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Nash equilibria of games with increasing best replies

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# NASH EQUILIBRIA OF GAMES WITH INCREASING BEST REPLIES

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Abstract. The intuitive idea of two activities being complements, for example tea and lemon, is that increasing the level of one makes somehow desirable to increase the level of the other (Samuelson, 1974). We introduce notions of increasingness for the joint best reply of a game which capture appropriately this intuitive idea of complementarity among players' strategies. We show, by generalizing the fixpoint theorems of Veinott (1992) and Zhou (1994), that the Nash sets of our games are nonempty complete lattices. Hence we extend the class of games with strategic complementarities.

**Keywords:** strategic complementarity, supermodular games, quasisupermodular games, fixpoint theorems, Nash equilibria.

*JEL Classification:* C60, C70, C72.

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This paper is a substantial extension of fixpoint results presented in Calciano (2007), and supersedes an earlier draft published as Calciano (2009).

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## 1. Introduction

This paper presents fixpoint theorems for correspondences defined on posets and lattices, and it is part of a project that aims at extending the scope and applicability of the current theory of games with strategic complementarities.

Games whose normal form exhibits an increasing joint best reply are usually referred to, in the economic literature, as games with strategic complementarities, and have plenty of important applications in economics.[1]

The current theory of games with strategic complementarities is a beautiful merge of fixpoint results and comparative statics results. With respect to the fixpoint part of the theory, the main notion of increasingness for the best reply that is adopted in the literature is due to A. Veinott.[2] Veinott (1992) and Zhou (1994) have independently proved that the fixpoint set of a Veinott-increasing correspondence mapping a complete lattice into itself is a nonempty complete lattice, thus extending to correspondences the celebrated fixpoint theorem of Tarski (1955).

Veinott and Zhou's fixpoint results allow to establish the main property of a game with strategic complementarities; namely, that Nash equilibria exist and that the Nash set is a complete lattice. The latter implies, in particular, that a least and a greatest Nash-equilibrium point exist; a property that is often useful in applications.

As for the comparative statics part of the theory, Topkis (1978), Veinott (1992) and Milgrom and Shannon (1994) have developed a lattice-based approach to monotone comparative statics, which allows establishing sufficient conditions on payoffs under which the joint best reply of a game is indeed Veinott-increasing. In this approach - which in the context of games must be applied to the individual decision problems in the normal form - payoffs are required to be supermodular or quasisupermodular, and to satisfy increasing differences or single crossing properties. Due to these sufficient conditions, games with strategic complementarities are also known, respectively, as supermodular or quasisupermodular games.

[1] See Amir (1996, 2005) and Vives (1999, 2005) for microeconomic applications, and Cooper (1999) for applications to coordination problems in macroeconomics.

[2] All the relevant definitions are given in Sections 2 and 3.

However Veinott's increasingness, which is the crucial ingredient in the fixpoint part of the theory of games with strategic complementarities, while convenient from a mathematical point of view and naturally arising from the comparative statics approach mentioned above, does not bear any direct connection with the intuitive idea of complementarity; and this is the starting point of the present paper.

Indeed, the intuitive idea of two activities being complements, for example tea and lemon, is that increasing the level of one makes somehow desirable to increase the level of the other (Samuelson, 1974)[3]. Hence complementarity is a sensitivity property of the solution set to an optimization problem, and as such, in the context of games, it should be appropriately captured by properties of the joint best reply.

We introduce in this paper notions of increasingness for the joint best reply of a game that are substantially weaker than Veinott-increasingness, and linked directly with the intuitive idea of complementarity.

We show that a game whose joint best reply satisfies our increasingness notions retains the main properties of a game with strategic complementarities; namely, that a least and a greatest Nash-equilibrium points exist and that, furthermore, the Nash set is a complete lattice. We achieve these results by generalizing the fixpoint theorems of Veinott and Zhou[4].

To outline our approach, consider for example a game played by two players whose individual strategy spaces are $S \subseteq \mathbb{R}$ and $T \subseteq \mathbb{R}$. Of course in the paper we will not restrict strategy spaces to be totally ordered sets. Let $F : T \to S$ be the (nonempty) best reply of, say, player I.

We claim that a good notion of increasingness for $F$, one that captures our intuitive idea of the two strategies $(s, t)$ being complements for player I, is that for all player II's strategies $t_1 \leq t_2$,

$$\forall s_1 \in F(t_1), \exists s_2 \in F(t_2) : s_1 \leq s_2.$$

Indeed, if after an increase in player II's action from $t_1$ to $t_2$, there is some best response $s_1 \in F(t_1)$ such that every best response $s_2 \in F(t_2)$ is strictly less than $s_1$, then we must conclude that the increase in player

[3]This form of complementarity is usually referred to as Pareto-Edgeworth complementarity.

[4]The generalization concerns not only the increasingness notions, but also the range of the correspondence. Vives (1990) proves fixpoint results using ideas similar to ours, but with a notion of increasingness stronger than Veinott-increasingness.

II's action has made desirable for player I *to decrease* her own action, against our intuition about complementarity.

This notion of increasingness, that we will call upper increasingness in the paper, once imposed on individual best replies is preserved under cartesian products, and hence inherited by the joint best reply of the game. Furthermore, it is much weaker than Veinott-increasingness. We will show that upper increasingness yields to the existence of a greatest Nash-equilibrium point and that its dual, which we will call lower increasingness and which represents complementarity by exactly the same argument as above, yields to the existence of a least Nash-equilibrium point.

Furthermore Calciano (2007) has shown that, if the restrictions of the joint best reply to certain subsets of its domain are all lower increasing, hence if all of these restrictions capture complementarity in our intuitive sense, then the Nash set is a nonempty complete lattice. However, these subsets could be in principle infinitely many, and no global condition on the best reply was given to assure that the required property would hold for all the restrictions involved.

This paper fix the problem. A new global notion of increasingness is introduced for the best reply which makes all the aforementioned restrictions be lower increasing, hence the Nash set be a nonempty complete lattice. We call this new notion of increasingness *strong lower increasingness*. It is still substantially weaker than Veinott-increasingness.

As it is the case for Veinott's and Zhou's theorems, our fixpoint results stand in the same relation to Tarski's fixpoint theorem as Kakutani's fixpoint theorem does to Brouwer's. However, we prove our results without using Tarski's theorem. Furthermore, we do not use selection arguments in our proofs. We will relate our theorems to increasing-selection arguments in the last section of the paper.

The paper is organized as follow. Section 2 contains the necessary background material and terminology. Section 3 introduces our increasingness notions. Section 4 presents our fixpoint results. Section 5 discusses increasing selections.

## 2. Background Material

Let $X$ be a nonempty set. A partial order on $X$ is a reflexive, antisymmetric and transitive binary relation $\leq$ on $X$. The set $X$ together with $\leq$ is called a partially ordered set, or a poset. Its dual, denoted by $X^d$, is the set $X$ endowed with the relation $\geq$ defined as $x \geq y \Leftrightarrow y \leq x$.

For any nonempty subsets $T \subseteq S \subseteq X$ of a poset $X$ we denote, when they exist, by $\wedge T$ and $\vee T$ respectively the inf and sup of $T$ in $X$, and by $\wedge_S T$ and $\vee_S T$ respectively the inf and sup of $T$ in $S$. We denote, when they exists, by $a \wedge b$ and $a \vee b$ respectively the inf and sup in $X$ of the subset $\{a, b\} \subseteq X$. A subset $S \subseteq X$ has a least (greatest) element if $\wedge S \in S$ (if $\vee S \in S$).

A poset $X$ is a join (meet) lattice if every nonempty and finite subset of $X$ has a supremum (an infimum). Dually, $X$ is a join (meet) lattice if and only if $X^d$ is a meet (join) lattice. $X$ is a lattice if it is both a join and meet lattice.

A poset $X$ is a join-complete (meet-complete) lattice if every nonempty subset of $X$ (not necessarily finite) has a supremum (an infimum). $X$ is join-complete (meet-complete) if and only if its dual is meet-complete (join-complete). A join-complete lattice has a greatest element $\mathbf{1}$, and a meet-complete lattice has a least element $\mathbf{0}$.

$X$ is a complete lattice if it is both a join-complete and a meet-complete lattice; or equivalently if it is join (meet) complete and has a least (greatest) element.

A subset $S \subseteq X$ of a poset $X$ is a (complete) lattice whenever it is so in the induced order; that is, whenever for every nonempty (not necessarily) finite subset $T \subseteq S$, $\vee_S T \in S$ and $\wedge_S T \in S$. Subset $S$ is a (subcomplete) sublattice of $X$ whenever for every nonempty (not necessarily) finite subset $T \subseteq S$, $\vee T \in S$ and $\wedge T \in S$.

A subcomplete sublattice of $X$ is a complete lattice, while the converse is false: a subset of $X$ can be a complete lattice without being not even a sublattice of $X$.

By a correspondence we mean a point-to-set mapping. A correspondence $F : X \to X$ is said to be nonempty if $F(x)$ is nonempty for every $x \in X$. Whenever we write a definition or theorem about correspondences in this paper, we always assume that these are nonempty.

A correspondence $F : X \to X$ is said to have a least (greatest) element if $F(x)$ has a least (greatest) element for every $x \in X$; that is, if for every $x \in X$ there exists some $0_x \in F(x)$ (some $1_x \in F(x)$) such that $0_x \leq y$ ($y \leq 1_x$) for every $y \in F(x)$.

A selection from $F$ is any function $f : X \to X$ such that $f(x) \in F(x)$ for every $x \in X$. Clearly, if correspondence $F$ has a least (greatest) element, this is a selection from $F$.

## 3. Increasing Correspondences

Let $X$ be a poset and $F : X \to X$ be a correspondence. We can consider $F$ as being the joint best reply of a game. Hence $X$ as being the set of strategy profiles, with no distinction among pure, mixed, correlated strategies etc.

We say that $F$ is *upper (lower) increasing* if for every $x, y \in X$, $x \leq y$ implies that for every $a \in F(x)$ (for every $b \in F(y)$), there is some $b \in F(y)$ (some $a \in F(x)$) such that $a \leq b$.

These properties are dual, in the sense that $F$ is upper (lower) increasing on $X$ if and only if it is lower (upper) increasing on $X^d$. Antoniadou (2007) uses upper and lower increasingness, which she calls - in terms of the corresponding set relations - the pathwise-lower-than relation, to study the comparative statics of consumer problems. Calciano (2007) uses upper and lower increasingness to study fixpoint problems for correspondences defined on posets and lattices.

Let now $X$ be a lattice. We say that $F$ is *Veinott-increasing*[5] if for every $x, y \in X$, $x \leq y$ implies that for every $a \in F(x)$ and every $b \in F(y)$, $a \vee b \in F(y)$ (called join increasingness), and $a \wedge b \in F(x)$ (called meet increasingness).

The two following properties are introduced in this paper for the first time, to be best of our knowledge[6]. We say that $F$ is *strongly upper (lower) increasing* if for every $x, y \in X$, $x \leq y$ implies that for every $a \in F(x)$ and $b \in F(y)$, there is some $q \in F(y)$ (some $p \in F(x)$) such that $a \leq q \leq a \vee b$ (such that $a \wedge b \leq p \leq b$). Notice that we do not require neither $a \vee b$ to belong to $F(y)$, nor $a \wedge b$ to belong to $F(x)$.

Clearly, $F$ is strongly upper (lower) increasing on $X$ if and only if it is strongly lower (upper) increasing on $X^d$.

If $F$ is a function, all the notions of increasingness introduced so far boil down to the function being increasing, by which we mean that if $x < y$, then $F(x) \leq F(y)$. Upper (lower) increasingness is implied by strong upper (lower) increasingness, which in turn is implied by join (meet) increasingness. But the converses do not hold.

## 4. Fixpoint Theorems

In this section we prove fixpoint theorems for increasing correspondences. Our theorems generalize those of Veinott (1992, Ch. 4, Th. 14)

[5]Topkis (1978) ascribes this notion of increasingness to Arthur Veinott.

[6]Calciano (2007, Def. 2) uses the same names but to denote different properties.

and Zhou (1994, Th. 1). Our main result is Theorem 3, where we establish conditions in terms of the increasingness of the correspondence and of existence of extremal elements that make the correspondence's fixpoint set be a nonempty complete lattice.

Our results stand in the same relation to Tarski's fixpoint theorem as Kakutani's fixpoint theorem does to Brouwer's. However, we prove our results without using Tarski's theorem. Zhou as well does not use Tarski's theorem, while Veinott uses it directly. Furthermore, we do not use selection arguments in our proofs. We will relate our theorems to increasing-selection arguments in the next section.

Some more notation is needed. Let $X$ be a poset and associate to a correspondence $F : X \to X$ the two sets

$$A := \{x \in X : F(x) \cap [x, +\infty) \neq \emptyset\},$$

$$B := \{x \in X : F(x) \cap (-\infty, x] \neq \emptyset\}.$$

$A$ is the set of elements $x$ of $X$ at which $F$ stays above the diagonal, by which we mean that some element of $F(x)$ is greater than or equal to $x$. Set $B$ is the dual of $A$. The fixpoint set of $F$ is exactly the intersection of $A$ and $B$.

We report here, for completeness and ease of comparison with our results, both Tarski's and Veinott-Zhou's fixpoint theorems[7].

Theorem 0 (Tarski 1955, Th. 1): *Let $X$ be a complete lattice and $f : X \to X$ be an increasing function. (a) $\vee A$ and $\wedge B$ are, respectively, the greatest and least fixpoint of $f$; and (b) the fixpoint set of $f$ is a complete lattice.*

Theorem 1 (Veinott 1992, Ch. 4, Th. 14. Zhou 1994, Th. 1): *Let $X$ be a complete lattice and $F : X \to X$ be a correspondence. If $F$ is Veinott-increasing and $F(x)$ is a subcomplete sublattice of $X$ for every $x \in X$, then (a) $\vee A$ and $\wedge B$ are, respectively, the greatest and least fixpoint of $F$; and (b) the fixpoint set of $F$ is a complete lattice.*

[7]Veinott's theorem proves indeed a broader result. For a correspondence $F : X \times T \to X$, where $X$ is a complete lattice, $T$ a poset, and $F$ is Veinott-increasing on $X \times T$ and subcomplete-sublattice-valued for each $t$, Veinott proves that the sets of selections and increasing selections from the associated fixpoint correspondence $E : T \to X$ are nonempty complete lattices with common least and greatest elements. For the special case of $T = \{t\}$, this amounts to the statement of Theorem 1 upon recognizing that, in such a case, the sets of selections and increasing selections from $E : T \to X$ both coincide with the set of fixpoints of $F(., t)$.

We prove now, in its general form, a simple result which will be used in proving the theorems that follow. Zhou, in his proof of Theorem 1, prove the result separately for the special cases of $A$ and of arbitrary subsets of the fixpoint set of $F$, while Veinott (1992, Ch. 4, Th. 11) proves the result for the special case of increasing functions.

LEMMA 1: *Let $X$ be a poset and $F : X \to X$ be a correspondence. (i) If $F$ is upper increasing and has a greatest element, then for every nonempty subset $S \subseteq A$, if $\vee S$ exists then it belongs to $A$. (ii) If $F$ is lower increasing and has a least element, then for every nonempty subset $T \subseteq B$, if $\wedge T$ exists then it belongs to $B$.*

PROOF: We prove (i). Point (ii) then follows by duality. Pick any $S \subseteq A$ and any $x \in S$. By the definition of $A$, there is $y \in F(x)$ such that $x \leq y$. Since $x \leq \vee S$, by upper increasingness of $F$ for such $y \in F(x)$ there exists $z \in F(\vee S)$ such that $y \leq z$. Let $1 \in F(\vee S)$ be the greatest element of $F(\vee S)$. Clearly $x \leq y \leq z \leq 1$. Hence 1 majorizes $S$ and so $\vee S \leq 1$. Thus $\vee S \in A$. Q.E.D.

By Lemma 1, when $X$ is a complete lattice $A$ is a join-subcomplete sublattice of $X$ and, having a least element (the least element of $X$), it is also a meet-complete lattice. Of course, $A$ it is not necessarily a subcomplete sublattice of $X$. Dually, $B$ is a meet-subcomplete and join-complete lattice[8].

Theorem 2 below generalizes point (a) of Theorem 1 in two respects. First, it requires the correspondence $F$ to be upper (lower) increasing, instead of Veinott increasing. Second, it requires for any $x \in X$ that $F(x)$ has a greatest (least) element, instead of requiring it to be a subcomplete sublattice of $X$.

We state Theorem 2 in the context of posets. This is to underscore the fact that, in point (a) of Theorems 0 and 1, the completeness of lattice $X$ plays the only role of guaranteeing that $A$ and $B$ are nonempty and that $\vee A$ and $\wedge B$ exist, but apart from this, completeness is not a crucial ingredient in the proofs[9].

[8]Notice that the intersection of $A$ and $B$, that is, the fixpoint set of $F$, needs not to be a complete lattice (indeed, not even a lattice). Otherwise we would not need Theorem 3 in this paper. Take for example any unordered $a, b$ in $X$ and set $A = \{\mathbf{0}, a, b, a \vee b\}$ and $B = \{\mathbf{1}, a, b, a \wedge b\}$. The intersection $A \cap B$ is not a lattice, notwithstanding that $A$ is join-subcomplete and meet-complete and that $B$ is meet-subcomplete and join-complete.

[9]Theorem 2 has been already proved in Calciano (2007, Th. 1), but without recurring to Lemma 2.

THEOREM 2: *Let $X$ be a poset and $F : X \to X$ be a correspondence. (i) If $F$ is upper increasing and has a greatest element then $\vee A$, whenever it exists, is the greatest fixpoint of $F$. (ii) If $F$ is lower increasing and has a least element then $\wedge B$, whenever it exists, is the least fixpoint of $F$.*

PROOF: Point (i). Let $1 \in F(\vee A)$ be the greatest element of $F(\vee A)$. By Lemma 1, $\vee A \in A$ and so there exists $y \in F(\vee A)$ such that $\vee A \leq y$. Thus, for every $x \in A$, $x \leq \vee A \leq y \leq 1$, were $1 \in F(\vee A)$ is the greatest element of $F(\vee A)$. By upper increasingness of $F$, from $\vee A \leq 1$ it follows that there is some $y \in F(1)$ such that $1 \leq y$. Thus $1 \in A$, and so $1 \leq \vee A$. Hence $1 = \vee A \in F(\vee A)$. Thus $\vee A$ is a fixpoint of $F$. It is the greatest fixpoint since the fixpoint set of $F$ is a subset of $A$.

Point (ii). Consider the dual poset $X^d$ of $X$. Since $F$ is lower increasing on $X$, it is upper increasing on $X^d$, and since $F(x)$ has a least element for every $x \in X$, it has a greatest element for every $x \in X^d$. Furthermore, $\wedge B$ in $X$ coincides with $\vee A$ in $X^d$. Hence, by applying to $X^d$ the result obtained in (i) for $X$, we have done. Q.E.D.

Theorem 3 below generalizes point (b) of Theorem 1 in a similar way as Theorem 2 generalized point (a) of Theorem 1. Namely, Theorem 3 assumes a form of increasingness for $F$ weaker than Veinott-increasingness, and requires existence of certain extremal elements for $F$ instead of requiring it to be subcomplete-sublattice-valued.

Some new notation is needed. Let $X$ be a poset with a greatest element $\mathbf{1}$ and $F : X \to X$ be a correspondence. For a fixed $h \in X$, define the correspondence $F_h : [h, \mathbf{1}] \to [h, \mathbf{1}]$ as

$$F_h(x) = F(x) \cap [h, \mathbf{1}].$$

Notice that this correspondence may well be empty.

THEOREM 3: *Let $X$ be a complete lattice and $F : X \to X$ be a correspondence. Assume that:*
*(i) $F$ is upper increasing and strongly lower increasing,*
*(ii) $F$ has a greatest element,*
*(iii) for every $h \in A$, the correspondence $F_h$ has a least element whenever nonempty.*
*Then the fixpoint set of $F$ is a nonempty complete lattice.*

*Remark.* In both Veinott's and Zhou's theorems, $F$ is assumed to be subcomplete-sublattice-valued. Under this assumption any $F_h$ is subcomplete-sublattice-valued as well, and hence it has - if nonempty

- all its infima and suprema, and in particular it has both a least and a greatest element.

PROOF OF THEOREM 3: Call $E$ the fixpoint set of $F$. Since $X$ is complete, then $B$ is nonempty and $\wedge B$ exists. Observe that the correspondence $F_{\mathbf{0}}$ coincides with $F$, which is assumed to be nonempty. So $F$ has a least element and, being strongly lower increasing, it is lower increasing. Thus by point (ii) of Theorem 1, $E$ is nonempty and $\wedge B$ is its least element.

We now show that $E$ is a join-complete lattice. Hence, having a least element, it is a complete lattice. We follow Tarski's route. Pick any nonempty subset $T \subseteq E$. We want to show that $\vee_E T$ exists, which is equivalent to show that the set $E \cap [\vee T, \mathbf{1}]$ has a least element. To prove the latter, consider the correspondence $F_{\vee T}$. Observe that the fixpoint set of $F_{\vee T}$ is exactly $E \cap [\vee T, \mathbf{1}]$. We show that $F_{\vee T}$ has a least fixpoint.

*Claim 1.* The correspondence $F_{\vee T}$ is nonempty. Indeed, we prove a little more, namely that for every $h \in A$, the correspondence $F_h$ is nonempty. Then, because by Lemma 1 we know that $\vee T \in A$, Claim 1 is proved. Indeed, pick any $h \in A$ and any $x \in [h, \mathbf{1}]$. Since $h \in A$ there is some $y \in F(h)$ such that $h \leq y$, and since $h \leq x$ and $F$ is upper increasing, for such $y \in F(h)$ there is some $z \in F(x)$ such that $y \leq z$. Hence $z \in F(x) \cap [h, \mathbf{1}]$. □

*Claim 2.* $F_{\vee T}$ is lower increasing. Again, we prove a little more, namely that for every $h \in X$ such that $F_h$ is nonempty, $F_h$ is lower increasing. Then by Claim 1, Claim 2 is proved. Pick indeed any $x, y \in [h, \mathbf{1}]$ such that $x \leq y$. Pick any $b \in F_h(y)$ and any $a \in F_h(x)$. If $a \leq b$ we are done. If $a$ is unordered with $b$, then by strong lower increasingness of $F$ there is some $p \in F(x)$ such that $a \wedge b \leq p \leq b$. Since $h$ minorizes the set $\{a, b\}$, then $h \leq a \wedge b$, and so $p$ is in $F_h(x)$. If $b < a$, by strong lower increasingness of $F$ there is some $p \in F(x)$ such that $b = a \wedge b \leq p \leq b$, and since $h \leq a \wedge b$, then $b$ is in $F_h(x)$. Hence $F_h$ is lower increasing. □

By Claim 1, the set $F_{\vee T}(\mathbf{1}) = F(\mathbf{1}) \cap [\vee T, \mathbf{1}]$ is nonempty, and since $F(\mathbf{1}) \cap [\vee T, \mathbf{1}] \equiv F_{\vee T}(\mathbf{1}) \cap [\vee T, \mathbf{1}]$, then $\mathbf{1}$ belongs to the set

$$B_T := \{x \in [\vee T, \mathbf{1}] : F_{\vee T}(x) \cap [\vee T, x] \neq \emptyset\},$$

which is then nonempty. Furthermore $\wedge B_T$ exists by completeness of $X$, and it clearly belongs to $[\vee T, \mathbf{1}]$ because $\vee T$ minorizes $B_T$. Finally, the correspondence $F_{\vee T}$ has a least element by assumption and hence satisfies all the assumptions of point (ii) of Theorem 1. As a result, for the least element $0 \in F_{\vee T}(\wedge B_T)$, $0 = \wedge B_T \in F_{\vee T}(\wedge B_T)$. Hence $\wedge B_T$ is the least fixpoint of $F_{\vee T}$. Since $T$ was arbitrary, $E$ is a join-complete lattice and the theorem is proved. Q.E.D.

## 5. Increasing Selections

We have proved our Theorems 2 and 3 without recurring to increasing-selection arguments. We want to pose here the following question: would there be any added-value in using increasing-selection arguments instead, or in other words, could we have assumed less in Theorems 2 and 3 had we tried to prove those theorems by identifying appropriate increasing selection to which one could then apply Tarski's fixpoint theorem?

We address these questions with respect to the increasingness assumptions for the correspondence $F$ that we have done in Theorems 2 and 3. We consider these assumptions for $F$ 'minimal' if they are equivalent to the increasingness of some selection $f$ from $F$ to which we can apply Tarski's result to prove our theorem.

In Theorem 4 below we show that, keeping fixed all the other assumptions of Theorem 2, assuming that $F$ is lower increasing is equivalent as assuming that a certain selection from $F$ is increasing, this selection having the same least fixpoint as $F$ (an analogous result holds for the greatest fixpoint of $F$).

In Theorem 5 we show that, keeping fixed all the other assumptions of Theorem 3, assuming that $F$ is strongly lower increasing implies that a certain selection from $F$ is increasing, this selection having the same fixpoint set of $F$.

Unfortunately, in Theorem 5 the equivalence does not hold. Hence the notion of increasingness for $F$ is minimal in Theorem 2, but it is not so in Theorem 3. The next two subsections present and discuss these results.

5.1. **Extremal Fixpoints.** If one is interested only in the existence of extremal fixed point, then the proof of Theorem 2 suggests how to pick selections from $F$ that have the same extremal fixpoints as $F$. In particular, if $F$ has a least (greatest) element and a least (greatest) fixpoint, then by Tarski's theorem the least (greatest) element of $F$, whenever it is an increasing function, has the same least (greatest) fixpoint as $F$, and so represents a proper selection in this context.

The next theorem shows that, assuming $F$ lower (upper) increasing, is equivalent as assuming that the least (greatest) element is an increasing selection from $F$, and hence it is the weakest form of increasingness for correspondences that implies this property for the least (greatest) element.

In this sense, according to our notion of minimality, we cannot relax the assumption of lower (upper) increasingness in Theorem 2 to show existence of extremal fixed points. However, our notion of minimality is weak: we cannot assure that there are no others selection from $F$ which share their extremal fixed points with $F$ and which would be increasing under assumptions on $F$ weaker than lower or upper increasingness.

THEOREM 4: *Let $X$ be a poset and $F : X \to X$ be a correspondence with a least (greatest) element. $F$ is lower (upper) increasing if and only if the least (greatest) element of $F$ is an increasing selection from $F$.*

PROOF: We prove the theorem for the least element. The case for the greatest element follows then by duality. For $z$ in $X$, let $0_z$ denote the least element of $F(z)$. Pick $x, y$ in $X$ with $x \leq y$. Assume that the least element of $F$ is an increasing selection. Hence, for any $b$ in $F(y)$, $0_x \leq 0_y \leq b$, hence $F$ is lower increasing. In the other direction, if $F$ is lower increasing, given $0_y$ there is some $a$ in $F(x)$ such that $0_x \leq a \leq 0_y$. Q.E.D.

5.2. **Structure of the Fixpoint Set.** Theorem 4 can be used also in the context of Theorem 3; that is, when one is interested in the ordering of the whole fixpoint set. In fact, the proof of Theorem 3 makes evident that what matters, in order for the (nonempty) fixpoint set of $F$ to be a complete lattice, is that for each subset $T$ of the fixpoint set of $F$, the correspondence $F_{\vee T}$ be lower increasing. In the presence of Theorem 4, this is equivalent to assuming that, for every $T$, the least element of $F_{\vee T}$ is an increasing selection. We have indeed introduced strong lower increasingness to provide for a global condition on $F$ which guarantees that each correspondence $F_{\vee T}$ is lower increasing.

However, according to this reasoning, we would need an ad-hoc selection for each fixpoint of $F$. As a partial remedy to this, we introduce now a selection from the restriction of $F$ to set $A$ that works for every fixpoint of $F$; that is, which has exactly the same fixpoint set of $F$. We show that this selection is increasing if $F$ is strongly lower increasing.

The vice-versa however is not true, that is, our selection can be increasing without $F$ being strongly lower increasing. In such sense, there may be some room to relax the assumption of strong lower increasingness for $F$ in Theorem 3.

Consider the set $A$ and the function $l : A \to X$ defined as

$$l(x) = min\{F(x) \cap [x, \mathbf{1}]\}.$$

Under the assumptions of Theorem 3, in particular that $F$ has a greatest element and is upper increasing and that, for each $h \in A$, the correspondence $F_h$ has a least element, the function $l$ is indeed well-defined (see Claim 1 of Theorem 3). It is furthermore evident that the fixpoint set of $l$ is exactly that of $F$. Function $l$ is a selection from the restriction of $F$ to the set $A$. Notice that when we study the fixpoint set of $F$, the behavior of $F$ outside set $A$ is irrelevant.

In order to apply Tarski theorem to $l$, we would need $A$ to be a complete lattice, $l$ to map $A$ into itself, and $l$ to be increasing. All these properties are guaranteed by the assumptions of Theorem 3.

In particular, if $X$ is a complete lattice and $F$ is upper increasing and has a greatest element, Lemma 1 guarantees that $A$ is a complete lattice. We address the other two conditions below. Add to the assumptions stated to prove the next results those, aforementioned, that make selection $l$ well-defined.

LEMMA 2: *If $F$ is upper increasing, then $l$ maps $A$ into itself.*

PROOF: Pick any $h \in A$. We want to show that $l(h) := 0_h$ is in $A$. By the definition of $l$, we know that $0_h \in F(h)$ and that $h \leq 0_h$. But then, since $F$ is upper increasing, for such $0_h$ there exists some $x \in F(0_h)$ such that $0_h \leq x$, which proves the claim. Q.E.D.

THEOREM 5: *If $F$ is strongly lower increasing, then $l$ is increasing.*

PROOF: Pick any $x, y \in A$ with $x \leq y$. Set $l(x) := 0_x$ and $l(y) = 0_y$. Pick any $a \in F(x) \cap [x, \mathbf{1}]$ and consider $0_y$. By strong lower increasingness of $F$, there is some $p \in F(x)$ such that $0_y \wedge a \leq p \leq 0_y$. But $x \leq a$ and $y \leq 0_y$, and so the inequality $x \leq y$ implies that $x$ minorizes $\{a, 0_y\}$. Hence $x \leq 0_y \wedge a$, which implies that $p \in F(x) \cap [x, \mathbf{1}]$. Thus $0_x \leq p$ and hence $0_x \leq 0_y$. Q.E.D.

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