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The stability of the roommate problem revisited

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The stability of the roommate problem revisited

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Abstract

The lack of stability in some matching problems suggests that alternative solution concepts to the core might be a step towards furthering our understanding of matching market performance. We propose absorbing sets as a solution for the class of roommate problems with strict preferences. This solution, which always exists, either gives the matchings in the core or predicts other matchings when the core is empty. Furthermore, it satisfies the interesting property of outer stability. We also determine the matchings in absorbing sets and find that in the case of multiple absorbing sets a similar structure is shared by all.

Keywords: roommate problem, core, absorbing sets.

JEL Classification: C78

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1 Introduction

Matching markets are of great interest in a variety of social and economic environments, ranging from marriages formation, through admission of students into colleges to matching firms with workers.¹ One of the aims pursued by the analysis of these markets is to find stable matchings. There are, however, some markets for which the set of stable matchings, *i.e.* the core, is empty. For these cases, we suggest that instead of using the common approach of restricting the preferences domain to deal with nonempty core matching markets,² other solution concepts may be applied to find “predictable” matchings. We argue that this alternative is a step towards furthering our understanding of matching market performance.

Our approach consists of associating each matching market with an abstract system (an abstract set endowed with a binary relation) and then applying one of the existing solution concepts to determine predictable matchings. The modeling of abstract systems deals with the problem of choosing a subset from a feasible set of alternatives. Various solution concepts have been proposed for solving abstract systems, such as the core, von Neumann-Morgenstern stable sets³ (von Neumann-Morgenstern [31]), subsolutions (Roth [21]), admissible sets (Kalai *et al.* [14]), and absorbing sets. The notion of absorbing sets, which is the solution concept selected in our work, was first introduced by Schwartz [27] and it coincides with the elementary dynamic solution (Shenoy [26]).

We focus our attention on one-sided matching markets where each agent is allowed to form at most one partnership. This kind of problem is known as the *roommate problem* and is a generalization of the marriage problem, see Gale and Shapley [10]. In these problems each agent in a set ranks all others (including herself) according to her preferences. In this seminal paper it is shown that this problem may not have a stable matching.

The abstract system associated with a roommate problem is the pair formed by the set of all matchings and a binary domination relation which represents the existence of a blocking pair of agents allowing transition from one matching to another. Matchings that are not blocked by any pair of agents are called stable. In this model the set of stable matchings equals the

¹See Roth and Sotomayor [24] for a comprehensive survey of two-sided matching models.

²See for example, Roth [22] and Kelso and Crawford [16].

³Ehlers [9] studies von Neumann-Morgenstern stable sets in two-sided matching markets.

core. Roommate problems that do not admit any such matchings are called *unsolvable*. Otherwise they are said to be *solvable*.

Core stability for solvable roommate problems has been studied by Gale and Shapley [10], Irving [12], Tan [29], Abeledo and Isaak [1], Chung [5], Diamantoudi *et al.* [7] and Klaus and Klijn [17] among others. With few exceptions, however, unsolvable roommate problems have not been so thoroughly studied. When there is no core stability, interest is rekindled in the application of other solution concepts to the class of roommate problems. Such interest is further enhanced from the empirical perspective in that as Pittel and Irving [19] observe, when the number of agents increases, the probability of a roommate problem being solvable decreases fairly steeply.

Here we propose *absorbing sets* as a solution concept for the class of roommate problems with strict preferences. In this context, an absorbing set is a set of matchings that satisfies the following two conditions: (i) any two distinct matchings inside the set (directly or indirectly) dominate each other and (ii) no matching in the set is dominated by a matching outside the set. We believe that the selection of this solution concept is well justified since for a solvable roommate problem it exactly provides the matchings in the core, and for an unsolvable roommate problem it gives a nonempty set of matchings with an interesting property of stability. Thus, the solution of absorbing sets may be considered as a generalization of the core.

The notion of an absorbing set may perhaps be better understood if it is illustrated with the following description: Consider matchings derived from an unstable matching by satisfying a blocking pair of agents. This can be seen as a dynamic process in which unstable matchings are adjusted when a blocking pair of agents mutually decide to become partners. Either this change gives a stable matching or a new blocking pair of agents will generate another matching and so on. If some stable matching exists this dynamic process eventually converges to one.⁴ Otherwise the process will lead to a set of matchings (an absorbing set) such that via this dynamic process (i) any matching in the set can be obtained from any other and (ii) it is impossible to escape from the absorbing set⁵. From this perspective it is easy to see that an absorbing set satisfies a property of *outer stability* in the sense that all matchings not in an absorbing set are (directly or indirectly) dominated

⁴For marriage problems Roth and Vande Vate [25] show that there exists a convergence domination path from any unstable matching to a stable one. This is also shown for solvable roommate problems by Diamantoudi *et al.* [7].

⁵For unsolvable roommate problems Inarra *et al.* [11] show that there is a domination path from any matching that reaches certain matchings called *P*-stable matchings.

by a matching that does belong to an absorbing set. As a result, matchings outside absorbing sets can be ruled out as reasonable matchings.

Among the scant literature on unsolvable roommate problems the papers by Tan [28] and Abraham *et al.* [2] are worthy of mention. The former investigates matchings with the maximum number of disjoint pairs of agents such that these pairs are “internally” stable and the latter looks at matchings with the smallest number of blocking pairs. For solvable roommate problems both proposals give the matchings in the core, but for unsolvable ones it is easy to check that neither satisfies the outer stability property.

The contribution of this work to the analysis of the stability of roommate problems can be summarized as follows:

First, we find that absorbing sets are determined by stable partitions. This notion, introduced by Tan [29] as a structure generalizing the notion of a stable matching, allowed him to establish a necessary and sufficient condition for the existence of a stable matching in roommate problems. By using the relation between absorbing sets and stable partitions we also show that if a roommate problem is solvable then an absorbing set consists of a unique matching, which is stable and the union of all absorbing sets coincides with the core.

Second, we characterize absorbing sets in terms of stable partitions. The characterization provided allows us to specify the stable partitions determining absorbing sets and its matchings, which are those with the greatest number of agents without incentives to change their current partner.

Third, we show that all matchings in an absorbing set share some common features. Furthermore, in the case of a roommate problem with multiple absorbing sets we prove some similarities among their (corresponding) matchings. Specifically in terms of the dynamic process mentioned above, we find that any two absorbing sets have the same set of blocking agents responsible for going from matching to matching within the set, and that the other (nonblocking) agents are paired in a stable way, though this pairing is different across absorbing sets.

The rest of the paper is organized into the following sections. Section 2 contains the preliminaries. In Section 3 we study absorbing sets of a roommate problem. Those sets are determined in Section 4. We study the structure of their matchings in Section 5, and Section 6 brings together some extensions. We also incorporate two appendices. Appendix A contains an iterative process which is useful to find the stable partitions determining

absorbing sets. Appendix B contains two parts, one with the lemmas used along the paper and their proofs and the other with the proofs of theorems and corollaries in the text.

2 Preliminaries

A *roommate problem* is a pair $(N, (\succsim_x)_{x \in N})$ where N is a finite set of agents and for each agent $x \in N$, \succsim_x is a complete, transitive preference relation defined over N . Let \succ_x be the strict preference associated with \succsim_x . In this paper we only consider roommate problems with strict preferences, which we denote by $(N, (\succ_x)_{x \in N})$.

A *matching* μ is a one to one mapping from N onto itself such that for all $x \in N$ $\mu(\mu(x)) = x$, where $\mu(x)$ denotes the partner of agent x under the matching μ . If $\mu(x) = x$, then agent x is single under μ . Given $S \subseteq N$, $S \neq \emptyset$, let $\mu(S) = \{\mu(x) : x \in S\}$. That is, $\mu(S)$ is the set of partners of the agents in S under μ . Let $\mu|_S$ be the mapping from S to N which denotes the restriction of μ to S . If $\mu(S) = S$ then $\mu|_S$ is a matching in $(S, (\succ_x)_{x \in S})$.

A pair of agents $\{x, y\} \subseteq N$ (possibly $x = y$) is a *blocking pair* of the matching μ if

$$y \succ_x \mu(x) \text{ and } x \succ_y \mu(y). \quad (1)$$

That is, x and y prefer each other to their current partners in μ . If $x = y$, (1) means that agent x prefers being alone to being matched with $\mu(x)$. An agent $x \in N$ *blocks* a matching μ if that agent belongs to some blocking pair of μ . A matching is called *stable* if it is not blocked by any pair $\{x, y\}$. Let $\{x, y\}$ be a blocking pair of μ . A matching μ' is obtained from μ by *satisfying* $\{x, y\}$ if $\mu'(x) = y$ and for all $z \in N \setminus \{x, y\}$,

$$\mu'(z) = \begin{cases} z & \text{if } \mu(z) \in \{x, y\} \\ \mu(z) & \text{otherwise.} \end{cases}$$

That is, once $\{x, y\}$ is formed, their partners (if any) under μ are alone in μ' , while the remaining agents are matched as in μ .

Tan [29] establishes a necessary and sufficient condition for the solvability of roommate problems with strict preferences in terms of stable partitions. This notion, which is crucial in the investigation of this paper, can be formally defined as follows:⁶

⁶See Biró *et al.* [4] for a clarifying interpretation of this notion.

Let $A = \{a_1, \dots, a_k\} \subseteq N$ be an ordered set of agents. The set A is a *ring* if $k \geq 3$ and for all $i \in \{1, \dots, k\}$, $a_{i+1} \succ_{a_i} a_{i-1} \succ_{a_i} a_i$ (subscript modulo k). The set A is a pair of mutually acceptable agents if $k = 2$ and for all $i \in \{1, 2\}$, $a_{i-1} \succ_{a_i} a_i$ (subscript modulo 2).⁷ The set A is a singleton if $k = 1$.

A *stable partition* is a partition P of N such that:

- (i) For all $A \in P$, the set A is a ring, a mutually acceptable pair of agents or a singleton, and
- (ii) For any sets $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_l\}$ of P (possibly $A = B$), the following condition holds:

$$\text{if } b_j \succ_{a_i} a_{i-1} \text{ then } b_{j-1} \succ_{b_j} a_i,$$

for all $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$ such that $b_j \neq a_{i+1}$.

Condition (ii) may be interpreted as a notion of stability over the partitions satisfying Condition (i).

Note that a stable partition is a generalization of a stable matching. To see this, consider a matching μ and a partition P formed by pairs of agents and/or singletons. Let $A = \{a_1, a_2 = \mu(a_1)\}$ and $B = \{b_1, b_2 = \mu(b_1)\}$ be sets of P . If P is a stable partition then Condition (ii) implies that if $b_1 \succ_{a_1} a_2$ then $b_2 \succ_{b_1} a_2$, which is the usual notion of stability. Hence μ is a stable matching.

The following assertions are proven by Tan [29].

Remark 1 (i) A roommate problem $(N, (\succ_x)_{x \in N})$ has no stable matchings if and only if there exists a stable partition with an odd ring. (ii) Any two stable partitions have exactly the same odd rings.⁸ (iii) Every even ring in a stable partition can be broken into pairs of mutually acceptable agents preserving stability.

Throughout the paper we only consider stable partitions which do not contain even rings. By Remark 1 (iii) this does not imply a loss of generality.

Using the notion of a stable partition, Inarra *et al.* [11] introduce some specific matchings, called P -stable matchings, defined as follows:

Definition 1 Let P be a stable partition. A P -stable matching is a matching μ such that for each $A = \{a_1, \dots, a_k\} \in P$, $\mu(a_i) \in \{a_{i+1}, a_{i-1}\}$ for all $i \in \{1, \dots, k\}$ except for a unique j where $\mu(a_j) = a_j$ if A is odd.

⁷Hereafter we omit subscript modulo k .

⁸A ring is odd (even) if its cardinality is odd (even).

These matchings are interesting not only because jointly with stable partitions constitute a useful tool for the analysis of roommate problems, but also because they satisfy the property of “outer stability” in the sense that from any matching there is a sequence of blocking pairs to a P -stable matching. The following theorem, introduced by Inarra *et al.* [11], proves this.

Theorem 1 (Inarra et al [11]) *Let $(N, (\succ_x)_{x \in N})$ be a roommate problem. Then, for any matching μ , there exists a finite sequence of matchings $(\mu = \mu_0, \mu_1, \dots, \mu_m = \bar{\mu})$ such that for all $i \in \{1, \dots, m\}$, μ_i is obtained from μ_{i-1} by satisfying a blocking pair of μ_{i-1} and $\bar{\mu}$ is a P -stable matching for some stable partition P .*

Given the use made of the notions of a P -stable matching and a stable partition in deriving our results, it may be helpful to illustrate them with the following example.

Example 1 *Consider the following 6-agent roommate problem:*

$$\begin{array}{l}
 2 \succ_1 3 \succ_1 1 \succ_1 4 \succ_1 5 \succ_1 6 \\
 3 \succ_2 1 \succ_2 2 \succ_2 4 \succ_2 5 \succ_2 6 \\
 1 \succ_3 2 \succ_3 3 \succ_3 4 \succ_3 5 \succ_3 6 \\
 5 \succ_4 4 \succ_4 1 \succ_4 2 \succ_4 3 \succ_4 6 \\
 4 \succ_5 5 \succ_5 1 \succ_5 2 \succ_5 3 \succ_5 6 \\
 6 \succ_6 1 \succ_6 2 \succ_6 3 \succ_6 4 \succ_6 5
 \end{array}$$

It is easy to verify that $P = \{\{1, 2, 3\}, \{4, 5\}, \{6\}\}$ is a stable partition where $A_1 = \{1, 2, 3\}$ is an odd ring, $A_2 = \{4, 5\}$ is a pair of mutually acceptable agents and $A_3 = \{6\}$ is a singleton. This partition can be represented graphically as follows:

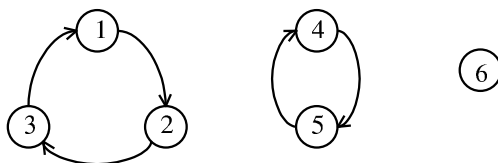


Figure 1.- A stable partition P .

The P -stable matchings associated with the stable partition P are: $\mu_1 = [\{1\}, \{2, 3\}, \{4, 5\}, \{6\}]$, $\mu_2 = [\{2\}, \{1, 3\}, \{4, 5\}, \{6\}]$ and $\mu_3 = [\{3\}, \{1, 2\}, \{4, 5\}, \{6\}]$.

3 Absorbing sets for the roommate problem

In this section we introduce the absorbing sets for the class of roommate problems with strict preferences. First, we find that absorbing sets are strongly related to stable partitions so that the notion of stable partition is converted into a useful tool for analyzing absorbing sets. To be specific, we show that each of these sets is determined by some stable partition. Second, by using this relation, we show that if a roommate problem is solvable then each absorbing set contains only one matching, and this matching is stable. Furthermore, the union of all of them coincides with the core. Thus, absorbing sets may be considered as a generalization of this solution concept in this framework.

An *abstract system* is a pair (X, R) where X is a finite set of alternatives and R is a binary relation on X . Two of the solution concepts put forward to solve an abstract system are the core and absorbing sets. In what follows, we associate a roommate problem with strict preferences with an abstract system and define these two solution concepts in this particular setting. Let \mathcal{M} denote the set of all matchings. Set $X = \mathcal{M}$ and define a binary relation R on \mathcal{M} as follows: Given two matchings $\mu, \mu' \in \mathcal{M}$, $\mu' R \mu$ if and only if μ' is obtained from μ by satisfying a blocking pair of μ . We say that μ' *directly dominates* μ if $\mu' R \mu$. Hereafter the system associated with the roommate problem $(N, (\succ_x)_{x \in N})$ is the pair (\mathcal{M}, R) . Let R^T denote the transitive closure of R . Then $\mu' R^T \mu$ if and only if there exists a finite sequence of matchings $\mu = \mu_0, \mu_1, \dots, \mu_m = \mu'$ such that, for all $i \in \{1, \dots, m\}$, $\mu_i R \mu_{i-1}$. We say that μ' *dominates* μ if $\mu' R^T \mu$.

As mentioned in the introduction, the conventional solution considered in matching problems is the *core*, which coincides with the set of stable matchings. In roommate problems, however, the core may be empty and *absorbing sets* stand out as a good candidate for an alternative solution concept. For these problems an absorbing set can be formally defined as follows:

Definition 2 *A nonempty subset \mathcal{A} of \mathcal{M} is an absorbing set of (\mathcal{M}, R) if the following conditions hold:*

- (i) *For any two distinct $\mu, \mu' \in \mathcal{A}$, $\mu' R^T \mu$.*
- (ii) *For any $\mu \in \mathcal{A}$ there is no $\mu' \notin \mathcal{A}$ such that $\mu' R \mu$.*

Condition (i) means that matchings of \mathcal{A} are symmetrically connected by the relation R^T . That is, every matching in an absorbing set is dominated

by any other matching in the same set. Condition (ii) means that the set \mathcal{A} is R -closed. That is, no matching in an absorbing set is directly dominated by a matching outside the set.

A nice property of this solution is that it always exists, although, in general, it may be not unique. Theorem 1 in Kalai and Schmeidler [15] states that if X is finite then the admissible set (the union of absorbing sets) is nonempty (see also Theorem 2.5 in Shenoy [26]). Thus either of these two results allows us to conclude that any (\mathcal{M}, R) has at least one absorbing set. Absorbing sets also satisfy the property of *outer stability*, which says that every matching not belonging to an absorbing set is dominated by a matching that does belong to an absorbing set.⁹

Let P be a stable partition. We denote by \mathcal{A}_P the set formed by all the P -stable matchings and those matchings that dominate them. The following remark follows directly from the definition of \mathcal{A}_P and the fact that P -stable matchings derived from the same stable partition form a cycle among them.¹⁰

Remark 2 *Let P be a stable partition and $\bar{\mu}$ be a P -stable matching. Then, $\mathcal{A}_P = \{\bar{\mu}\} \cup \{\mu \in \mathcal{M} : \mu R^T \bar{\mu}\}$.*

Next theorem establishes that stable partitions may be considered as structures generating the matchings in absorbing sets. It states that an absorbing set is one of these sets \mathcal{A}_P .

Theorem 2 *Let $(N, (\succ_x)_{x \in N})$ be a roommate problem. If \mathcal{A} is an absorbing set then $\mathcal{A} = \mathcal{A}_P$ for some stable partition P .*

The previous relation between absorbing sets and stable partitions is useful to prove that in solvable roommate problems each absorbing set consists of a unique matching, which is stable. This is shown by the following theorem.

Theorem 3 *If the roommate problem $(N, (\succ_x)_{x \in N})$ is solvable then \mathcal{A} is an absorbing set if and only if $\mathcal{A} = \{\mu\}$ for some stable matching μ .*

As a result, we have that the union of all absorbing sets coincides with the core.

To clarify the notion of absorbing sets we consider the following example, which is also used elsewhere in the paper to illustrate other results.

⁹This is shown in Kalai *et al.* [14].

¹⁰Example 1 illustrates this fact. See Lemma 1 in Appendix B.1 for its proof.

Example 2 Consider the following 10-agent roommate problem¹¹:

$2 \succ_1 3 \succ_1 4 \succ_1 5 \succ_1 6 \succ_1 7 \succ_1 8 \succ_1 9 \succ_1 1 \succ_1 10$
 $3 \succ_2 1 \succ_2 4 \succ_2 5 \succ_2 6 \succ_2 7 \succ_2 8 \succ_2 9 \succ_2 2 \succ_2 10$
 $1 \succ_3 2 \succ_3 4 \succ_3 5 \succ_3 6 \succ_3 7 \succ_3 8 \succ_3 9 \succ_3 3 \succ_3 10$
 $7 \succ_4 8 \succ_4 9 \succ_4 5 \succ_4 6 \succ_4 1 \succ_4 2 \succ_4 3 \succ_4 4 \succ_4 10$
 $8 \succ_5 9 \succ_5 7 \succ_5 4 \succ_5 6 \succ_5 5 \succ_5 1 \succ_5 2 \succ_5 3 \succ_5 10$
 $9 \succ_6 7 \succ_6 8 \succ_6 4 \succ_6 5 \succ_6 6 \succ_6 1 \succ_6 2 \succ_6 3 \succ_6 10$
 $5 \succ_7 6 \succ_7 1 \succ_7 4 \succ_7 9 \succ_7 8 \succ_7 7 \succ_7 2 \succ_7 3 \succ_7 10$
 $6 \succ_8 4 \succ_8 5 \succ_8 7 \succ_8 9 \succ_8 8 \succ_8 1 \succ_8 2 \succ_8 3 \succ_8 10$
 $4 \succ_9 5 \succ_9 6 \succ_9 7 \succ_9 8 \succ_9 9 \succ_9 1 \succ_9 2 \succ_9 3 \succ_9 10$
 $10 \succ_{10} 1 \succ_{10} \dots$

There are three stable partitions: $P_1 = \{\{1, 2, 3\}, \{4, 7\}, \{5, 8\}, \{6, 9\}, \{10\}\}$, $P_2 = \{\{1, 2, 3\}, \{4, 8\}, \{5, 9\}, \{6, 7\}, \{10\}\}$ and $P_3 = \{\{1, 2, 3\}, \{4, 9\}, \{5, 7\}, \{6, 8\}, \{10\}\}$. Consider the stable partition P_2 . The associated P_2 -stable matchings are: $\mu_1 = [\{1\}, \{2, 3\}, \{4, 8\}, \{5, 9\}, \{6, 7\}, \{10\}]$, $\mu_2 = [\{2\}, \{1, 3\}, \{4, 8\}, \{5, 9\}, \{6, 7\}, \{10\}]$ and $\mu_3 = [\{3\}, \{1, 2\}, \{4, 8\}, \{5, 9\}, \{6, 7\}, \{10\}]$ and the set $\mathcal{A}_{P_2} = \{\mu_1, \mu_2, \mu_3\}$. Notice that any of these matchings dominates any other but they are not directly dominated by any matching outside \mathcal{A}_{P_2} . Therefore \mathcal{A}_{P_2} is an absorbing set. In addition, matching $\mu_1 = [\{1\}, \{2, 3\}, \{4, 8\}, \{5, 9\}, \{6, 7\}, \{10\}]$ can be derived from the P_1 -stable matching $\mu = [\{1\}, \{2, 3\}, \{4, 7\}, \{5, 8\}, \{6, 9\}, \{10\}]$ by satisfying the following sequence of blocking pairs: $\{1, 7\}$, $\{4, 8\}$, $\{5, 9\}$, $\{6, 7\}$. Hence μ_1 belongs to \mathcal{A}_{P_1} . It is easy to verify, however, that μ does not dominate μ_1 . Thus \mathcal{A}_{P_1} is not an absorbing set since it does not satisfy Condition (i) of Definition 2.

4 Matchings in absorbing sets

In the previous section we have shown the existence of a link between absorbing sets and stable partitions. This link is straightforward when the roommate problem is solvable, since each stable partition induces an absorbing set¹². But this result is not maintained when the roommate problem is unsolvable. In this case from Theorem 2 we know that absorbing sets are determined by stable partitions but, as shown in Example 2, stable partitions with odd rings may not yield absorbing sets. These results suggest that we

¹¹In this example, absorbing sets select 6 matchings out of 9496.

¹²Tan [29] establishes the relation between stable matchings and stable partitions.

should investigate what the stable partitions determining the absorbing sets are. Thus, in this section, we start by characterizing the absorbing sets in terms of stable partitions.

For the characterization pursued we define two types of agents for each stable partition P (hence, the set \mathcal{A}_P is defined): “Dissatisfied” agents who move from one matching to another over the matchings in \mathcal{A}_P without finding a permanent partner, and “satisfied” ones, agents who lack any incentive to change their current partner over these matchings. As we shall see, satisfied agents play a crucial role in this characterization since the stable partitions determining the absorbing sets are those with the greatest number of them.

The investigation conducted proves to be useful in identifying matchings in absorbing sets. Notice that if P is the stable partition giving rise to the absorbing set \mathcal{A}_P then, by Theorem 2, this set is formed by the set of P -stable matchings and by the matchings that dominate them. The results of this section are illustrated using Example 2.

Our first theorem gives a characterization for absorbing sets in terms of stable partitions. To obtain it, some additional definitions are introduced. Given a stable partition P , let D_P denote the set of dissatisfied agents that block some matching in \mathcal{A}_P , and let $S_P = N \setminus D_P$ be the set of satisfied ones. In Appendix A we give an *iterative process* for calculating these two sets. From Remark 4 of this appendix we learn that for any set A of the stable partition P , either $A \subseteq D_P$ or $A \subseteq S_P$.

Let $P|_{S_P} = \{A \in P : A \subseteq S_P\}$ denote the stable partition P restricted to the set of satisfied agents S_P . Given that the elements in $P|_{S_P}$ are pairs and/or singletons matched in a stable manner, (see again the iterative process in Appendix A) it is immediate that $P|_{S_P}$ is also a stable partition for the roommate problem $(S_P, (\succ_x)_{x \in S_P})$. Thus, $P|_{S_P}$ may be interpreted as a “partial” matching for the roommate problem $(N, (\succ_x)_{x \in N})$.

We denote by $\mathcal{P} = \{P|_{S_P} : P \text{ is a stable partition}\}$ the set of all such partial matchings for a roommate problem $(N, (\succ_x)_{x \in N})$. We say that $P|_{S_P}$ is maximal in \mathcal{P} if there is not a stable partition P' such that $P|_{S_P} \subset P'|_{S_{P'}}$.

Theorem 4 *Let $(N, (\succ_x)_{x \in N})$ be a roommate problem. \mathcal{A} is an absorbing set if and only if $\mathcal{A} = \mathcal{A}_P$ for some stable partition P such that $P|_{S_P}$ is maximal in \mathcal{P} .*

Then, the number of absorbing sets in a roommate problem can be determined straightforwardly as the following corollary of Theorem 4 shows.¹³

Corollary 5 *Let $(N, (\succ_x)_{x \in N})$ be a roommate problem. The number of absorbing sets is equal to the number of distinct maximal partitions of \mathcal{P} .*

The following theorem specifies a property satisfied by some partial matchings for the roommate problem $(N, (\succ_x)_{x \in N})$. Specifically, it proves that any two stable partitions that determine two absorbing sets have the same set of satisfied agents.

Theorem 6 *Let $(N, (\succ_x)_{x \in N})$ be a roommate problem. If P and P' are two stable partitions such that $P \upharpoonright_{S_P}$ and $P' \upharpoonright_{S_{P'}}$ are maximal in \mathcal{P} then $S_P = S_{P'}$.*

The following remark, which follows immediately from Theorem 4 and Theorem 6, states that absorbing sets are determined by those stable partitions with the maximum number of satisfied agents.

Remark 3 *Let $(N, (\succ_x)_{x \in N})$ be a roommate problem. \mathcal{A} is an absorbing set if and only if $\mathcal{A} = \mathcal{A}_P$ for some stable partition P , such that $|S_P| \geq |S_{P'}|$ for every stable partition P' .*

Therefore, if P is the stable partition yielding the absorbing set \mathcal{A}_P , then by Theorem 2 we know that this set is formed by the P -stable matchings and those matchings that dominate them, and from this remark we also know that the set of satisfied agents of these matchings has cardinality greater than or equal to the corresponding set for the matchings in $\mathcal{A}_{P'}$.

To conclude this section, let us illustrate the above results with the roommate problem from Example 2.

Example 3 *(Example 2 continued)*

Applying the iterative process in Appendix A to the stable partitions of this problem gives the following information: For the stable partition P_1 , we have that the sets of dissatisfied and satisfied agents are $D_{P_1} = \{1, 2, \dots, 9\}$ and $S_{P_1} = \{10\}$ respectively. For the stable partitions P_2 and P_3 we have $D_{P_2} = D_{P_3} = \{1, 2, 3\}$ and $S_{P_2} = S_{P_3} = \{4, \dots, 10\}$. Hence, the partial matchings of \mathcal{P} are $P_1 \upharpoonright_{S_{P_1}} = \{\{10\}\}$, $P_2 \upharpoonright_{S_{P_2}} = \{\{4, 8\}, \{5, 9\}, \{6, 7\}, \{10\}\}$ and $P_3 \upharpoonright_{S_{P_3}} =$

¹³Note that two different stable partitions with the same maximal “partial” matchings provide the same absorbing set (See Lemma 6 in Appendix B.1.)

$\{\{4, 9\}, \{5, 7\}, \{6, 8\}, \{10\}\}$. Notice that $P_2 \upharpoonright_{S_{P_2}}$ and $P_3 \upharpoonright_{S_{P_3}}$ are the maximal partitions of \mathcal{P} with the greatest set of satisfied agents. Therefore, by Theorem 4 and Corollary 5, this roommate problem has exactly two absorbing sets \mathcal{A} and \mathcal{A}' where $\mathcal{A} = \mathcal{A}_{P_2}$ containing the P_2 -stable matchings, which are: $\mu_1 = [\{1\}, \{2, 3\}, \{4, 8\}, \{5, 9\}, \{6, 7\}, \{10\}]$, $\mu_2 = [\{2\}, \{1, 3\}, \{4, 8\}, \{5, 9\}, \{6, 7\}, \{10\}]$ and $\mu_3 = [\{3\}, \{1, 2\}, \{4, 8\}, \{5, 9\}, \{6, 7\}, \{10\}]$ and $\mathcal{A}' = \mathcal{A}_{P_3}$ containing the P_3 -stable matchings, which are: $\mu'_1 = [\{1\}, \{2, 3\}, \{4, 9\}, \{5, 7\}, \{6, 8\}, \{10\}]$, $\mu'_2 = [\{2\}, \{1, 3\}, \{4, 9\}, \{5, 7\}, \{6, 8\}, \{10\}]$ and $\mu'_3 = [\{3\}, \{1, 2\}, \{4, 9\}, \{5, 7\}, \{6, 8\}, \{10\}]$.

5 Structure of matchings in absorbing sets

In this section we investigate the structure of the matchings of absorbing sets. First, we show that all matchings in an absorbing set share certain common features. Furthermore, in the case of a roommate problem with multiple absorbing sets we also find similarities among their matchings.

Let \mathcal{A}_P be an absorbing set associated with the stable partition P and set $\mathcal{A}_P = \mathcal{A}$. Then, as in the previous section, the sets $D_{\mathcal{A}}$ and $S_{\mathcal{A}}$ will denote respectively the sets of dissatisfied and satisfied agents for the absorbing set \mathcal{A} .

The following theorem proves that all matchings in an absorbing set \mathcal{A} have some identical pairings formed by the satisfied agents which, in addition, are a stable matching for the roommate problem $(S_{\mathcal{A}}, (\succ_x)_{x \in S_{\mathcal{A}}})$.

Theorem 7 *Let $(N, (\succ_x)_{x \in N})$ be a roommate problem. For any absorbing set \mathcal{A} such that $S_{\mathcal{A}} \neq \emptyset$ the following conditions hold:*

- (i) *For any $\mu \in \mathcal{A}$, $\mu(S_{\mathcal{A}}) = S_{\mathcal{A}}$ and $\mu \upharpoonright_{S_{\mathcal{A}}}$ is stable for $(S_{\mathcal{A}}, (\succ_x)_{x \in S_{\mathcal{A}}})$.*
- (ii) *For any $\mu, \mu' \in \mathcal{A}$, $\mu \upharpoonright_{S_{\mathcal{A}}} = \mu' \upharpoonright_{S_{\mathcal{A}}}$.*

For an illustration of the result above see Example 3 at the end of Section 4.

Next, we investigate the structure of absorbing sets in case of multiplicity. For this purpose, some additional definitions are required. Given an absorbing set \mathcal{A} such that $D_{\mathcal{A}} \neq \emptyset$, let $\mathcal{A} \upharpoonright_{D_{\mathcal{A}}} = \{\mu \upharpoonright_{D_{\mathcal{A}}} : \mu \in \mathcal{A}\}$ denote the set of “partial” matchings of the absorbing set \mathcal{A} restricted to the set of dissatisfied agents $D_{\mathcal{A}}$. Analogously, if $S_{\mathcal{A}} \neq \emptyset$, let $\mathcal{A} \upharpoonright_{S_{\mathcal{A}}} = \{\mu \upharpoonright_{S_{\mathcal{A}}} : \mu \in \mathcal{A}\}$. The following theorem shows that there are similarities among matchings belonging to different absorbing sets.

Theorem 8 Let $(N, (\succ_x)_{x \in N})$ be a roommate problem. For any two absorbing sets \mathcal{A} and \mathcal{A}' , the following conditions hold:

- (i) $D_{\mathcal{A}} = D_{\mathcal{A}'}$ and $S_{\mathcal{A}} = S_{\mathcal{A}'}$.
- (ii) $\mathcal{A}|_{D_{\mathcal{A}}} = \mathcal{A}'|_{D_{\mathcal{A}'}}$.
- (iii) $\mathcal{A}|_{S_{\mathcal{A}}}$ and $\mathcal{A}'|_{S_{\mathcal{A}'}}$ are singletons consisting of a stable matching in $(S, (\succ_x)_{x \in S})$, where $S = S_{\mathcal{A}} = S_{\mathcal{A}'}$.

Thus, for a roommate problem $(N, (\succ_x)_{x \in N})$, all its absorbing sets have the following coincidences: (i) The set of dissatisfied agents is the same for all matchings across all absorbing sets and so is the set of satisfied agents. (ii) The roommate problem of the dissatisfied agents $(D, (\succ_x)_{x \in D})$ has a unique absorbing set. (iii) Satisfied agents form stable matchings for the roommate problem $(S, (\succ_x)_{x \in S})$. Hence, the two absorbing sets \mathcal{A} and \mathcal{A}' only differ in how the satisfied agents are matched.

The three conditions above provide all absorbing sets of a roommate problem with strict preferences with a similar structure, as illustrated in the following figure:

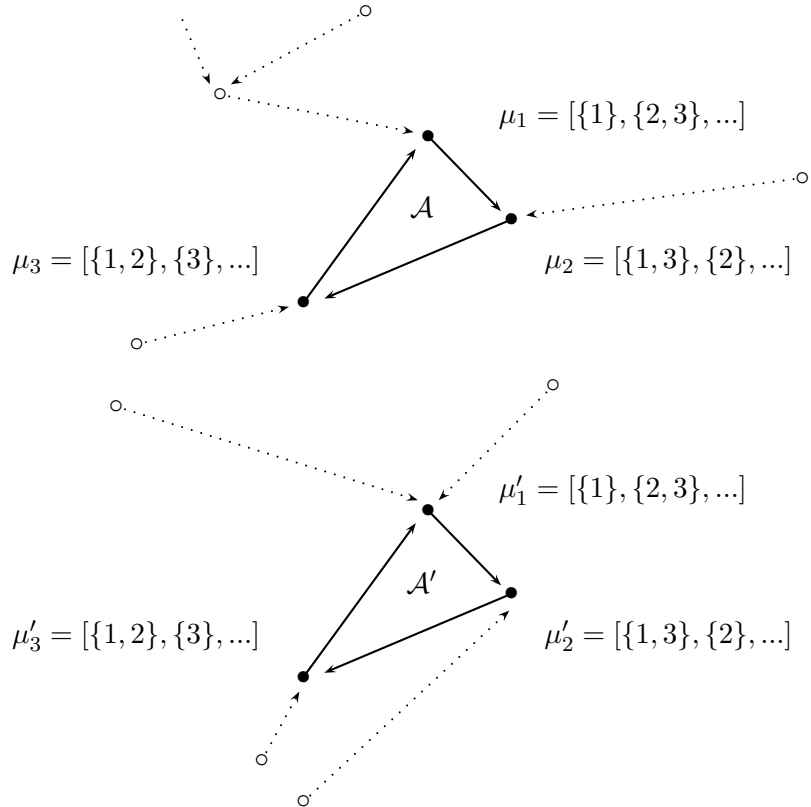


Figure 2.- The two absorbing sets of the roommate problem in Example 3.

Example 4 (*Example 2 continued*)

To explain this last result, consider the two absorbing sets $\mathcal{A} = \mathcal{A}_{P_2}$ and $\mathcal{A}' = \mathcal{A}_{P_3}$. Since $D_{\mathcal{A}} = D_{\mathcal{A}'} = \{1, 2, 3\}$ we have $\mathcal{A} = \{\mu_1, \mu_2, \mu_3\}$ and $\mathcal{A}' = \{\mu'_1, \mu'_2, \mu'_3\}$ where μ_1, μ_2, μ_3 are the P_2 -stable matchings and μ'_1, μ'_2, μ'_3 are the P_3 -stable matchings (see Figure 2). Additionally, $\mathcal{A} |_{D_{\mathcal{A}}} = \{\mu_1 |_{D_{\mathcal{A}}}, \mu_2 |_{D_{\mathcal{A}}}, \mu_3 |_{D_{\mathcal{A}}}\}$ and $\mathcal{A}' |_{D_{\mathcal{A}'}} = \{\mu'_1 |_{D_{\mathcal{A}'}} , \mu'_2 |_{D_{\mathcal{A}'}} , \mu'_3 |_{D_{\mathcal{A}'}}\}$ where $\mu_1 |_{D_{\mathcal{A}}} = \mu'_1 |_{D_{\mathcal{A}'}} = [\{1\}, \{2, 3\}]$, $\mu_2 |_{D_{\mathcal{A}}} = \mu'_2 |_{D_{\mathcal{A}'}} = [\{1, 3\}, \{2\}]$ and $\mu_3 |_{D_{\mathcal{A}}} = \mu'_3 |_{D_{\mathcal{A}'}} = [\{1, 2\}, \{3\}]$. Furthermore, $\mathcal{A} |_{S_{\mathcal{A}}}$ and $\mathcal{A}' |_{S_{\mathcal{A}'}}$ are respectively singletons consisting of the stable matchings $\mu = [\{4, 8\}, \{5, 9\}, \{6, 7\}, \{10\}]$ and $\mu' = [\{4, 9\}, \{5, 7\}, \{6, 8\}, \{10\}]$ in $(S, (\succ_x)_{x \in S})$ where $S = \{4, 5, 6, 7, 8, 9, 10\}$.

To conclude, some of the above results reveal an interesting property of stability verified by *all* matchings of absorbing sets. First, let us introduce an additional definition.

Definition 3 *Given a matching μ we say that a pair of agents $\{x, y\}$ matched under μ is strongly stable if, for any matching $\mu' \in \mathcal{M}$ such that $\mu' R^T \mu$, agents x and y are also matched under μ' .*

From Theorem 7, we know that for every matching μ in an absorbing set \mathcal{A} those pairs matched under $\mu |_{S_{\mathcal{A}}}$ are strongly stable. Furthermore, from Theorem 8 (iii) is easy to verify that all matchings in all absorbing sets have the same number of strongly stable pairs. These two results jointly with the outer stability property guarantee that all matchings in all absorbing sets have the greatest number of strongly stable pairs among the matchings in \mathcal{M} .

6 Conclusions and extensions

Let us emphasize that starting from an arbitrary matching, the process of allowing randomly chosen blocking pairs to match will converge to an absorbing set with probability one. This enhances the interest of applying this solution concept to matching markets, since it allows to predict which matchings may be reached when there is not a centralized matching procedure.

A potential extension of the results in this paper may be the application of our approach to more general choice problems such as hedonic games¹⁴ (See Dreze and Greenberg [8]), or network formation models (see, for instance, Jackson and Wolinsky [13]). An arbitrary hedonic game can be associated with an abstract system where the set of alternatives is the set of all coalitional partitions that can be formed by the agents involved in the problem. Analogously, for network formation models, Page *et al.* [18] define abstract systems associated with these problems where the set of alternatives is formed by a set of networks. In these two specific systems the binary relation represents transitions from one alternative to another and, as in our chapter, absorbing sets could be proposed as a solution for them whenever their corresponding cores are empty.

¹⁴Diamantoudi and Xue [6] and Barber and Gerber [3] have pointed out that roommate problems can be considered as a special case of hedonic games.

Appendix A

An iterative process to determine the sets of dissatisfied and satisfied agents

Given a stable partition P (hence the set \mathcal{A}_P is immediately defined) the following process determines the set D_P of dissatisfied agents, those agents that block some matching in \mathcal{A}_P , and the set S_P of satisfied agents, those agents that do not block any matching in \mathcal{A}_P .

The set D_P can be determined by an iterative process in a finite number of steps. To that end, we define inductively a sequence of sets $\langle D_t \rangle_{t=0}^\infty$ as follows:

- (i) for $t = 0$, D_0 is the union of all odd rings of P .
- (ii) for $t \geq 1$, $D_t = D_{t-1} \cup B_t$ where $B_t = \{b_1(t), \dots, b_{l_t}(t)\} \in P$ ($l_t = 1$ or 2), $B_t \not\subseteq D_{t-1}$, and there is a set $A_t = \{a_1(t), \dots, a_{k_t}(t)\} \in P$ such that $A_t \subseteq D_{t-1}$ and

$$b_j(t) \succ_{a_i(t)} a_i(t) \text{ and } a_i(t) \succ_{b_j(t)} b_{j-1}(t), \quad (2)$$

for some $i \in \{1, \dots, k_t\}$ and $j \in \{1, \dots, l_t\}$.¹⁵

Given that P contains a finite number of sets, then $D_t = D_{t-1}$ for some t . Let r be the minimum number such that $D_{r+1} = D_r$. Then, $D_r = D_P$.¹⁶

From this iterative process the following remark easily follows.

Remark 4 For any set $A \in P$, either $A \subseteq D_P$ or $A \subseteq S_P$.

Example 5 (Example 2 continued)

To illustrate the iterative process above, consider the stable partition $P_1 = \{\{1, 2, 3\}, \{4, 7\}, \{5, 8\}, \{6, 9\}, \{10\}\}$. Note that P_1 contains a unique odd ring. Then $D_0 = \{1, 2, 3\}$. Let $B_1 = \{4, 7\}$ and $A_1 = \{1, 2, 3\}$. Since $7 \succ_1 1$ and $1 \succ_7 4$, then $D_1 = D_0 \cup B_1 = \{1, 2, 3, 4, 7\}$. Consider now the sets $B_2 = \{5, 8\}$ and $A_2 = \{4, 7\}$. As $8 \succ_4 4$ and $4 \succ_8 5$, then $D_2 = D_1 \cup B_2 = \{1, 2, 3, 4, 7, 5, 8\}$. Finally, let $B_3 = \{6, 9\}$ and $A_3 = \{5, 8\}$. Since $9 \succ_5 5$ and $5 \succ_9 6$, then $D_3 = D_2 \cup B_3 = \{1, 2, 3, 4, 7, 5, 8, 6, 9\}$ and the

¹⁵If no such set exists then $D_t = D_{t-1}$.

¹⁶This is proven by Lemma 2 in Appendix B.1.

process is completed. Hence $D_{P_1} = D_3$. Repeating the process for $P_2 = \{\{1, 2, 3\}, \{4, 8\}, \{5, 9\}, \{6, 7\}, \{10\}\}$ and $P_3 = \{\{1, 2, 3\}, \{4, 9\}, \{5, 7\}, \{6, 8\}, \{10\}\}$ we have $D_{P_2} = D_{P_3} = \{1, 2, 3\}$. Therefore the sets of satisfied agents are $S_{P_1} = \{10\}$ and $S_{P_2} = S_{P_3} = \{4, 5, 6, 7, 8, 9, 10\}$

Appendix B

B.1 Lemmas

Lemma 1 *Let P be a stable partition. For any two distinct P -stable matchings μ and μ' , $\mu' R^T \mu$.*

Proof. If P does not contain any odd rings then there exists a unique P -stable matching and we are done. Suppose that P contains some odd ring.

Let A_1, \dots, A_r be the odd rings of P and $T = \bigcup_{i=1}^r A_i$.

Set $A_1 = \{a_1, \dots, a_k\}$. As A_1 is a ring then

$$a_{i+1} \succ_{a_i} a_{i-1} \succ_{a_i} a_i, \quad (3)$$

for all $i = \{1, \dots, k\}$. By Definition 1, since μ and μ' are P -stable matchings, there are two agents $a_l, a_s \in A_1$ such that $\mu(a_l) = a_l$ and $\mu'(a_s) = a_s$. If $a_l = a_s$, then we set a different ring. If $a_l \neq a_s$, then, since $\mu(a_l) = a_l$ and $\mu(a_{l-1}) = a_{l-2}$, by condition (3), $\{a_l, a_{l-1}\}$ blocks μ , inducing a P -stable matching μ_1 for which $\mu(a_{l-2}) = a_{l-2}$. Repeating the process, we obtain a sequence of P -stable matchings $\mu_0, \mu_1, \dots, \mu_i, \dots$ as follows:

(i) $\mu_0 = \mu$.

(ii) For $i \geq 1$, μ_i is the P -stable matching obtained from μ_{i-1} by satisfying the blocking pair $\{a_{l-2(i-1)}, a_{l-2(i-1)-1}\}$.

Let $m_1 \in \{1, \dots, k\}$ such that $a_{l-2m_1} = a_s$. Then $\mu = \mu_0, \mu_1, \dots, \mu_{m_1}$ is a finite sequence of P -stable matchings such that, for all $i \in \{1, \dots, m_1\}$, $\mu_i R \mu_{i-1}$ and $\mu_{m_1} |_{A_1} = \mu' |_{A_1}$.

Consider now the ring A_2 . Reasoning in the same way as before, for μ_{m_1} and μ' we obtain a finite sequence of P -stable matchings $\mu_{m_1}, \mu_{m_1+1}, \dots, \mu_{m_1+m_2}$ such that, for all $i \in \{m_1+1, \dots, m_1+m_2\}$, $\mu_i R \mu_{i-1}$ and $\mu_{m_1+m_2} |_{(A_1 \cup A_2)} = \mu' |_{(A_1 \cup A_2)}$.

Repeating the same procedure for the remaining odd rings, eventually we obtain a finite sequence of P -stable matchings $\mu = \mu_0, \mu_1, \dots, \mu_m$, where

$m = \sum_{i=1}^r m_i$, and such that, for all $i \in \{1, \dots, m\}$, $\mu_i R \mu_{i-1}$ and $\mu_m |_T = \mu' |_T$.

Now, since $\mu_m |_{(N \setminus T)} = \mu' |_{(N \setminus T)}$, then $\mu_m = \mu'$ and the proof is complete.

■

Lemma 2 $D_r = D_P$

Proof. (\subseteq): First we prove that $D_0 \subseteq D_P$. Let $A = \{a_1, \dots, a_k\}$ be an odd ring of P . We must show that $a_i \in D_P$ for all $i \in \{1, \dots, k\}$. Consider the P -stable matching μ such that $\mu(a_i) = a_i$. As $\mu(a_{i-1}) = a_{i-2}$ and $a_i \succ_{a_{i-1}} a_{i-2}$ and $a_{i-1} \succ_{a_i} a_i$ then $\{a_i, a_{i-1}\}$ is a blocking pair of μ and therefore $a_i \in D_P$. Now we prove that, for each $t \in \{1, \dots, r\}$, the following conditions hold:

- a) $B_t \subseteq D_P$.
- b) There exists a matching $\mu_t \in \mathcal{A}_P$ such that

$$\mu_t(x) = \begin{cases} x & \text{if } x \in B_t \\ \bar{\mu}_t(x) & \text{otherwise,} \end{cases}$$

where $\bar{\mu}_t$ is a P -stable matching.

We argue by induction on t .

If $t = 1$, we have $A_1 = \{a_1(1), \dots, a_{k_1}(1)\}$ and $B_1 = \{b_1(1), \dots, b_{l_1}(1)\}$.¹⁷ Since $A_1 \subseteq D_0$ then A_1 is an odd ring of P . Consider the P -stable matching μ such that $\mu(a_i) = a_i$. Since $\mu(b_j) = b_{j-1}$, by (2)¹⁸, we have $b_j \succ_{a_i} \mu(a_i)$ and $a_i \succ_{b_j} \mu(b_j)$. Hence $\{a_i, b_j\}$ is a blocking pair of μ and therefore $b_j \in D_P$. Let μ' be the matching obtained from μ by satisfying this blocking pair. Now, since $a_i \succ_{b_j} b_{j-1}$, by the stability of P , $a_{i-1} \succ_{a_i} b_j$. As $\mu'(a_{i-1}) = a_{i-2}$ and $a_i \succ_{a_{i-1}} a_{i-2}$, then $\{a_i, a_{i-1}\}$ is a blocking pair of μ' which induces a matching $\tilde{\mu} \in \mathcal{A}_P$ such that $\tilde{\mu}(x) = x$ if $x \in B_1$ and $\tilde{\mu}(x) = \bar{\mu}(x)$ otherwise, where $\bar{\mu}$ is the P -stable matching such that $\bar{\mu}(a_{i-2}) = a_{i-2}$. Let $\mu_1 = \tilde{\mu}$ and $\bar{\mu}_1 = \bar{\mu}$. Then, if $l = 1$ we are done. If $l = 2$, to complete the proof we need to show that $b_{j-1} \in D_P$. But this is trivial because as agents b_j and b_{j-1} are alone under μ_1 , $\{b_j, b_{j-1}\}$ is a blocking pair of μ_1 and therefore $b_{j-1} \in D_P$. Now assume that $t \geq 2$. We consider two cases:

Case 1. A_t is an odd ring. Reasoning in the same way as before for the sets A_t and B_t , the result follows.

Case 2. A_t is not an odd ring. Then $A_t = B_s$ for some $s < t$. By the inductive hypothesis, there exists $\mu_s \in \mathcal{A}_P$ such that $\mu_s(x) = x$ if $x \in B_s$ and $\mu_s(x) = \bar{\mu}_s(x)$ otherwise, where $\bar{\mu}_s$ is a P -stable matching. As $\mu_s(a_i) = a_i$ and $\mu_s(b_j) = \bar{\mu}_s(b_j) = b_{j-1}$, by (2), we have $b_j \succ_{a_i} \mu_s(a_i)$ and $a_i \succ_{b_j} \mu_s(b_j)$. Hence $\{a_i, b_j\}$ is a blocking pair of μ_s and therefore $b_j \in D_P$. Let μ'_s be the matching obtained from μ_s by satisfying this blocking pair. Since $a_i \succ_{b_j} b_{j-1}$, by the stability of P , $a_{i-1} \succ_{a_i} b_j$ and as $\mu'_s(a_{i-1}) = a_{i-1}$ then $\{a_i, a_{i-1}\}$ is a blocking pair of μ'_s , which induces a matching $\tilde{\mu}_s \in \mathcal{A}_P$

¹⁷Abusing notation, we write a_i and b_j instead of $a_i(t)$ and $b_j(t)$ for all t .

¹⁸See condition (2) in the Appendix A.

such that $\tilde{\mu}_s(x) = x$ if $x \in B_t$ and $\tilde{\mu}_s(x) = \bar{\mu}_s(x)$ otherwise. Then, choosing $\mu_t = \tilde{\mu}_s$ and $\bar{\mu}_t = \bar{\mu}_s$ and reasoning in the same way as before, the result follows. Finally, as $D_0 \subseteq D_P$ and, for each $t \in \{1, \dots, r\}$, $B_t \subseteq D_P$ we conclude that $D_r \subseteq D_P$.

(\supseteq): We prove that D_r contains all the blocking pairs of the matchings in \mathcal{A}_P . Now, by Remark 2, $\mathcal{A}_P = \{\bar{\mu}\} \cup \{\mu \in \mathcal{M} : \mu R^T \bar{\mu}\}$ where $\bar{\mu}$ is a P -stable matching. Hence it suffices to show that, for any finite sequence of matchings $\bar{\mu} = \mu_0, \mu_1, \dots, \mu_m$ such that, for all $s \in \{1, \dots, m\}$, μ_s is obtained from μ_{s-1} by satisfying the blocking pair $\{x_s, y_s\}$, therefore $\{x_s, y_s\} \subseteq D_r$. We argue by induction on s .

If $s = 1$, then $\{x_1, y_1\}$ is a blocking pair of $\bar{\mu}$. Let $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_l\}$ be the sets of P such that $x_1 \in A$ and $y_1 \in B$. Then $x_1 = a_i$ and $y_1 = b_j$ for some i and j . As $\{x_1, y_1\}$ blocks $\bar{\mu}$ we have $y_1 \succ_{x_1} \bar{\mu}(x_1)$ and $x_1 \succ_{y_1} \bar{\mu}(y_1)$, i.e., $b_j \succ_{a_i} \bar{\mu}(a_i)$ and $a_i \succ_{b_j} \bar{\mu}(b_j)$. Suppose, by contradiction, that $\{a_i, b_j\} \not\subseteq D_r$. If $\{a_i, b_j\} \cap D_r = \emptyset$ then A and B are not odd rings hence, by Definition 1, we have $\bar{\mu}(a_i) = a_{i-1}$ and $\bar{\mu}(b_j) = b_{j-1}$. But then $b_j \succ_{a_i} a_{i-1}$ and $a_i \succ_{b_j} b_{j-1}$, contradicting the stability of P . If $a_i \in D_r$ and $b_j \notin D_r$ since $\bar{\mu}(b_j) = b_{j-1}$ we have $b_j \succ_{a_i} a_i$ and $a_i \succ_{b_j} b_{j-1}$. Hence, by (2), $b_j \in D_r$, and we reach a contradiction. If we assume that $a_i \notin D_r$ and $b_j \in D_r$, a similar contradiction is reached.

Suppose now that $s \geq 2$. Then $\{x_s, y_s\}$ blocks μ_{s-1} . Consider the sets $A' = \{a'_1, \dots, a'_{k'}\}$ and $B' = \{b'_1, \dots, b'_{l'}\}$ of P such that $x_s = a'_i$ and $y_s = b'_j$ for some i and j . First we prove that if $x_s \notin D_r$ then $\mu_{s-1}(x_s) = \bar{\mu}(x_s)$. We argue by contradiction. If $\mu_{s-1}(x_s) \neq \bar{\mu}(x_s)$ we have $\{x_s, \bar{\mu}(x_s)\} \cap \{x_i, y_i\} \neq \emptyset$. ■

Lemma 3 *Let P be a stable partition. Then, there exists $\mu^* \in \mathcal{A}_P$ such that*

$$\mu^*(x) = \begin{cases} x & \text{if } x \in D_P \setminus D_0 \\ \bar{\mu}(x) & \text{otherwise,} \end{cases}$$

where $\bar{\mu}$ is a P -stable matching.

Proof. By Lemma 2 we have $D_r = D_P$. We argue by induction on r .

If $r = 0$, consider $\mu^* = \bar{\mu}$, where $\bar{\mu}$ is any P -stable matching.

For $r \geq 1$, by Lemma 2 (see its proof), there exists $\mu_r \in \mathcal{A}_P$ such that $\mu_r(x) = x$ if $x \in B_r$ and $\mu_r(x) = \bar{\mu}_r(x)$ otherwise, where $\bar{\mu}_r$ is a P -stable matching. Let $N' = N \setminus B_r$. Then $P' = P \setminus \{B_r\}$ is a stable partition of N' for which $D_{P'} = D_{r-1}$. Therefore, by the inductive hypothesis, there exists $\mu' \in \mathcal{A}_{P'}$ such that $\mu'(x) = x$ if $x \in D_{P'} \setminus D_0$ and $\mu'(x) = \bar{\mu}'(x)$ otherwise, where $\bar{\mu}'$ is a P' -stable matching. Let μ^* and $\bar{\mu}$ be such that $\mu^*|_{N'} = \mu'$, $\mu^*|_{B_r} = \mu_r|_{B_r}$, $\bar{\mu}|_{N'} = \bar{\mu}'$ and $\bar{\mu}|_{B_r} = \bar{\mu}_r|_{B_r}$. Clearly, $\bar{\mu}$ is a P -stable

matching. Now, we show that $\mu^* \in \mathcal{A}_P$. If $\mu^* = \mu_r$ since $\mu_r \in \mathcal{A}_P$ we are done. Otherwise, as $\mu^*|_{N'} \in \mathcal{A}_{P'}$ and $\mu_r|_{N'}$ is a P' -stable matching we have $\mu^*|_{N'} R^T \mu_r|_{N'}$. Hence $\mu^* R^T \mu_r$ and since $\mu_r \in \mathcal{A}_P$ then $\mu^* \in \mathcal{A}_P$. Thus μ^* satisfies the assertion in this lemma. ■

Lemma 4 *Let P be a stable partition such that $S_P \neq \emptyset$. The following conditions hold:*

- (i) *For any $\mu \in \mathcal{A}_P$, $\mu(S_P) = S_P$ and $\mu|_{S_P}$ is stable for $(S_P, (\succ_x)_{x \in S_P})$.*
- (ii) *For any $\mu, \mu' \in \mathcal{A}_P$, $\mu|_{S_P} = \mu'|_{S_P}$.*

Proof. By Remark 2, $\mathcal{A}_P = \{\bar{\mu}\} \cup \{\mu \in \mathcal{M} : \mu R^T \bar{\mu}\}$ where $\bar{\mu}$ is a P -stable matching.

(i) Let $\mu \in \mathcal{A}_P$. We prove that, for each $x \in S_P$, $\mu(x) \in S_P$. Let $x \in S_P$ and $A \in P$ such that $x \in A$. Then $A \subseteq S_P$. If $\mu = \bar{\mu}$, as $\bar{\mu}$ is a P -stable matching, by Definition 1, $\mu(x) \in A$ and since $A \subseteq S_P$ we have $\mu(x) \in S_P$. If $\mu \neq \bar{\mu}$ then $\mu R^T \bar{\mu}$ and since $\{x, \bar{\mu}(x)\} \subseteq S_P$ it follows that $\mu(x) = \bar{\mu}(x)$ and therefore $\mu(x) \in S_P$. Clearly $\mu|_{S_P}$ is stable.

(ii) Since $\mu|_{S_P} = \bar{\mu}|_{S_P}$ for all $\mu \in \mathcal{A}_P$, the result follows directly. ■

Lemma 5 *Let P and P' be two distinct stable partitions and let μ and μ' be a P -stable matching and a P' -stable matching respectively. Then, $\mu' R^T \mu$ if and only if $P|_{S_P} \subseteq P'|_{S_{P'}}$.*

Proof. (\implies): This is trivial if $P|_{S_P} = \emptyset$. Suppose that $P|_{S_P} \neq \emptyset$. Let $A \in P$ such that $A \subseteq S_P$. We must prove that $A \in P'$ and $A \subseteq S_{P'}$. As $\mu' R^T \mu$ then $\mu' \in \mathcal{A}_P$ hence $\mathcal{A}_{P'} \subseteq \mathcal{A}_P$. Therefore $S_P \subseteq S_{P'}$. Now, by Lemma 4, we have $\mu'|_{S_P} = \mu|_{S_P}$ and since $\mu(A) = A$, it follows that $\mu'(A) = A$. Hence $A \in P'$. Moreover, as $A \subseteq S_P$ and $S_P \subseteq S_{P'}$ then $A \subseteq S_{P'}$.

(\impliedby): By Lemma 3, there exists $\mu^* \in \mathcal{A}_P$ such that $\mu^*(x) = x$ if $x \in D_P \setminus D_0$ and $\mu^*(x) = \bar{\mu}(x)$ otherwise, where $\bar{\mu}$ is a P -stable matching. First we prove that there exists a P' -stable matching $\tilde{\mu}$ such that $\tilde{\mu} R^T \mu^*$. Consider the P' -stable matching $\tilde{\mu}$ such that $\tilde{\mu}|_{D_0} = \bar{\mu}|_{D_0}$. As $\mu^*(x) = \bar{\mu}(x)$ for all $x \in D_0$ then $\tilde{\mu}|_{D_0} = \mu^*|_{D_0}$. Furthermore, if $S_P \neq \emptyset$ since $P|_{S_P} \subseteq P'|_{S_{P'}}$ we have $\tilde{\mu}|_{S_P} = \bar{\mu}|_{S_P}$ and as $\mu^*(x) = \bar{\mu}(x)$ for all $x \in S_P$, it follows that $\tilde{\mu}|_{S_P} = \mu^*|_{S_P}$. Then, for each $x \in D_P \setminus D_0$, we have $\tilde{\mu}(x) \in D_P \setminus D_0$ (otherwise, $\tilde{\mu}(x) = \mu^*(x) = x$ hence $x \notin D_P \setminus D_0$). Let $(D_P \setminus D_0)' = \{x \in D_P \setminus D_0 : \tilde{\mu}(x) \neq x\}$. First of all, note that $(D_P \setminus D_0)' \neq \emptyset$ (if $(D_P \setminus D_0)' = \emptyset$ then $\mu^* = \tilde{\mu} = \bar{\mu}$ and therefore $P = P'$). Now we can write $(D_P \setminus D_0)' = \cup_{i=1}^s \{x_i, y_i\}$ where $y_i = \tilde{\mu}(x_i)$. Since agents x_i and y_i are alone

under μ^* we can consider the finite sequence of matchings $\mu^* = \mu_0, \mu_1, \dots, \mu_s$ where, for all $i \in \{1, \dots, s\}$, μ_i is obtained from μ_{i-1} by satisfying the blocking pair $\{x_i, y_i\}$. Then we have $\mu_s = \tilde{\mu}$. Therefore $\tilde{\mu}R^T\mu^*$ and since $\mu^*R^T\bar{\mu}$ we conclude that $\tilde{\mu}R^T\bar{\mu}$. Finally, the result follows directly by Lemma 1. ■

Lemma 6 *Let P and P' be two stable partitions. $\mathcal{A}_P = \mathcal{A}_{P'}$ if and only if $P \downarrow_{S_P} = P' \downarrow_{S_{P'}}$.*

Proof. Suppose that $\mathcal{A}_P = \mathcal{A}_{P'}$. Let $\bar{\mu}$ and $\tilde{\mu}$ be a P -stable matching and a P' -stable matching respectively. By Remark 2, we have $\mathcal{A}_P = \{\bar{\mu}\} \cup \{\mu \in \mathcal{M} : \mu R^T \bar{\mu}\}$ and $\mathcal{A}_{P'} = \{\tilde{\mu}\} \cup \{\mu \in \mathcal{M} : \mu R^T \tilde{\mu}\}$. As $\mathcal{A}_P = \mathcal{A}_{P'}$ then $\tilde{\mu} \in \mathcal{A}_P$ and $\bar{\mu} \in \mathcal{A}_{P'}$. If $\tilde{\mu} = \bar{\mu}$ then $P = P'$ and we are done. If $\tilde{\mu} \neq \bar{\mu}$ we have $\tilde{\mu}R^T\bar{\mu}$ and $\bar{\mu}R^T\tilde{\mu}$. Hence, by Lemma 5, $P \downarrow_{S_P} = P' \downarrow_{S_{P'}}$.

The converse is analogous. ■

Lemma 7 *Let P and P' be two stable partitions. Then for each $A \in P$ either $A \subseteq D_{P'}$ or $A \subseteq S_{P'}$.*

Proof. Let $A \in P$. If A is an odd ring then $A \subseteq D_{P'}$. If A is a singleton the result is trivial. Assume, therefore, that A is a pair of mutually acceptable agents. Let $A = \{x, y\}$. Assume, by contradiction, and without loss of generality, that $x \in S_{P'}$ and $y \in D_{P'}$. By Lemma 3, we know that there exists a matching $\mu' \in \mathcal{A}_{P'}$ such that $\mu'(x) = x$ if $x \in D_{P'} \setminus D_0$ and $\mu'(x) = \tilde{\mu}(x)$ otherwise, where $\tilde{\mu}$ is a P' -stable matching. To reach a contradiction we prove that $\{x, y\}$ blocks μ' by using a proposal-rejection procedure intuitively described as follows. Let $y_0 = y$. Let x_1 denote the predecessor of y_0 in P ¹⁹ and $y_1 = \mu'(x_1)$. As agent y_0 prefers x_1 to being alone, y_0 proposes x_1 . If x_1 accepts the proposal (that is, x_1 prefers y_0 to his partner under μ') the pair $\{x_1, y_0\}$ blocks μ' and the procedure concludes. Otherwise, let x_2 be the predecessor of y_1 in P and $y_2 = \mu'(x_2)$. Since agent x_1 prefers y_1 to y_0 , then, by the stability of P , agent y_1 prefers x_2 to x_1 . So y_1 becomes a new proposer in the procedure and offers x_2 the possibility of forming a new pair. Then if x_2 accepts the proposal, the pair $\{x_2, y_1\}$ blocks μ' and the procedure concludes. Otherwise, it may continue iteratively in this manner.

Formally, we define inductively a sequence of pairs $\langle \{x_n, y_n\} \rangle_{n=0}^\infty$, that are matched under μ' as follows:

¹⁹Given $x \in N$, we say that y is the predecessor of x in P if y is the immediate predecessor of x in A , where $A \in P$ such that $x \in A$.

(i) $x_0 = \mu'(y)$ and $y_0 = y$.

(ii) For $n \geq 1$, x_n is the predecessor of y_{n-1} in P and $y_n = \mu'(x_n)$.

Given that N is finite there exists $r \in \mathbb{N}$ such that $y_n \succ_{x_n} y_{n-1}$ for all $n = 1, \dots, r-1$ and $y_{r-1} \succ_{x_r} y_r$. Thus the procedure generates the blocking pair $\{x_r, y_{r-1}\}$ of μ' and therefore agents x_r and y_{r-1} are in $D_{P'}$. We now show that $r = 1$. If, on the contrary, $r \geq 2$ since $y_{r-1} \in D_{P'} \setminus D_0$ then agent y_{r-1} is single under μ' . Hence $x_{r-1} = y_{r-1}$. But then $y_{r-2} \succ_{x_{r-1}} y_{r-1}$, contradicts the choice of r (x_{r-1} would accept the proposal of y_{r-2}). So, $r = 1$ and since $x_1 = x$ and $y_0 = y$ we have $\{x, y\}$ blocks μ' . Hence $x \in D_{P'}$ and we have reached a contradiction. ■

B.2 Main results and their proofs

Proof of Theorem 2. First, we prove that there exists a P -stable matching $\bar{\mu}$ such that $\bar{\mu} \in \mathcal{A}$. Let μ be an arbitrary matching of \mathcal{A} . If μ is a P -stable matching for some stable partition P then $\bar{\mu} = \mu$ and we are done. Otherwise, by Theorem 1, there exists a P -stable matching $\bar{\mu}$ such that $\bar{\mu} R^T \mu$ and by Condition (ii) of Definition 2 we have $\bar{\mu} \in \mathcal{A}$.

Now, we prove that $\mathcal{A} = \mathcal{A}_P$. By Remark 2, we have $\mathcal{A}_P = \{\bar{\mu}\} \cup \{\mu \in \mathcal{M} : \mu R^T \bar{\mu}\}$.

(\subseteq): Let $\mu \in \mathcal{A}$. We must show that $\mu \in \mathcal{A}_P$. If $\mu = \bar{\mu}$ and given that $\bar{\mu} \in \mathcal{A}_P$ we are done. Assume that $\mu \neq \bar{\mu}$. Since $\bar{\mu} \in \mathcal{A}$, by Condition (i) of Definition 2, we have $\mu R^T \bar{\mu}$. Hence $\mu \in \mathcal{A}_P$ as desired.

(\supseteq): Let $\mu \in \mathcal{A}_P$. We must show that $\mu \in \mathcal{A}$. If $\mu = \bar{\mu}$ since $\bar{\mu} \in \mathcal{A}$ we are done. If $\mu \neq \bar{\mu}$ then $\mu R^T \bar{\mu}$. As $\bar{\mu} \in \mathcal{A}$, by Condition (ii) of Definition 2 it follows that $\mu \in \mathcal{A}$. ■

Proof of Theorem 3. If \mathcal{A} is an absorbing set then, by Theorem 2, $\mathcal{A} = \mathcal{A}_P$ for some stable partition P . Now, as the roommate problem is solvable, by Remark 1 (i) the stable partition P contains no odd rings. Hence there exists a unique P -stable matching μ which is stable by the stability of P . Then $\mathcal{A}_P = \{\mu\}$ and therefore $\mathcal{A} = \{\mu\}$. Conversely, if $\mathcal{A} = \{\mu\}$ for some stable matching μ , then \mathcal{A} satisfies Conditions (i) and (ii) of Definition 2. Hence \mathcal{A} is an absorbing set. ■

Proof of Theorem 4. (\implies): Let \mathcal{A} be an absorbing set. Then, by Theorem 2, $\mathcal{A} = \mathcal{A}_P$ for some stable partition P . We prove that $P \upharpoonright_{S_P}$ is maximal in \mathcal{P} . Assume that $P \upharpoonright_{S_P}$ is not maximal, *i.e.*, there exists a stable partition P' such that $P \upharpoonright_{S_P} \subset P' \upharpoonright_{S_{P'}}$. Let μ and μ' be a P -stable matching and a P' -stable matching respectively. Thus, by Lemma 5,

$\mu'R^T\mu$. Now, since $\mu \in \mathcal{A}_P$ and $\mathcal{A} = \mathcal{A}_P$ we have $\mu \in \mathcal{A}$. Hence, by Condition (ii) of Definition 2 $\mu' \in \mathcal{A}$. But then, by Condition (i), $\mu R^T \mu'$ and therefore, by Lemma 5, $P' \upharpoonright_{S_{P'}} \subseteq P \upharpoonright_{S_P}$, contradicting the assumption that $P \upharpoonright_{S_P} \subset P' \upharpoonright_{S_{P'}}$.

(\Leftarrow): Let P be a stable partition such that $P \upharpoonright_{S_P}$ is maximal in \mathcal{P} . We prove that \mathcal{A}_P is an absorbing set, *i.e.*, \mathcal{A}_P satisfies Conditions (i) and (ii) of Definition 2. By Remark 2, $\mathcal{A}_P = \{\bar{\mu}\} \cup \{\mu \in \mathcal{M} : \mu R^T \bar{\mu}\}$ where $\bar{\mu}$ is a P -stable matching. Let $\mu \in \mathcal{A}_P$. If there exists $\mu' \in \mathcal{M}$ such that $\mu'R\mu$ then $\mu'R^T\bar{\mu}$. Hence $\mu' \in \mathcal{A}_P$ and Condition (ii) follows.

Now we show that \mathcal{A}_P satisfies Condition (i). It suffices to prove that $\bar{\mu}R^T\mu$ for all $\mu \in \mathcal{A}_P$ such that $\mu \neq \bar{\mu}$. If μ is not a P' -stable matching for any stable partition P' , by Theorem 1, there exists a P' -stable matching μ' such that $\mu'R^T\mu$. Since $\mu R^T \bar{\mu}$ we have $\mu'R^T\bar{\mu}$ (if μ is a P' -stable matching for some stable partition P' then $\mu' = \mu$ can be considered). Thus, by Lemma 5, $P \upharpoonright_{S_P} \subseteq P' \upharpoonright_{S_{P'}}$ and since $P \upharpoonright_{S_P}$ is maximal in \mathcal{P} , it follows that $P \upharpoonright_{S_P} = P' \upharpoonright_{S_{P'}}$. But then $\bar{\mu}R^T\mu'$ and since $\mu'R^T\mu$ we conclude that $\bar{\mu}R^T\mu$ as desired. ■

Proof of Corollary 5. This corollary is an immediate consequence of Theorem 4 and Lemma 6 ■

Proof of Theorem 6. Suppose, for a contradiction, that $S_P \neq S_{P'}$. Then $S_P \cap D_{P'} \neq \emptyset$ or $S_{P'} \cap D_P \neq \emptyset$. We assume, without loss of generality, that $S_P \cap D_{P'} \neq \emptyset$ (otherwise, the argument will be identical except for the roles of P and P' , which are interchanged). By Lemma 7, for each $A \in P$ either $A \subseteq D_{P'}$ or $A \subseteq S_{P'}$. Let $P^* = \{A \in P : A \subseteq D_{P'}\} \cup \{A' \in P' : A' \subseteq S_{P'}\}$ be a partition of N . It is easy to verify that P^* is stable. Now we prove that $D_{P^*} \subseteq D_P \cap D_{P'}$. By the iterative process described in Appendix A, there exists a finite sequence of sets $\langle D_t^* \rangle_{t=0}^{r^*}$ such that:

- (i) D_0^* is the union of all odd rings of P^* .
- (ii) For $t \geq 1$, $D_t^* = D_{t-1}^* \cup B_t^*$ where $B_t^* = \{b_1^*(t), \dots, b_{l_t^*}^*(t)\} \in P^*$ ($l_t^* = 1$ or 2), $B_t^* \not\subseteq D_{t-1}^*$, for which there is a set $A_t^* = \{a_1^*(t), \dots, a_{k_t^*}^*(t)\} \in P^*$, $A_t^* \subseteq D_{t-1}^*$ and

$$b_j^*(t) \succ_{a_i^*(t)} a_i^*(t) \text{ and } a_i^*(t) \succ_{b_j^*(t)} b_{j-1}^*(t), \quad (4)$$

for some $i \in \{1, \dots, k_t^*\}$ and $j \in \{1, \dots, l_t^*\}$.

Then, by Lemma 2, $D_{P^*} = D_{r^*}^*$. We prove by induction on t that, for each $t = 0, \dots, r^*$, $D_t^* \subseteq D_P \cap D_{P'}$. If $t = 0$, this is trivial. Assume that $t \geq 1$. It suffices to prove that $B_t^* \subseteq D_P \cap D_{P'}$. By Lemma 7, we only

need to show that $b_j^*(t) \in D_P \cap D_{P'}$. Since $A_t^* \subseteq D_{t-1}^*$, by the inductive hypothesis, $a_i^*(t) \in D_P \cap D_{P'}$. Clearly $b_j^*(t) \in D_{P'}$ (otherwise, $B_t^* \in P'$ and since $a_i^*(t) \in D_{P'}$, by (4), $b_j^*(t) \in D_{P'}$). So $B_t^* \in P$ and since $a_i^*(t) \in D_P$, from (4) it follows that $b_j^*(t) \in D_P$, as desired.

Finally, since $D_{P^*} \subseteq D_P \cap D_{P'}$ we have $S_{P' \cup (S_P \cap D_{P'})} \subseteq S_{P^*}$ and therefore $P' \upharpoonright_{S_{P'}} \subset P^* \upharpoonright_{S_{P^*}}$, contradicting the maximality of $P' \upharpoonright_{S_{P'}}$. ■

Proof of Theorem 7. This theorem is easily derived from Theorem 4 and Lemma 4 ■

Proof of Theorem 8. Let \mathcal{A} and \mathcal{A}' be two absorbing sets. Then, by Theorem 4, there are stable partitions P and P' such that $\mathcal{A} = \mathcal{A}_P$, $\mathcal{A}' = \mathcal{A}_{P'}$ where $P \upharpoonright_{S_P}$ and $P' \upharpoonright_{S_{P'}}$ are maximal in \mathcal{P} .

(i) Since $S_{\mathcal{A}} = S_P$ and $S_{\mathcal{A}'} = S_{P'}$ and, by Theorem 6, $S_P = S_{P'}$, then $S_{\mathcal{A}} = S_{\mathcal{A}'}$. Therefore $D_{\mathcal{A}} = D_{\mathcal{A}'}$.

(ii) It is very easy to verify that $\mathcal{A} \upharpoonright_{D_{\mathcal{A}}}$ and $\mathcal{A}' \upharpoonright_{D_{\mathcal{A}'}}$ are absorbing sets in $(D, (\succ_x)_{x \in D})$ where $D = D_{\mathcal{A}} = D_{\mathcal{A}'}$ such that $\mathcal{A} \upharpoonright_{D_{\mathcal{A}}} = \mathcal{A}_P \upharpoonright_{D_P}$ and $\mathcal{A}' \upharpoonright_{D_{\mathcal{A}'}} = \mathcal{A}_{P'} \upharpoonright_{D_{P'}}$. Since $S_{P \upharpoonright_{D_P}} = S_{P' \upharpoonright_{D_{P'}}} = \emptyset$, from Lemma 6, we conclude that $\mathcal{A} \upharpoonright_{D_{\mathcal{A}}} = \mathcal{A}' \upharpoonright_{D_{\mathcal{A}'}}$.

(iii) This follows directly from Theorem 7. ■

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