϕ^a .

6 Discussion

In Section 2, we proposed an approach for strengthening coefficients of inequality based on a composite lifting and fill-in process. Observe that the mixed integer program (2) corresponding to the fill-in coefficients need not be solved to optimality to obtain a valid inequality. The lifting and fill-in process applied together represents a compromise between the strength of inequalities and the difficulty in deriving them.

Theorem 8 presents some necessary conditions for the inequality (ϕ^a, π) to be minimal. Observe that the second and third conditions of Theorem 8 restrict the choice of a significantly. The first condition requires that the choice of a should be such that the lifting problem (7) has at least three distinct integer solutions. Thus these conditions severely restrict the potential candidates for which lifting of a single variable (followed by fill-in of other variables) yield an extreme inequality. More generally, one can ask the question, given a fixed positive number k , whether it is possible to come up with a lifting sequence involving k integer variables (followed by fill-in of other variables) so as to obtain a strong inequality.

In Section 5, we illustrated the application of the composite lifting and fill-in process. By the appropriate choice of a , we obtained a new family of extreme inequalities for the two-row mixed integer infinite group relaxation. Since $MI(I^2, \mathbb{R}^2, r)$ is a relaxation of two rows of a simplex tableau, $(\phi^a, \pi^{\text{MIX}})$ can be applied to any two rows of a simplex tableau whenever $\mathbb{P}(b) = r$, where b is the right-hand-side of the simplex tableau. We note here that Dash and Günlük [6] also recently considered the question of using mixing inequalities to generate cuts for general simplex tableau using a different approach. The key component in the proof of Theorem 23 is the verification of

subadditivity of the function $\bar{\phi}$ in Proposition 22. Typically proving the subadditivity of functions is difficult. One possible approach to proving that $\bar{\phi}$ is a subadditive function is presented in Dey and Richard [8]. However, a proof using this approach requires verification of approximately 350 different cases corresponding to 12 edges (some parallel) and 8 so-called vertices of $\bar{\phi}$. The bounds in Section 3 significantly simplified the calculation of $\phi^a(a)$ (Proposition 20), which in turn was used to prove the validity and consequently the subadditivity of $\bar{\phi}$. We hope these bounds will prove to be a useful tool in proving subadditivity in general.

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