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An extension of disjunctive programming and its impact for compact tree formulations

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An extension of disjunctive programming and its impact for compact tree formulations

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Abstract

In the 1970's, Balas [2, 4] introduced the concept of disjunctive programming, which is optimization over unions of polyhedra. One main result of his theory is that, given linear descriptions for each of the polyhedra to be taken in the union, one can easily derive an extended formulation of the convex hull of the union of these polyhedra. In this paper, we give a generalization of this result by extending the polyhedral structure of the variables coupling the polyhedra taken in the union. Using this generalized concept, we derive polynomial size linear programming formulations (*compact formulations*) of a well-known spanning tree approximation of Steiner trees and flow equivalent trees for node- as well as edge-capacitated undirected networks. We also present a compact formulation for Gomory-Hu trees, and, as a consequence, of the minimum T-cut problem (but not for the associated T-cut polyhedron). Recently, Kaibel and Loos [10] introduced a more involved framework called *polyhedral branching systems* to derive extended formulations. The most of our model can be expressed in terms of their framework. The value of our model can be seen in the fact that it completes their framework with an interesting algorithmic aspect.

Keywords: disjunctive programming, compact formulation, flow-equivalent trees, Gomory-Hu trees.

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1 Introduction

Let $Q := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q \mid Ax + By \geq c\}$ be a polyhedron. The *projection of Q onto the x-space* is the polyhedron

$$\operatorname{Proj}_x(Q) := \{x \in \mathbb{R}^p \mid \exists y \in \mathbb{R}^q : (x, y) \in Q\}.$$

Conversely, Q is said to be an *extension* of the projected polyhedron $P := \operatorname{Proj}_x(Q)$, and the system $Ax + By \geq c$ is an *extended formulation* for P. The extended formulation $Ax + By \geq c$ is *compact* if q, the number of rows, and the input length of each entry of the inequality system is polynomial in p.

In the 1970's, Balas [2, 4] introduced the concept of disjunctive programming, which is optimization over unions of polyhedra. Below we restate a well-known result saying that, given linear descriptions for each of the polyhedra to be taken in the union, one can easily derive an extended formulation of the convex hull of the union of these polyhedra. Recently, Kaibel and Loos [10] introduced a powerful framework called *polyhedral branching systems* that generalizes Balas' result as well as the framework of Martin, Rardin, and Campbell [13] to derive extended formulations from dynamic programming algorithms for combinatorial optimization problems.

In this paper, we consider a generalization of the concept of disjunctive programming whose polyhedral aspects are covered by the framework of Kaibel and Loos [10]. The motivation of our model lies in the algorithmic interpretation of disjunctive programming as a two-level approach to solve a linear optimization problem. Since the generalization is straightforward, while the framework of polyhedral branching systems is quite involved (at least, it would need some space to explain it), we pass on a description of that framework and directly start our considerations with disjunctive programming, its interpretation as two-level optimization model, and its consequences for extended formulations.

Given a finite collection of polyhedra P^i, $i \in \mathcal{I}$, where $\mathcal{I}$ is a finite index set, a *disjunctive program* is a mathematical program of the form

$$\begin{array}{ll} \max & w^T x \\ \text{s.t.} & x \in \bigcup_{i \in \mathcal{I}} P^i. \end{array} \tag{1}$$

By a well-known result of Balas [3], given complete linear descriptions of each of the polyhedra P^i to be taken in the union, one can describe the convex combination of the union of the polyhedra by an extended formulation.

Theorem 1.1 (Balas [3]). *Given polyhedra*

$$P^i = \{x \in \mathbb{R}^n \mid A^i x \geq b^i\} \;=\; \operatorname{conv} V^i + \operatorname{cone} R^i,\; i \in \mathcal{I},$$

the following system:

$$
\begin{aligned}
x - \sum_{i\in\mathcal{I}} x^i &= 0, & & \\
A^i x^i - \lambda_i b^i &\geq 0, & & i \in \mathcal{I}, \\
\sum_{i\in\mathcal{I}} \lambda_i &= 1, & & \\
\lambda_i &\geq 0, & & i \in \mathcal{I}
\end{aligned}
\tag{2}
$$

provides an extended formulation for the polyhedron

$$P_{\mathcal{I}} := \operatorname{conv} \bigcup_{i\in\mathcal{I}} V^i + \operatorname{cone} \bigcup_{i\in\mathcal{I}} R^i.$$

In particular, denoting by P the set of vectors $(x, \{x^i, \lambda_i\}_{i\in\mathcal{I}})$ satisfying (2)*,*

(i) if $x^\star$ is a vertex of $P_{\mathcal{I}}$, then $(\bar{x}, \{\bar{x}^i, \bar{\lambda}_i\}_{i\in\mathcal{I}})$ is a vertex of P, with $\bar{x} = x^\star$, $(\bar{x}^k, \bar{\lambda}^k) = (x^\star, 1)$ for some $k \in \mathcal{I}$, and $(\bar{x}^i, \bar{\lambda}_i) = (0, 0)$ for $i \in \mathcal{I} \setminus \{k\}$;

(ii) if $(\bar{x}, \{\bar{x}^i, \bar{\lambda}_i\}_{i\in\mathcal{I}})$ is a vertex of P, then $(\bar{x}^k, \bar{\lambda}^k) = (\bar{x}, 1)$ for some $k \in \mathcal{I}$, $(\bar{x}^i, \bar{\lambda}_i) = (0, 0)$ for $i \in \mathcal{I} \setminus \{k\}$, and $\bar{x}$ is a vertex of $P_{\mathcal{I}}$. □

By Theorem 1.1, the disjunctive program (1) can be solved by solving the linear program $\max\{w^T x \mid (x, \{x^i, \lambda_i\}_{i\in\mathcal{I}}) \text{ satisfies } (2)\}$, provided we are given linear descriptions of each polyhedron P^i as required.

From an algorithmic viewpoint, to solve (1), one usually would compute an optimal solution of each subproblem $\max\{w^T x \mid x \in P^i\}$, and then one would choose the best (or a best) among them. This two-level approach is reflected in the extended formulation. For simplicity, let us assume that, in the first phase, for each subproblem $\max\{w^T x \mid x \in P^i\}$ an optimal solution $\bar{x}^i$ exists. Let $\lambda \in \mathbb{R}^{\mathcal{I}}$ be the (variable) vector whose components are the λ_i. Then, one defines a vector $\bar{w}$ by $\bar{w}_i := w^T \bar{x}^i$, $i \in \mathcal{I}$, and solves, in the second phase, the linear program $\max\{\bar{w}^T \lambda \mid \lambda \in \Delta\}$ over the simplex

$$\Delta := \{\lambda \in \mathbb{R}^{\mathcal{I}} : \sum_{i\in\mathcal{I}} \lambda_i = 1,\ \lambda_i \geq 0\ \forall\, i \in \mathcal{I}\}.$$

Given linear programs $\max\{w^T x \mid x \in P^i\}$, it is, however, not always intended to optimize over the union of these polyhedra, but sometimes the subproblems are part of a more complex optimization problem.

As an example, let us consider (the polyhedral version of) the minimum spanning tree problem over the metric closure of a weighted graph. This problem has relevance for the approximation of the Steiner tree problem, which will be discussed in more detail in Section 2. We denote the node and edge set of a graph G by $V(G)$ and $E(G)$, respectively. Given an edge weighting $w : E(G) \to \mathbb{R}$ and a subset F of $E(G)$, we define $w(F) := \sum_{e\in F} w(e)$. Given a graph

G with a nonnegative edge weighting $w : E(G) \to \mathbb{R}_+$, the *metric closure* of (G, w) is the pair $(K, \bar{w})$, where K is a complete graph on node set $V(G)$ and $\bar{w}(e)$, for $e = \{s, t\}$, is defined to be the length of a shortest path connecting s and t in G w.r.t. to w if there is any such path, and otherwise $\bar{w}(e) := +\infty$. The aim is now to find a spanning tree T of K minimizing $\bar{w}(E(T))$. The approach obviously consists of a two-level model. In the first step, we solve a so-called *all-pairs shortest path problem*, and in the second step, a minimum spanning tree problem whose input data are given by the output data of the first problem. For each of the two problems, several linear programming formulations are known and, among these, even compact formulations. The question now arises whether or not these formulations can be brought together to provide a linear programming formulation for the entire problem. The answer to this question is surprisingly quite easy and can be given using a modification of (2). Let $\mathcal{I} := E(K)$ be given by the edge set of K, and let $\min\{\sum w(e)x_e^{\{s,t\}} \mid A^{\{s,t\}}x^{\{s,t\}} \geq b^{\{s,t\}}\}$ be a linear programming formulation of a shortest s,t-path problem in G for each edge $\{s,t\} \in E(K) = \mathcal{I}$. Then, replacing Δ by a linear characterization $\Pi := \{\lambda \in \mathbb{R}^{E(K)} : C\lambda \geq d\}$ of the spanning tree polytope, we propose the following model:

$$
\begin{aligned}
\min \quad & \sum_{e \in E(G)} w(e)x_e \\
\text{s.t.} \quad & x - \sum_{\{s,t\} \in E(K)} x^{\{s,t\}} = 0 \\
& A^{\{s,t\}}x^{\{s,t\}} - \lambda_{\{s,t\}}b^{\{s,t\}} \geq 0 \qquad \text{for all } \{s,t\} \in E(K), \\
& C\lambda \geq d.
\end{aligned}
$$

In any optimal solution $(\bar{x}, \{\bar{x}^{\{s,t\}}, \bar{\lambda}_{\{s,t\}}\}_{\{s,t\} \in E(K)})$, $\bar{\lambda}$ is then a convex combination of the characteristic vectors of minimum weight spanning trees w.r.t. $\bar{w} : E(K) \to \mathbb{R}_+$.

The aim of the remainder of this section is to prove the correctness of this approach in general.

For any polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$ and any $\alpha \in \mathbb{R}_+$, let $P(\alpha) := \{x \in \mathbb{R}^n \mid Ax \geq \alpha b\}$. This implies that

$$
\begin{aligned}
& P = \operatorname{conv}\{v_1, v_2, \ldots, v_k\} + \operatorname{cone}\{r_1, r_2, \ldots, r_m\} \\
\Leftrightarrow \quad & P(\alpha) = \operatorname{conv}\{\alpha v_1, \alpha v_2, \ldots, \alpha v_k\} + \operatorname{cone}\{r_1, r_2, \ldots, r_m\}.
\end{aligned}
$$

Theorem 1.2. *Given pointed polyhedra $P^i = \{x \in \mathbb{R}^n \mid A^i x \geq b^i\}$, $i \in \mathcal{I}$ and a 0/1-polytope $\Pi = \{\lambda \in \mathbb{R}^{\mathcal{I}} \mid C\lambda \geq d\}$, that is, $\Pi = \operatorname{conv} V$ for some $V \subseteq \{0,1\}^{\mathcal{I}}$.*

(i) $(\bar{x}, \{\bar{x}^i\}_{i \in \mathcal{I}}, \bar{\lambda})$ is a vertex of the polyhedron Q defined as the set of vectors $(x, \{x^i\}_{i \in \mathcal{I}}, \lambda)$ satisfying

$$
\begin{aligned}
x - \sum_{i \in \mathcal{I}} x^i &= 0, \\
A^i x^i - \lambda_i b^i &\geq 0, \qquad i \in \mathcal{I}, \qquad (3) \\
C\lambda &\geq d
\end{aligned}
$$

if and only if $\bar{\lambda}$ is a vertex of Π, and for each $i \in \mathcal{I}$, if $\bar{\lambda}_i = 1$, then $\bar{x}^i$ is a vertex of P^i while if $\bar{\lambda}_i = 0$, then $\bar{x}^i = 0$.

(ii) For any $w \in \mathbb{R}^n$, consider the linear programs

$$\max\{w^T x \mid (x, \{x^i\}_{i\in\mathcal{I}}, \lambda) \in Q\}, \tag{4}$$

$$\bar{w}_i := \max\{w^T x \mid x \in P^i\} \tag{5i}$$

for $i \in \mathcal{I}$, and

$$\max\{\bar{w}^T \lambda \mid \lambda \in \Pi\}. \tag{6}$$

Then, (4) is unbounded if and only if $\bar{w}_i = \infty$ for some $i \in \mathcal{I}$. Next, let (4) be bounded. Then, $(\bar{x}, \{\bar{x}^i\}_{i\in\mathcal{I}}, \bar{\lambda})$ is an optimal solution of (4) if and only if $\bar{\lambda}_i^{-1}\bar{x}^i$ is an optimal solution of (5i) for each $i \in \mathcal{I}$ with $\bar{\lambda}_i > 0$ and $\bar{\lambda}$ is an optimal solution of (6).

Proof. (i) To show the necessity, let $(\bar{x}, \{\bar{x}^i\}_{i\in\mathcal{I}}, \bar{\lambda})$ be a vertex of Q. First suppose, for the sake of contradiction, that $\bar{\lambda}$ is not a vertex of Π. Then, there are μ, $\nu \in \Pi$ and $0 < \alpha < 1$ such that $\bar{\lambda} = \alpha\mu + (1-\alpha)\nu$. Let $\mathcal{I}'$ be the set of indices $i \in \mathcal{I}$ with $\bar{\lambda}_i = 0$. This implies that $\mu_i = \nu_i = 0$ for $i \in \mathcal{I}'$. Define now vectors $y^i := z^i := \bar{x}^i$ for $i \in \mathcal{I}'$, $y^i := \frac{\mu_i}{\bar{\lambda}_i}\bar{x}^i$, $z^i := \frac{\nu_i}{\bar{\lambda}_i}\bar{x}^i$ for $i \in \mathcal{I} \setminus \mathcal{I}'$, as well as $y := \sum_{i\in\mathcal{I}} y^i$ and $z := \sum_{i\in\mathcal{I}} z^i$. Then, one easily verifies that, on the one hand,

$$\alpha(y, \{y^i\}_{i\in\mathcal{I}}, \mu) + (1-\alpha)(z, \{z^i\}_{i\in\mathcal{I}}, \nu) = (\bar{x}, \{\bar{x}^i\}_{i\in\mathcal{I}}, \bar{\lambda}),$$

and on the other hand, $(y, \{y^i\}_{i\in\mathcal{I}}, \mu)$, $(z, \{z^i\}_{i\in\mathcal{I}}, \nu) \in Q$. Thus, $(\bar{x}, \{\bar{x}^i\}_{i\in\mathcal{I}}, \bar{\lambda})$ is not a vertex, a contradiction.

Next, suppose that $(\bar{x}, \{\bar{x}^i\}_{i\in\mathcal{I}}, \bar{\lambda})$ is a vertex of Q and $\bar{\lambda}$ a vertex of Π. The latter implies that $\bar{\lambda} \in \{0,1\}^{\mathcal{I}}$. Assume now that $\bar{x}^j$ is not a vertex of P^j for some j with $\bar{\lambda}_j = 1$. Then, $\bar{x}^j$ is the convex combination of two vectors y^j, $z^j \in P^j$. This, in turn, implies that $(\bar{x}, \{\bar{x}^i\}_{i\in\mathcal{I}}, \bar{\lambda})$ is a convex combination of the two vectors obtained from $(\bar{x}, \{\bar{x}^i\}_{i\in\mathcal{I}}, \bar{\lambda})$ by replacing the vector $\bar{x}^j$ by y^j and z^j. Moreover, assuming that $\bar{x}^j \neq 0$ for some j with $\bar{\lambda}_j = 0$, we see that $\bar{x}^j$ is a ray of the cone $\{x \in \mathbb{R}^n \mid A^j x \geq 0\}$. This immediately implies that, also in this case, $(\bar{x}, \{\bar{x}^i\}_{i\in\mathcal{I}}, \bar{\lambda})$ is not a vertex of Q, a contradiction.

To show the sufficiency, suppose that $(\bar{x}, \{\bar{x}^i\}_{i\in\mathcal{I}}, \bar{\lambda})$ is not a vertex of Q. Then, there are two different vectors $(y, \{y^i\}_{i\in\mathcal{I}}, \mu)$, $(z, \{z^i\}_{i\in\mathcal{I}}, \nu) \in Q$ and $0 < \alpha < 1$ such that

$$\alpha(y, \{y^i\}_{i\in\mathcal{I}}, \mu) + (1-\alpha)(z, \{z^i\}_{i\in\mathcal{I}}, \nu) = (\bar{x}, \{\bar{x}^i\}_{i\in\mathcal{I}}, \bar{\lambda}).$$

First, assume that $\mu \neq \nu$. Since both vectors are in Π, this immediately implies that $\bar{\lambda}$ is a convex combination of μ and ν, a contradiction. Consequently, we may assume that $\mu = \nu = \bar{\lambda} \in \{0,1\}^{\mathcal{I}}$. However, since $(y, \{y^i\}_{i\in\mathcal{I}}, \mu)$ and $(z, \{z^i\}_{i\in\mathcal{I}}, \nu)$ have to be distinct, and since the vectors y and z are just sums of the vectors y^i and z^i, respectively, we conclude that $y^j \neq z^j$ for some $j \in \mathcal{I}$. If $\bar{\lambda}_j = 1$, it follows that $\bar{x}^j$ was not a vertex of P^i. Finally, if $\bar{\lambda}_j = 0$, then $\alpha y^j + (1-\alpha)z^j = \bar{x}^j \neq 0$

or $\alpha y^j + (1-\alpha)z^j = 0$, and hence P^j is not a pointed polyhedron. In either case, this yields a contradiction.

(ii) If (4) is unbounded, then there exists a ray $(r, \{r^j\}_{j\in\mathcal{I}}, \rho) \neq 0$ of P with $w^T r > 0$, which implies $r \neq 0$. Since Π is a polytope, it follows that $\rho = 0$. Moreover, since r is the sum of the vectors r^j, $w^T r^i > 0$ for at least one $i \in \mathcal{I}$. Since, by definition, $A^i r^i \geq 0$, this implies that r^i is a ray of P^i and (5i) is unbounded. Conversely, if (5i) is unbounded for some $i \in \mathcal{I}$, there exists a ray $\tilde{r}$ of P^i with $w^T\tilde{r} > 0$. Define $(r, \{r^j\}_{j\in\mathcal{I}}, \rho)$ by $r := r^i := \tilde{r}$, $r^j := 0$ for $j \in \mathcal{I}\setminus\{i\}$, and $\rho := 0$. Then, we conclude that $(r, \{r^j\}_{j\in\mathcal{I}}, \rho)$ is a ray of Q and $w^T r > 0$. Therefore, (4) is unbounded.

Next, suppose that (4) is bounded, which means that (5i) is bounded for each $i \in \mathcal{I}$. This, in turn, justifies the definition of $\bar{w}$.

For any $i \in \mathcal{I}$ and any $\alpha \in \mathbb{R}_+$, $x^\star$ is an optimal solution of $\max\{w^T x \mid x \in P^i\}$ if and only if $\alpha x^\star$ is one of $\max\{w^T x \mid x \in P^i(\alpha)\}$. Now let $(\bar{x}, \{\bar{x}^j\}_{j\in\mathcal{I}}, \bar{\lambda})$ be an optimal solution of (4). Assume that $w^T x^\star > w^T(\bar{\lambda}_i^{-1}\bar{x}^i)$ for some $x^\star \in P^i$, with $i \in \mathcal{I}$ and $\bar{\lambda}_i > 0$. Then, $(x, \{x^j\}_{j\in\mathcal{I}}, \bar{\lambda}) \in Q$, where $x^i := \bar{\lambda}_i x^\star$, $x^j := \bar{x}^j$ for $j \in \mathcal{I}\setminus\{i\}$, and $x := \sum_{i\in\mathcal{I}} x^i$. Moreover, $w^T x > w^T\bar{x}$, a contradiction. Next, for any $\lambda \in \Pi$ and any optimal solutions $\tilde{x}^i$ of (5i) for each $i \in \mathcal{I}$, we derive that $(x, \{x^j\}_{j\in\mathcal{I}}, \lambda) \in Q$, where $x := \sum_{i\in\mathcal{I}} x^i$ and $x^i := \lambda_i\tilde{x}^i$ for $i \in \mathcal{I}$. Thus, $w^T x \leq w^T\bar{x}$. Since

$$\begin{array}{rc} & w^T x \leq w^T\bar{x} \\ \Leftrightarrow & \sum_{i\in\mathcal{I}} w^T x^i \leq \sum_{i\in\mathcal{I}} w^T\bar{x}^i \\ \Leftrightarrow & \sum_{i\in\mathcal{I}} w^T(\lambda_i\tilde{x}^i) \leq \sum_{i\in\mathcal{I}} w^T\bar{x}^i \\ \Leftrightarrow & \sum_{i\in\mathcal{I}} \lambda_i w^T\tilde{x}^i \leq \sum_{i\in\mathcal{I}} \bar{\lambda}_i w^T\tilde{x}^i \\ \Leftrightarrow & \sum_{i\in\mathcal{I}} \lambda_i\bar{w}_i \leq \sum_{i\in\mathcal{I}} \bar{\lambda}_i\bar{w}_i \\ \Leftrightarrow & \bar{w}^T\lambda \leq \bar{w}^T\bar{\lambda}, \end{array}$$

it follows that $\bar{\lambda}$ is an optimal solution of (6).

For the same reasons, if $\bar{x}^i$ is an optimal solution of (5i) for each $i \in \mathcal{I}$ and $\bar{\lambda}$ is one of (6), then $(\bar{x}, \{\bar{x}^j\}_{j\in\mathcal{I}}, \bar{\lambda})$ is optimal for (4), where $\bar{x} := \sum_{i\in\mathcal{I}} \bar{x}^i$. □

Theorem 1.2 can be generalized in several ways. Of course, it can be easily extended for the case that Π is not a 0/1-polytope but any other polytope contained in $\mathbb{R}^{\mathcal{I}}_+$. More important for the following applications is the case in which the polyhedra P^i and Π are given themselves by extended formulations. The following theorem only generalizes those results of Theorem 1.2 that are relevant for the following applications.

Theorem 1.3. *Given extensions $\Theta := \{(\lambda, \mu) \in \mathbb{R}^{\mathcal{I}} \times \mathbb{R}^q \mid C\lambda + D\mu \geq d\}$ of a 0/1-polytope $\Pi \subseteq \mathbb{R}^{\mathcal{I}}$ as well as $Q^i = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{p_i} \mid A^i x + B^i y \geq b^i\}$ of pointed polyhedra $P^i \subseteq \mathbb{R}^n$,*

$i \in \mathcal{I}$. Moreover, for any $w \in \mathbb{R}^n$, consider the linear programs

$$\begin{array}{lll} \max & w^T \sum\limits_{i\in\mathcal{I}} x^i & \\ s.t. & A^i x^i + B^i y^i - \lambda_i b^i \geq 0, & i \in \mathcal{I}, \\ & \quad C\lambda + D\mu \geq d, & \end{array} \tag{7}$$

$$\bar{w}_i := \max\{w^T x \,|\, (x,y) \in Q^i\} \tag{8i}$$

for $i \in \mathcal{I}$, and

$$\max\{\bar{w}^T \lambda \,|\, (\lambda,\mu) \in \Theta\}. \tag{9}$$

Then, (7) is unbounded if and only if $\bar{w}_i = \infty$ for some $i \in \mathcal{I}$. Moreover, in case that $\bar{w}_i < \infty$ for all $i \in \mathcal{I}$, $(\{\bar{x}^i, \bar{y}^i\}_{i\in\mathcal{I}}, \bar{\lambda}, \bar{\mu})$ is an optimal solution of (7) if and only if $(\bar{x}^i, \bar{y}^i) \in Q^i$, $i \in \mathcal{I}$, are optimal solutions of (8i) and $(\bar{\lambda}, \bar{\mu}) \in \Theta$ is an optimal solution of (9). □

Let us call the linear program (7) an *extended LP-formulation* for the collection $V^{\bar{w}}$ of all optimal vertex solutions $\pi \in \Pi$ w.r.t. $\bar{w}$. Using standard linear programming techniques, an extension for the convex hull of $V^{\bar{w}}$, $\Pi^{\bar{w}} := \operatorname{conv} V^{\bar{w}}$, can be easily derived from (7). Recall that if each of a pair of dual linear programs

$$(P) \quad \min\{\mathfrak{c}^T \mathfrak{x} \,|\, \mathfrak{A}\mathfrak{x} \geq \mathfrak{b}\} \qquad\qquad (D) \quad \max\{\mathfrak{y}^T \mathfrak{b} \,|\, \mathfrak{y}^T \mathfrak{A} = \mathfrak{c}^T,\, \mathfrak{y} \geq 0\}$$

have feasible solutions, then

$$Q := \{(\mathfrak{x},\mathfrak{y}) \,|\, \mathfrak{A}\mathfrak{x} \geq \mathfrak{b},\, \mathfrak{y}^T\mathfrak{A} = \mathfrak{c}^T, \mathfrak{c}^T\mathfrak{x} - \mathfrak{y}^T\mathfrak{b} = 0,\, \mathfrak{y} \geq 0\}$$

is a polyhedron that consists of all vectors $(\mathfrak{x},\mathfrak{y})$ such that $\mathfrak{x}$ and $\mathfrak{y}$ are optimal for (P) and (D), respectively. Thus, if we identify (7) with (P), then we conclude that $\Pi^{\bar{w}}$ is the projection of Q onto the λ-space.

An application of Theorem 1.3 can be found in Pochet and Wolsey [15] for a variant of the classical lot sizing problem with constant production capacities in which the capacity in each period is an integer multiple of a basic capacity unit.

In the following sections, we give further applications of our model. All of them have in common that (7) provides an extended LP-fomulation of the set $\Pi^{\bar{w}}$. Since these applications are based on compact formulations for spanning trees, we recall two well-known spanning tree formulations.

Given a graph G, the *spanning tree polytope* $\mathrm{P}_{\mathrm{Tree}}(G)$ is the convex hull of the characteristic vectors $\chi^{E(T)} \in \mathbb{R}^{E(G)}$ of spanning trees $T \subseteq G$. Recall that the characteristic vector χ^F of any edge set $F \subseteq E(G)$ is a 0/1-vector with $\chi^F_e = 1$ if and only if $e \in F$. As it is well known, the spanning tree polytope is the set of all $\lambda \in \mathbb{R}^{E(G)}$ satisfying the nonnegativity constraints $\lambda_e \geq 0$, $e \in E$, the equation $\lambda(E(G)) = |V(G)| - 1$, and the inequalities

$$\lambda(E(G[S])) \leq |S| - 1 \qquad\qquad \text{for all } S \subset V(G),\, 2 \leq |S| \leq |V(G)| - 1.$$

Here, $G[S]$ denotes the subgraph of G induced by $S \subseteq V$. This linear description of $\mathrm{P}_{\mathrm{Tree}}(G)$ has an exponential number of inequalities.

According to [5, 12, 18], a compact formulation for $\mathrm{P}_{\mathrm{Tree}}(G)$ can be given as follows. For each edge $e \in E(G)$, we introduce a variable λ_e, and for each for each edge $e \in E(G)$, each node $u \in e$, and each node $v \in V(G)$, we introduce a variable $\mu_{e,u,v}$. In the extended formulation, the edge set of a spanning tree T of G will be represented by the vector (λ^T, μ^T) with $\lambda^T = \chi^{E(T)}$ and $\mu^T_{e,u,v} = 1$ if and only if $e \in E(T)$ and v belongs to the same component of u in $T - e$. Then, $\mathrm{P}_{\mathrm{Tree}}(G)$ is the projection of the polyhedron Π defined as the set of all (λ, μ) satisfying

$$\begin{aligned} \lambda(E) &= |V| - 1, && \\ \mu_{\{u,v\},u,v} = \mu_{\{u,v\},v,u} &= 0 && \text{for all } \{u,v\} \in E, \\ \lambda_e - \sum_{v \in e} \mu_{e,v,w} &= 0 && \text{for all } e \in E,\ w \in V, \\ \lambda_{\{u,v\}} + \sum_{w \in V \setminus \{u,v\}} \mu_{\{u,w\},w,v} &= 1 && \text{for all } \{u,v\} \in E, \\ \lambda_e &\geq 0 && \text{for all } e \in E, \\ \mu_{e,v,w} &\geq 0 && \text{for all } e \in E, v \in e, w \in V, \end{aligned} \tag{10}$$

onto the λ-space. Here, $V := V(G)$ and $E := E(G)$.

For later reference, we need the following result which can be found in [12].

Lemma 1.4. Π *is an integer polyhedron. Moreover,* $(\lambda, \mu) \in \Pi$ *is a vertex of* Π *if and only if* λ *is the characteristic vector of a spanning tree* T *of* G'*, and for each* $e \in E(T)$ *and each* $u \in e$*, the vector* $(\mu_{e,u,v})_{v \in V(G')}$ *is the characteristic vector of the component of* $T - e$ *containing* u*, and for each* $e \in E(G') \setminus E(T)$*,* $\mu_{e,u,v} = 0$ *for all* $u \in e$*,* $v \in V$*.* □

The following directed formulation is due to Wong [17] and Maculan [11]. Let D be a directed graph, with node set $V(D)$ and arc set $A(D)$. For a fixed node $r \in V(D)$, the *r-arborescence polytope*, that is, the convex hull of the characteristic vectors of r-arborescences, is the projection of the system

$$\begin{aligned} \nu(A) &= |V| - 1, && \\ \nu_a &\geq \sigma^w_a && \text{for all } a \in A, w \in V \setminus \{r\}, \\ \sigma^w(\delta^+(r)) &= 1 && \text{for all } w \in V \setminus \{r\}, \\ \sigma^w(\delta^+(v)) - \sigma^w(\delta^-(v)) &= 0 && \text{for all } v, w \in V \setminus \{r\}, v \neq w, \\ \sigma^w_a &\geq 0 && \text{for all } a \in A, w \in V \setminus \{r\} \end{aligned} \tag{11}$$

onto the ν-space, where $V := V(D)$, $A := A(D)$, and for any $v \in V$, $\delta^+(v)$ and $\delta^-(v)$ are the set of arcs leaving and entering v, respectively. This formulation is motivated by the fact that an arborescence with root r contains an (r, w)-path for each node $w \neq r$. Let now G be a graph and D the digraph obtained from G by replacing each edge $\{u, v\}$ of G by the arcs (u, v) and (v, u). Then, $\mathrm{P}_{\mathrm{Tree}}(G)$ is the projection of (11) extended by the inequalities

$$\lambda_{\{u,v\}} = \nu_{uv} + \nu_{vu} \qquad \text{for all } \{u,v\} \in E(G) \tag{12}$$

onto the λ-space.

We note that both extended formulations are of size $\mathcal{O}(|V|^3)$.

The remainder of this paper is organized as follows. Using the framework outlined in this introduction, we provide, in Section 2, a compact linear programming formulation of a well-known spanning tree approximation of the minimum Steiner tree problem, and, in Section 3, compact formulations of flow equivalent trees for edge- as well as node-capacitated graphs. In the same section, we give a compact formulation for Gomory-Hu trees, which however does not exactly follows from the framework. Based on this formulation, we derive one for the minimum T-cut problem whose inequalities depend, however, on the objective function. Moreover, we discuss the relevance of these formulations for finding a compact formulation for the perfect matching polytope. In Section 4, we briefly summarize our findings and point out some interesting open questions.

2 A compact formulation for the approximation of the Steiner tree problem

Let G be an undirected graph and $S \subseteq V(G)$. A *Steiner tree* for S in G is a tree $T \subseteq G$ whose node set $V(T)$ contains S. Given a cost function $c : E(G) \to \mathbb{R}_+$, in the *Steiner tree problem* for (G, c, S), one wants to find a Steiner tree $T \subseteq G$ minimizing $c(E(T))$.

Let $(K, \bar{c})$ be the metric closure of (G, c). By a well-known result of Gilbert and Pollak [6], if T is a minimum Steiner tree for (G, c, S), and M is a minimum spanning tree in $K[S]$ w.r.t. $\bar{c}$, then $\bar{c}(E(M)) \le 2c(E(T))$.

For the all-pairs shortest path problem (for the computation of the metric closure) as well as for the minimum spanning tree problem there exist compact formulations. Hence, using Theorem 1.3, it is easy to derive a compact formulation approximating the Steiner problem.

For any digraph D, we denote by $V(D)$ and $A(D)$ the node and arc set of D, respectively. Let G be a graph and $c : E(G) \to \mathbb{R}_+$. In what follows, let D be the digraph obtained from G by replacing each edge $e = \{v, w\} \in E(G)$ by the arcs (v, w) and (w, v). Define $\tilde{c} : A(D) \to \mathbb{R}$ by $\tilde{c}((v, w)) := c(\{v, w\})$ for $(v, w) \in A(D)$. Then, a shortest s, t-path problem in (G, c) can be modeled as a minimum cost flow problem in $(D, \tilde{c})$. Consequently, for each edge $e = \{s, t\} \in E(K[S])$,

$$\begin{array}{lll} \min & \sum\limits_{\{u,v\} \in E(G)} c(\{u, v\})(x^e_{uv} + x^e_{vu}) & \\ \text{s.t.} & x^e(\delta^+(s)) - x^e(\delta^-(s)) = 1, & \\ & x^e(\delta^+(v)) - x^e(\delta^-(v)) = 0 & \text{for all } v \in V(D) \setminus \{s, t\}, \\ & 0 \le x^e_a \le 1 & \text{for all } a \in A(D) \end{array} \tag{13e}$$

is an $\mathcal{O}(\langle c \rangle + |G|)$ formulation of the shortest s, t-path problem in (G, c). Here, $|G| := |V(G)| + |E(G)|$, and $\langle c \rangle$ denotes the input size of c.

Combining the shortest path formulations (13e) for each edge $e \in E(K[S])$ with a compact formulation for the spanning tree polytope $\mathrm{P}_{\mathrm{Tree}}(K[S])$ defined on the complete graph on S, we obtain, by Theorem 1.3, a compact formulation approximating the Steiner tree problem defined for (G, c):

$$\min \sum_{e \in E} \sum_{\{u,v\} \in E(G)} c(\{u,v\})(x^e_{uv} + x^e_{vu}) \tag{14}$$

$$\begin{aligned} x^e(\delta^+(s)) - x^e(\delta^-(s)) - \lambda_e &= 0 && \text{for all } e = \{s,t\} \in E, \\ x^e(\delta^+(v)) - x^e(\delta^-(v)) &= 0 && \text{for all } v \in V(D) \setminus \{e\},\ e \in E, \\ x^e_a - \lambda_e &\leq 0 && \text{for all } a \in A(D),\ e \in E, \\ x^e_a &\geq 0 && \text{for all } a \in A(D),\ e \in E, \end{aligned} \tag{15}$$

$$(\lambda, \mu) \text{ satisfies (10)} \quad (\text{or } (\lambda, \nu, \sigma) \text{ satisfies (11), (12)}),$$

where $E := E(K[S])$.

Theorem 2.1. *Let G be a connected graph, let $c : E(G) \to \mathbb{R}_+$ be a cost function, and let $S \subseteq V(G)$. Moreover, let $(K[S], \bar{c}|_S)$ be the restriction of the metric closure $(K, \bar{c})$ of (G, c) to $K[S]$, and let $\mathcal{T}(K[S], \bar{c}|_S)$ be the collection of all maximum weight spanning trees of $K[S]$ w.r.t. $\bar{c}|_S$. Then, there exists an $\mathcal{O}((p(G,c)|S|^2 + q(S))$ extended LP-formulation of $\mathcal{T}(K[S], \bar{c}|_S)$, where $p(G,c)$ is the input size of a linear program for solving a shortest s,t-path problem in G and $q(S)$ that of an extended formulation of the spanning tree polytope defined on $K[S]$. By construction, each optimal vertex solution of such a linear program provides a feasible solution of the Steiner tree problem for (G, c, S) whose cost is at most twice the cost of an optimal Steiner tree.* □

3 Compact formulations for flow equivalent trees and Gomory-Hu trees

Flow equivalent trees provide compact representations of minimum s,t-cut values for all pairs of nodes s,t of an edge- or node-capacitated graph. For edge-capacitated graphs, Gomory-Hu trees are special flow equivalent trees that specify not only the minimum cut values but also the corresponding cuts. Moreover, they can be used to determine minimum T-cuts and play an important role in the design of minimum cost communication networks.

For any subset of nodes $U \subseteq V(G)$ of a graph G, let $\delta(U)$ be the set of edges of G connecting U and $V \setminus U$. A *cut* in G is an edge set of type $\delta(U)$ for some $\varnothing \neq U \subset V(G)$. U and $V(G) \setminus U$ are called the *shores* of $\delta(U)$. For two distinct nodes s and t of G, an s,t*-cut* is an edge set of type $\delta(U)$ such that $s \in U$ and $t \in V(G) \setminus U$, or vice versa. We sometimes write $\delta_G(U)$ instead of $\delta(U)$ to indicate that $\delta_G(U)$ induces a cut of G. Given a capacity function $c : E(G) \to \mathbb{R}_+$ and a cut $K \subseteq E(G)$, the number $c(K)$ is called the *capacity* of K. The *minimum s,t-cut problem* asks for an s,t-cut $K \subseteq E(G)$ of minimum capacity.

As it is well known, minimum s,t-cut problems in undirected graphs can be represented by compact linear programs. Let G be a graph, let $c : E(G) \to \mathbb{R}_+$ be a capacity function, and let s and t two distinct nodes of G. For each node $v \in V(G)$, introduce a variable z_v, and for each edge $e \in E(G)$, a variable x_e. Then, the model reads:

$$\begin{aligned} \min \quad & \sum_{e \in E(G)} c(e) x_e \\ \text{s.t.} \quad & z_s = 0, \\ & z_t = 1, \\ & x_{\{u,v\}} + z_u - z_v \geq 0, \\ & x_{\{u,v\}} + z_v - z_u \geq 0 \qquad \text{for all } \{u,v\} \in E(G). \end{aligned} \tag{16st}$$

Given a directed graph D and a subset U of $V(D)$, we denote by $\delta^{\text{out}}(U)$ and $\delta^{\text{in}}(U)$ the set of arcs of D leaving and entering U, respectively. Let $s, t \in V(D)$ and $U \subseteq V(D)$. Then, $\delta^{\text{out}}(U)$ is an *(s,t)-cut* if $s \in U$ and $t \in V(D) \setminus U$. Given arc capacities $c : A(D) \to \mathbb{R}_+$, the *minimum (s,t)-cut problem* is to find an $(s,t) - cut$ $\delta^{\text{out}}(U)$ minimizing $c(\delta(U))$. Introducing for each node $v \in V(D)$ a variable z_v and for each arc $(u,v) \in A(D)$, a variable x_{uv}, the minimum (s,t)-cut problem can be formulated as a compact linear program as follows:

$$\begin{aligned} \min \quad & \sum_{(u,v) \in A(D)} c((u,v)) x_{uv} \\ \text{s.t.} \quad & z_s = 0, \\ & z_t = 1, \\ & x_{uv} \geq 0,\ x_{uv} + z_u - z_v \geq 0 \qquad \text{for all } (u,v) \in A(D). \end{aligned} \tag{17st}$$

3.1 Flow equivalent trees for edge-capacitated graphs

Given a graph G with edge capacities $c : E(G) \to \mathbb{R}_+$, in the *all-pairs minimum cut problem*, one wants to find a minimum s,t-cut for all pairs $s,t \in V(G)$. Although one has $\frac{1}{2}|V(G)||V(G)-1|$ pairs of nodes, there exist at most $|V(G)-1|$ different minimum cut values, which can be represented by a tree. An edge-capacitated spanning tree H on $V(G)$ is called a *flow equivalent tree* for (G,c) if for each pair of nodes $s,t \in V(G)$, the value of a minimum s,t-cut in G is equal to that of a minimum s,t-cut in H, or equivalently, both maximum flow values are the same. If, in addition, for each edge $e = \{s,t\}$ of H, the cut determined by the two components of $H - e$ is a minimum s,t-cut of G, then H is said to be a *Gomory-Hu tree* or *cut tree*. The condition for H being a flow equivalent tree implies that the minimum edge weight in the unique path between s and t equals the value of a minimum s,t-cut in G. Thus, letting K be the complete graph on $V(G)$ and defining $r(\{s,t\})$ as the capacity of a minimum s,t-cut of G for all $\{s,t\} \in E(K)$, we see that the flow equivalent trees for (G,c) are exactly the maximum weight spanning trees of (K, r). However, not every maximum spanning tree for (K, r) is a Gomory-Hu tree, see Schrijver [16, Section 15.4].

By applying Theorem 1.3 one now can easily write down a compact LP-formulation of flow equivalent trees. In this context, note that this theorem presumes to couple either maximization or minimization problems. Since the linear programs (16st) for $e \in E(K)$ are minimization problems over whose outputs we have to solve a maximization problem, we switch to the edge complements of spanning trees called *co-trees*. Clearly, $H \subseteq K$ is a maximum weight spanning tree if and only if its edge complement is a minimum weight co-tree. Thus, we obtain the following compact LP-formulation for flow equivalent trees:

$$\min \sum_{\{s,t\} \in E(K)} \sum_{e \in E(G)} c(e) x_e^{\{s,t\}} \tag{18}$$

$$\begin{aligned} z_s^{\{s,t\}} = 0,\ z_t^{\{s,t\}} - \vartheta_{\{s,t\}} &= 0 && \text{for all } \{s,t\} \in E(K), \\ x_{\{u,v\}}^{\{s,t\}} + z_u^{\{s,t\}} - z_v^{\{s,t\}} &\geq 0, && \\ x_{\{u,v\}}^{\{s,t\}} + z_v^{\{s,t\}} - z_u^{\{s,t\}} &\geq 0 && \text{for all } \{u,v\} \in E(G), \{s,t\} \in E(K), \end{aligned} \tag{19}$$

$$\vartheta = \mathbb{1} - \lambda, \tag{20}$$

$$(\lambda, \mu) \text{ satisfies (10)} \quad (\text{or } (\lambda, \nu, \sigma) \text{ satisfies (11), (12)}).$$

Here, $\mathbb{1}$ denotes the vector of all ones, and hence, equation (20) ensures that ϑ is a convex combination of the characteristic vectors of co-trees. Thus, in any optimal vertex solution of the above linear program, λ is the characteristic vector of a flow-equivalent tree.

Theorem 3.1. *Let G be a graph with edge capacities $c : E(G) \to \mathbb{R}_+$. Then, there exists an extended LP-formulation for flow-equivalent trees for (G, c) of input size $\mathcal{O}((p(G, c)|V(G)|^2 + q(V(G)))$, where $p(G, c)$ is the input size of an extended LP-formulation of a minimum s, t-cut problem in G and $q(V(G))$ that of an extended formulation of the spanning tree polytope defined on the complete graph on $V(G)$.* □

3.2 Flow equivalent trees for node-capacitated graphs

Consider a graph G and two distinct nodes $s, t \in V(G)$. A set $S \subseteq V(G)$ is said to be an *s, t-node cut* if s and t are in distinct connected components of the graph induced by $V(G) \setminus S$ in G. In particular, if S is an s, t-node cut, then $S \subseteq V(G) \setminus \{s, t\}$. $S \subseteq V(G)$ is called an *s, t-separation* if either $s \in S$ or $t \in S$ or S is an s, t-node cut. Given node capacities $d : V(G) \to \mathbb{R}_+$, the *minimum s, t-node cut problem* is to find an s, t-node cut S of minimum capacity $d(S) = \sum_{v \in S} d(v)$. The *minimum s, t-separation problem* is defined accordingly.

The concept of flow-equivalent trees and cut trees has also been introduced for node cuts and separations. Some wrong statements on the existence of these trees have been made in previous papers, and corrections followed. For an overview, see Hassin and Levin [8].

Let G be a graph with node capacities $d : V(G) \to \mathbb{R}_+$. A *flow-equivalent tree* for (G, d) is a spanning tree H with edge capacities $r : E(H) \to \mathbb{R}_+$ on $V(G)$ such that for each pair of nodes $s, t \in V(G)$, the value of a minimum s,t-separation in G is equal to that of a minimum s,t-cut in H. Granot and Hassin [7] proved the existence of flow-equivalent trees for separations.

Two key results of Granot and Hassin [7] lead to a compact formulation of flow equivalent trees for (G, d). The first is that the flow equivalent trees for (G, d) are exactly the maximum spanning trees of (K, r), where K denotes the complete graph on $V(G)$ and $r(\{s,t\})$ is defined as the value of a minimum s,t-separation in G for all $\{s,t\} \in E(K)$. The second is that each s,t-separation problem can be represented by a compact linear program. The model involves a standard construction of a digraph D as follows. For each node $v \in V(G)$ introduce two nodes v^1, v^2 and an arc (v^1, v^2) of capacity $d(v)$ in D. Moreover, each edge $\{u, v\} \in E(G)$ will be represented by the two arcs (u^2, v^1) and (v^2, u^1) with capacities $+\infty$ in D. One readily checks that every directed minimum (s^1, t^2)-cut $\delta^{\text{out}}(U)$ in D provides a minimum s,t-separation $S := \{v \in V(G) : v^1 \in U, v^2 \in V(D) \setminus U\}$ in G. Thus, to derive a compact formulation of flow-equivalent trees for (G, d), we write for each edge $\{s,t\} \in E(K)$ a linear program of the form (17st). Index the variables by $\{s,t\}$. Since the capacities of the arcs $(u^2, v^1), (v^2, u^1)$ are infinity for each edge $\{u, v\}$ of G, we can set the variables $x^{\{s,t\}}_{u^2v^1}, x^{\{s,t\}}_{v^2u^1}$ to zero. This results into some easily checkable simplifications of the linear objective function and the inequalities of (17st):

$$
\begin{array}{lll}
\min & \sum\limits_{v \in V(G)} d(v) x^{\{s,t\}}_{v^1v^2} & \\
\text{s.t.} & z^{\{s,t\}}_s = 0,\ z^{\{s,t\}}_t = 1, & \\
& x^{\{s,t\}}_{v^1v^2} \geq 0,\ x^{\{s,t\}}_{v^1v^2} + z^{\{s,t\}}_{v^1} - z^{\{s,t\}}_{v^2} \geq 0 & \text{for all } v \in V(G), \\
& z^{\{s,t\}}_{u^2} - z^{\{s,t\}}_{v^1} \geq 0,\ z^{\{s,t\}}_{v^2} - z^{\{s,t\}}_{u^1} \geq 0 & \text{for all } \{u, v\} \in E(G).
\end{array}
$$

Scaling the simplified versions of (17st) for $\{s,t\} \in E(K)$ by the components $\vartheta_{\{s,t\}}$ of a convex combination of co-trees, we obtain:

$$\min \sum_{\{s,t\}\in E(K)} \sum_{v\in V(G)} d(v) x^{\{s,t\}}_{v^1v^2} \tag{21}$$

$$\left\{ \begin{array}{rr} z^{\{s,t\}}_{s^1} = 0,\ z^{\{s,t\}}_{t^2} - \vartheta_{\{s,t\}} = 0, & \\ x^{\{s,t\}}_{v^1v^2} \geq 0,\ x^{\{s,t\}}_{v^1v^2} + z^{\{s,t\}}_{v^1} - z^{\{s,t\}}_{v^2} \geq 0 & \text{for all } v \in V(G), \\ z^{\{s,t\}}_{u^2} - z^{\{s,t\}}_{v^1} \geq 0,\ z^{\{s,t\}}_{v^2} - z^{\{s,t\}}_{u^1} \geq 0 & \text{for all } \{u,v\} \in E(G), \end{array} \right\} \tag{22st}$$

$$\text{for all } \{s,t\} \in E(K)$$

$$\vartheta = \mathbb{1} - \lambda, \tag{23}$$

(λ, μ) satisfies (10)   (or (λ, ν, σ) satisfies (11), (12)).

Theorem 3.2. *Let G be a graph with node capacities $d : V(G) \to \mathbb{R}_+$. Then, there exists an extended LP-formulation for flow-equivalent trees for (G, d) of input size $\mathcal{O}((p(G, d)|V(G)|^2 + q(V(G)))$, where $p(G, d)$ is the input size of an extended formulation of a minimum s, t-separation problem in G and $q(V(G))$ that of one of the spanning tree polytope defined on the complete graph on $V(G)$.* □

3.3 Gomory-Hu trees

We start by considering minimum-requirement trees studied by Hu [9] that turn out to be the same as Gomory-Hu trees.

Let G be a graph with "requirement" function $c : E(G) \to \mathbb{R}_+$ (say the data volume to be sent between two nodes of a network), and let K be the complete graph on $V(G)$. A *minimum-requirement tree* is a spanning tree H of K minimizing

$$\sum_{e\in E(G)} c(e)\, \mathrm{dist}_H(e), \tag{24}$$

where $\mathrm{dist}_H(e)$ denotes the length of the path in H connecting the end nodes of e.

For any edge f of H, define $r_H(f)$ to be the capacity (=requirement) of the cut in G determined by the two components of $H - f$. This cut is called the *fundamental cut induced by f*. Then, (24) is equal to

$$\sum_{f\in E(H)} r_H(f). \tag{25}$$

For a fixed spanning tree H of K and a fixed edge f of H, the capacity $r_H(f)$ can be easily expressed as the optimal objective value of the linear program

$$\begin{aligned}
\min \quad & \sum_{e \in E(G)} c(e) x_e^f \\
\text{s.t.} \quad & z_v^f = 0 && \text{for all } v \in U, \\
& z_v^f = 1 && \text{for all } v \in V(G) \setminus U, \\
& x_{\{u,v\}}^f + z_u^f - z_v^f \geq 0, \\
& x_{\{u,v\}}^f + z_v^f - z_u^f \geq 0 && \text{for all } \{u,v\} \in E(G),
\end{aligned}$$

where U is either of the components of $H - f$. This linear program can be expressed in terms of the compact spanning tree formulation (10) by identifying z- and μ-variables and fixing λ-variables as follows:

$$\begin{aligned}
\min \quad & \sum_{e \in E(G)} c(e) x_e^f \\
\text{s.t.} \quad & x_{\{u,v\}}^f + \mu_{f,t,u} - \mu_{f,t,v} \geq 0, \\
& x_{\{u,v\}}^f + \mu_{f,t,v} - \mu_{f,t,u} \geq 0 && \text{for all } \{u,v\} \in E(G), \\
& \lambda_e = 1 && \text{for all } e \in E(H), \\
& \lambda_e = 0 && \text{for all } e \in E(K) \setminus E(H), \\
& (\lambda, \mu) \text{ satisfies (10)},
\end{aligned}$$

where $t \in f$ is fixed. Thus, fixing for each edge f of H a node $t_f \in f$, (25) is determined by

$$\begin{aligned}
\min \quad & \sum_{f \in E(H)} \sum_{e \in E(G)} c(e) x_e^f \\
\text{s.t.} \quad & x_{\{u,v\}}^f + \mu_{f,t_f,u} - \mu_{f,t_f,v} \geq 0, \\
& x_{\{u,v\}}^f + \mu_{f,t_f,v} - \mu_{f,t_f,u} \geq 0 && \text{for all } f \in E(H),\ \{u,v\} \in E(G), \\
& \lambda_e = 1 && \text{for all } e \in E(H), \\
& \lambda_e = 0 && \text{for all } e \in E(K) \setminus E(H), \\
& (\lambda, \mu) \text{ satisfies (10)}.
\end{aligned}$$

Hence, the linear program

$$\min \sum_{f \in E(K)} \sum_{e \in E(G)} c(e) x_e^f \tag{26}$$

$$\begin{aligned}
\text{s.t.}\ & x_{\{u,v\}}^f + \mu_{f,t_f,u} - \mu_{f,t_f,v} \geq 0, \\
& x_{\{u,v\}}^f + \mu_{f,t_f,v} - \mu_{f,t_f,u} \geq 0 && \text{for all } f \in E(K),\ \{u,v\} \in E(G), \qquad (27) \\
& (\lambda, \mu) \text{ satisfies (10)}.
\end{aligned}$$

is a compact LP-formulation for minimum-requirement trees.

Theorem 3.3. *Let G be a graph with capacity function $c : E(G) \to \mathbb{R}_+$, and let K be the complete graph on $V(G)$. Moreover, let the polyhedron P be the set of all $(\{x^f\}_{f\in E(K)}, \lambda, \mu)$ satisfying* (10) *and* (27). *Then, for any vector $(\{\bar{x}^f\}_{f\in E(K)}, \bar{\lambda}, \bar{\mu}) \in P$, $(\{\bar{x}^f\}_{f\in E(K)}, \bar{\lambda}, \bar{\mu})$ is a vertex of P minimizing* (26) *if and only if $\bar{\lambda}$ is the characteristic vector of a minimum-requirement tree H for (G, c), for each $f \in E(H)$, the vector $(\bar{\mu}_{f,t_f,v})_{v\in V(K)}$ is the characteristic vector of the component of $H - f$ containing t_f, $\bar{x}^f$ is the characteristic vector of the fundamental cut in G induced by f, and for each $f \in E(K) \setminus E(T)$, $\mu_{f,u,v} = \bar{x}^f_v = 0$ for all $u \in f$, $v \in V$.*

Proof. 0 obviously is a lower bound of the objective values of the linear program to be considered, and hence, always an optimal vertex of P exists.

Any vector $(\{x^f\}_{f\in E(K)}, \lambda, \mu) \in P$ is a vertex of P if and only if each component $x^f_{\{u,v\}}$ is chosen to be minimal, that is,

$$x^f_{\{u,v\}} = \max\{\mu_{f,t_f,v} - \mu_{f,t_f,u}, \mu_{f,t_f,u} - \mu_{f,t_f,v}\}$$

and (λ, μ) is a vertex of the extension Π (constituted by (10)) of the spanning tree polytope $\mathrm{P}_{\mathrm{Tree}}(K)$. So by Lemma 1.4, $(\{x^f\}_{f\in E(K)}, \lambda, \mu)$ is vertex if and only if it satisfies the conditions mentioned in Theorem 3.3.

To conclude, for any vertex $(\{x^f\}_{f\in E(K)}, \lambda, \mu)$ of P and any spanning tree H of K, $\lambda = \chi^{E(H)}$ implies $\sum_{e\in E(G)} c(e)x^f_e = r_H(f)$ if $f \in E(H)$ and equals 0 otherwise. Thus, $(\{x^f\}_{f\in E(K)}, \lambda, \mu)$ minimizes (26) if and only if H is a c-minimum-requirement tree. □

Let us now return to Gomory-Hu trees. Hu [9] showed, based on an earlier result of Adolphson and Hu [1], that Gomory-Hu trees are minimum-requirement trees. Considering the proofs of the results in [1, 9] it turns out that, conversely, minimum-requirement trees are Gomory-Hu trees. Below we restate Hu's result taking into consideration this equivalence.

Theorem 3.4. *Let G be a graph with capacity function $c : E \to \mathbb{R}_+$, and let K be the complete graph on $V(G)$. Then, $H \subseteq K$ is a Gomory-Hu tree for (G, c) if and only if H is a minimum-requirement tree for (G, c).*

Proof. Let H and H' be a Gomory-Hu and a minimum-requirement tree for (G, c), respectively. By definition of H, for each edge $f = \{s, t\} \in E(H)$ and each edge f' on the s, t-path in H' one has

$$r_{H'}(f') \geq r_H(f), \tag{28}$$

as $r_H(f)$ is the capacity of a minimum s, t-cut and the components of $H' - f'$ determine an s, t-cut. There exists a bijection $\varphi : E(H) \to E(H')$ such that for each $f = \{s, t\} \in E(H)$, $\varphi(f)$ is an edge on the s, t-path in H', since H and H' are spanning trees. For details, see [1] or [16, Section 15.4a]. So (28) implies

$$\sum_{f'\in E(H')} r_{H'}(f') = \sum_{f\in E(H)} r_{H'}(\varphi(f)) \geq \sum_{f\in E(H)} r_H(f). \tag{29}$$

Since (24) and (25) are equal, H' minimizes (25), and hence all sums in (29) are equal. This implies that H is a minimum-requirement tree. Moreover, because of (28), $r_{H'}(f')) = r_H(\varphi^{-1}(f'))$ for all $f' \in E(H')$. Hence, H' is a Gomory-Hu tree. □

Corollary 3.5. *Let G be a graph, and let $c : E(G) \to \mathbb{R}_+$ be a capacity function. The linear program* (26) *subject to* (10) *and* (27) *is a compact LP-formulation for Gomory-Hu trees for* (G, c).

In what follows, we describe a consequence of Theorem 3.4 for the minimum T-cut problem.

3.4 The minimum T-cut problem

Let G be a graph with capacities on the edges $c : E(G) \to \mathbb{R}_+$, and let T be a subset of the nodes of G of even size. A *T-cut* is a cut $\delta(U)$ of G such that $|T \cap U|$ is odd. In the *minimum T-cut problem* one wants to find a T-cut $\delta(U)$ of G minimizing $c(\delta(U))$. The polyhedral counterpart to the minimum T-cut problem is the *T-cut polyhedron*, defined as the set of all $x \in \mathbb{R}^{E(G)}$ for which there exists a convex combination y of the characteristic vectors of T-cuts such that $x \geq y$.

The minimum T-cut problem can be solved in polynomial time. Various algorithms are available and, among them, the famous algorithm of Padberg and Rao [14] that computes a Gomory-Hu tree for (G, c), and selects among the fundamental cuts a T-cut of minimum capacity. This T-cut has minimum capacity among all T-cuts of G.

This algorithm can be used as as basis to derive a compact formulation for the minimum T-cut problem. However, the inequalities of our formulation will depend on the input c, and as a consequence, this is only a formulation for the minimum T-cut problem but not for the T-cut polyhedron.

Let K be the complete graph on $V(G)$, let H be a Gomory-Hu tree for (G, c), and let λ its characteristic vector. Recall that for $f \in E(H)$, the capacity of the fundamental cut induced by f is the optimal objective value of the linear program

$$\min \sum_{e \in E(G)} c(e) x_e^f \tag{30f}$$

$$\text{s.t. } \lambda_e = 1 \qquad \text{for all } e \in E(H), \tag{31}$$

$$\lambda_e = 0 \qquad \text{for all } e \in E(K) \setminus E(H), \tag{32}$$

$$\left.\begin{array}{l} x_{\{u,v\}}^f + \mu_{f,t_f,u} - \mu_{f,t_f,v} \geq 0, \\ x_{\{u,v\}}^f + \mu_{f,t_f,v} - \mu_{f,t_f,u} \geq 0 \end{array}\right. \qquad \text{for all } \{u, v\} \in E(G), \tag{33f}$$

$$(\lambda, \mu) \text{ satisfies (10)},$$

where $t_f \in f$ is fixed, while for $f \in E(K) \setminus E(H)$ the optimal value is zero. Introduce for each $f \in E(K)$ a (binary) variable ν_f such that $\nu_f = 1$ if and only if $f \in E(H)$ and the fundamental cut induced by f is a T-cut. Denote the edge set corresponding to T-cuts by F. Then, the following mathematical program is a disjunctive programming approach for finding a minimum T-cut among the fundamental cuts of H:

$$\min \sum_{f\in E(K)} \sum_{e\in E(G)} c(e)y_e^f \tag{30}$$

$$\begin{aligned}
\text{s.t. } & (\lambda,\mu) \text{ satisfies } (10),(31),(32), \\
& (\{x^f\}_{f\in E(K)},\mu) \text{ satisfies (33f)} && \text{for all } f\in E(K), && (33)\\
& \nu_f = \begin{cases} 1 & \text{if } f\in F \\ 0 & \text{otherwise,}\end{cases} && && (34)\\
& \sum_{f\in E(K)} \vartheta_f = 1,\ 0\le \vartheta \le \nu, && && (35)\\
& y_e^f \ge x_e^f + \vartheta_f - 1,\ y_e^f \ge 0 && \text{for all } e\in E(G), f\in E(K). && (36)
\end{aligned}$$

To see the correctness of this formulation, let $(\{x^f,y^f\}_{f\in E(K)},\lambda,\mu,\nu,\vartheta)$ be a vertex of the polyhedron determined by the inequalities of this formulation. Recall that λ is the characteristic vector of H. Moreover, if $f=\{s,t\}\in E(H)$, $\mu_{f,s,\star},\mu_{f,t,\star}$ are the characteristic vectors of the components of $H-f$, and x^f represents the associated fundamental cut. Otherwise, that is, in case $\lambda_f=0$, $x^f=\mu_{f,s,\star}=\mu_{f,t,\star}=0$. Moreover, since $(\{x^f,y^f\}_{f\in E(K)},\lambda,\mu,\nu,\vartheta)$ is a vertex, inequalities (34), (35) imply that ϑ is a unit vector with $\vartheta_g=1$ for some $g\in F$. For the same reason, $(\{x^f,y^f\}_{f\in E(K)},\lambda,\mu,\nu,\vartheta)$ satisfies at least one of the two equations in (36) at equality for each pair of edges $e\in E(G), f\in E(K)$. Hence, $y^f=0$ for all $f\in E(K)\setminus\{g\}$, while y^g is the characteristic vector of a T-cut.

In an instance of the minimum T-cut problem neither H nor F are part of the input. Therefore, to derive a compact linear formulation from the above mathematical program we have to remove the fixing equations (30), (31) and to replace the fixing equations (34) by an appropriate system of inequalities.

In a first step, let us assume that H but only H is part of the input. One way to determine ν is as follows. Fix any node $r\in T$, and for each $s\in T\setminus\{r\}$, denote by P_s the unique r,s-path in H. Then, for any edge f of H, $f\in F$ if and only if the number of paths P_s using f is odd. This, in turn, is the case if and only if the symmetric difference of the paths P_s contain f.

Let $s\in T\setminus\{r\}$. Introducing for each edge $e\in E(K)$ a variable π_e^s, one easily checks that the characteristic vector of P_s is, for instance, determined by the system

$$\begin{aligned}
y^s(\delta(r)) = y^s(\delta(s)) &= 1, \\
y^s(\delta(v)\setminus e) - y_e^s &\ge 0 \qquad \text{for all } v\in V(K),\ e\in\delta(v), \qquad (37s)\\
0\le y^s &\le \lambda.
\end{aligned}$$

Next, we have to express the symmetric difference of the paths P_s by linear inequalities. As it is well-known, the characteristic vector of the symmetric difference $X \Delta Y := (X \cup Y) \setminus (X \cap Y)$ of any sets $X, Y \subseteq E(K)$ is determined by the system

$$\begin{aligned}
&\alpha_f \le \chi_f^X + \chi_f^Y && \text{for all } f \in E(K),\\
&\alpha_f \ge \chi_f^X - \chi_f^Y && \text{for all } f \in E(K),\\
&\alpha_f \ge \chi_f^Y + \chi_f^X && \text{for all } f \in E(K),\\
&\alpha_f \le 2 - \chi_f^X - \chi_f^Y && \text{for all } f \in E(K).
\end{aligned}$$

Since the Δ-operator is associative, the symmetric difference of the paths P_s can be determined sequentially. For this, let $(s_0, s_1, \ldots, s_k)$ be any order of the nodes in $T \setminus \{r\}$. Then, defining $D_0 := P_{s_0}$ and $D_i := D_{i-1} \Delta P_{s_i}$ for $i = 1, \ldots, k$, it follows that $F = D_k$. Thus, introducing for each step $i \in \{0, 1, \ldots, k\}$, an edge variable vector α^i to represent D_i, we see that equations (34) can be replaced by the system

$$\begin{aligned}
(y^{s_i}, \lambda) \text{ satisfies (37s)} & \qquad \text{for } s = s_i,\ i = 0, 1, \ldots, k, && (37)\\
\alpha^0 - y^{s_0} = 0 & && (38)\\
-\alpha^i + \alpha^{i-1} + y^{s_i} \ge 0 & \qquad \text{for } i = 1, \ldots, k, && (39)\\
\alpha^i - \alpha^{i-1} + y^{s_i} \ge 0 & \qquad \text{for } i = 1, \ldots, k, && (40)\\
\alpha^i + \alpha^{i-1} - y^{s_i} \ge 0 & \qquad \text{for } i = 1, \ldots, k, && (41)\\
\alpha^i + \alpha^{i-1} + y^{s_i} \le 2 & \qquad \text{for } i = 1, \ldots, k, && (42)\\
\alpha^k - \nu = 0. & && (43)
\end{aligned}$$

In the remainder of this section we consider the linear program (30) subject to (10), (31)-(33), (35)-(43). Removing the fixing variables (31), (32), we obtain, of course, a compact formulation for a relaxation of the minimum T-cut problem. This implies that the system (10), (33), (35)-(43) characterizes a polyhedron that contains the T-cut polyhedron, but we conjecture that this system does not determine the T-cut polyhedron. However, using standard techniques, a compact formulation for a particular instance of the minimum T-cut problem can be derived as follows. Denote by $\mathcal{GH}(G, c)$ the collection of all Gomory-Hu trees for (G, c) and by $\mathrm{P}_{\text{GH-Tree}}(G, c) := \mathrm{conv}\{\chi^{E(H)} \in \mathbb{R}^{E(K)} | H \in \mathcal{GH}(G, c)\}$ the *Gomory-Hu tree polytope.* By Corollary 3.5, the linear program (26) subject to (10) and (27) is a compact LP-formulation for $\mathcal{GH}(G, c)$. Thus, there exists a compact extension Q of $\mathrm{P}_{\text{GH-Tree}}(G, c)$ as we have outlined in the introduction. We assume that Q is given in λ, μ, η- space. Let $(f_1, f_2, \ldots, f_m)$ be any ordering of the edges of K. Then, the unique optimal solution of the linear program

$$\begin{aligned}
\min \sum_{j=1}^{m} 2^{j-1} \lambda_{e_j}\\
\text{s.t. } (\lambda, \mu, \eta) \in Q
\end{aligned}$$

is the characteristic vector of the minimal lexicographic Gomory-Hu tree with respect to the ordering. Q is determined by a compact system of the form

$$A \begin{pmatrix} \lambda \\ \mu \\ \eta \end{pmatrix} \geq b,$$

where we may assume that this system contains (10) as subsystem. The characteristic vector of the minimal lexicographic Gomory-Hu tree is the projection of the polytope determined by

$$A \begin{pmatrix} \lambda \\ \mu \\ \eta \end{pmatrix} \geq b, \pi^T A = d^T, d^T \begin{pmatrix} \lambda \\ \mu \\ \eta \end{pmatrix} - \pi^T b = 0, \pi \geq 0 \tag{44}$$

onto the λ-space, where $d^T := (1, 2, 4, \ldots, 2^{m-1}, 0, 0, \ldots, 0)$, and hence, the linear program (30) subject to (33), (35)-(44) is a compact formulation for the minimum T-cut problem. However, we note that (44) contains inequalities some of whose coefficients have input length $|E(K)|$, and more important, some depend on c, and thus this is not a formulation for the T-cut polyhedron.

4 Conclusion

Motivated by a well-known result of Balas on disjunctive programming, we studied extended formulations in connection with a simple two-level optimization scheme that also can be derived as a special case from the more general framework *branched polyhedral systems* of Kaibel and Loos [10]. Using this scheme, we gave compact formulations for the spanning tree approximation of the Steiner tree problem, flow equivalent trees, and Gomory-Hu trees. Using the Gomory-Hu tree formulation, we also derived a compact formulation for the minimum T-cut problem whose inequalities, however, depend on the objective function.

A very interesting question in this context is if the compact formulation for the minimum T-cut problem can be modified in such a way that it extends to a compact formulation for the T-cut polyhedron. However, answering this question seems to be a complicated undertaking, since the separation problem for the perfect matching polytope defined on a graph G can be modeled as linear optimization problem over the $V(G)$-cut polyhedron. Such a formulation would also imply a compact formulation for the perfect matching polytope – a long standing unsolved task.

A powerful theory on coupling extended formulations has been developed, especially due to the contribution of Kaibel and Loos [10]. In future research, we are interested to find further examples, where the coupling of extended formulations yields to compact formulations for polynomially solvable combinatorial optimization problems.

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