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and robust sets of equilibria

Vladyslav Nora and Hiroshi Uno

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Voie du Roman Pays, 34
B-1348 Louvain-la-Neuve
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Saddle functions and robust sets of equilibria

Vladyslav NORA [1] and Hiroshi UNO[2]

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Abstract

This paper introduces games with a saddle function. A saddle function is a real valued function on the set of action profiles such that, for one player, minimizing the function implies choosing her best-response, and, for the other players, maximizing it implies choosing their best-responses. We provide a new sufficient condition for robustness to incomplete information of sets of equilibria in a sense of Kajii and Morris (1997, Econometrica), Morris and Ui (2005, J. of Econ. Theory) for games with a saddle function. Our result unifies and generalizes sufficient conditions for zero-sum and best-response potential games.

Keywords: incomplete information, robust equilibrium, potential games, zero-sum games, team-maximin equilibrium.

JEL classification: C72

[1] Université catholique de Louvain, CORE, B-1348 Louvain-la-Neuve, Belgium.
E-mail: vladyslav.nora@uclouvain.be
[2] Graduate School of Economics, Osaka Prefecture University, Osaka 599-8531, Japan. E-mail: uno@eco.osakafu-u.ac.jp

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1 Introduction

We often model a strategic situation as a complete information game. However, it is well known now that equilibrium outcomes of a complete information game may be very different from the outcomes of an arbitrarily "close" incomplete information game.[1] When can we justify an assumption of common knowledge of payoffs, implicit in the definition of complete information games? One answer, proposed by Kajii and Morris [5], is to require an equilibrium to be robust to incomplete information. An equilibrium of a complete information game is robust if every incomplete information game with payoffs almost always as in the original game has a Bayesian Nash equilibrium that induces an observed behavior close to the equilibrium of complete information game. If an equilibrium is robust, then the simplifying assumption of common knowledge can be justified. Unfortunately, robust equilibrium may not exists. To guarantee existence, Morris and Ui [8] extend such a robustness to a set-valued notion. A set of equilibria is robust if every incomplete information game close to the the original game has a Bayesian Nash equilibrium that induces an observed behavior close to some equilibrium in the set.

To give a sufficient condition for robustness of sets of equilibria we introduce games with a saddle function. A saddle function is a real valued function on the set of action profiles such that, for one player, *minimizing* the function implies choosing her best-response, and, for the other players, *maximizing* it implies choosing their best-responses. The value of a game with a saddle function is a minimax value attained when the saddle function is minimized over strategies of one player and maximized over distributions on action profiles of the other players. Our condition says that the set of correlated equilibria that induce an expectation of a saddle function equal to the value of the game is robust if it contains a Nash equilibrium.

A game with a saddle function can be viewed as "strategically equivalent" to a zero-sum game where a set of players with identical payoffs plays against a single adversary. We call such zero-sum games, studied by von Stengel and Koller [14], team vs. adversary games. As an illustrative example consider a three-person game below.

	L	R
U	$1, 1, -1$	$-1, -1, 1$
D	$-2, -2, 2$	$0, 0, 0$

T

	L	R
U	$2, 2, -2$	$-2, -2, 2$
D	$-1, -1, 1$	$3, 3, -3$

B

(a) Three-person team vs. adversary game.

	(U, L)	(U, R)	(D, L)	(D, R)
T	1	-1	-2	0
B	2	-2	-1	3

(b) Auxiliary two-person zero-sum game.

[1]See for example Rubinstein [10] or Carlsson and van Damme [2].

In game (a) on the left team players are Row and Column choosing a row and a column respectively; adversary chooses a matrix. A payoff function of the auxiliary two-person zero-sum game (b) where we regard Row and Column as a *single* maximizing player and adversary as a minimizing player is a saddle function of the original three-person game (a). The minimax value of the auxiliary zero-sum game and thus the value of game (a) is 1. Observe that a Nash equilibrium profile (U, L, T) is a unique correlated equilibrium of game (a) that induces the expectation of the saddle function equal to 1. Our result implies that (U, L, T) is robust.

The notion of robust equilibrium is introduced by Kajii and Morris [5] who also give first sufficient conditions in terms of a unique correlated equilibrium and **p**-dominant equilibrium. As a corollary of the robustness of a unique correlated equilibrium, in two-person zero-sum games, a unique Nash equilibrium is robust to incomplete information. Next, Ui [12] proves the robustness of a unique maximizer of a potential function defined by Monderer and Shapley [7]. Two approaches are unified and generalized by Morris and Ui [8] with the notion of a generalized potential function. Although a generalized potential provides a powerful tool to study robust sets of equilibria, finding it may be difficult. This leads Morris and Ui [8] to develop three special but tractable classes of generalized potential functions: best response potentials, monotone potentials and local potentials. Similarly, Tercieux [11] obtains a simple condition for games with **p**-best response sets, which is a special case of a generalized potential result. Further, Oyama and Tercieux [9] generalize the sufficient condition in terms of monotone potential maximizers using iterated monotone potential functions. All these conditions are generally not applicable to games with a saddle function, in particular to team vs. adversary games. In this paper we exploit a saddle function to find nontrivial robust sets of equilibria in games outside the scope of the existing results.

Our contribution is to provide a new and tractable sufficient condition for robustness. The condition unifies and generalizes those in terms of zero-sum and best-response potential games. Although the condition is easier than the one in terms of generalized potential maximizers, it can guarantee the robustness of strictly smaller sets of equilibria. This implies that our condition is independent of the other conditions in the literature.[2]

The rest of the paper is organized as follows. In the next section we provide basic definitions. In Section 3, we introduce a saddle function and state the main result. In Section 4, we prove the main result. We conclude with the discussion of the related literature in Section 5.

[2]We also show that the notion of a saddle function can be extended in a similar way the notion of a potential function is extended to a generalized potential by Morris and Ui [8].

2 Robust sets of equilibria

A complete information game consists of a finite set of players N and, for each i in N, a finite set of actions A_i and a payoff function $g_i : A \to \mathbb{R}$, where $A = \prod_{i\in N} A_i$. Since we fix the set N of players and the set A of action profiles, we simply denote a complete information game by $\mathbf{g} := (g_i)_{i\in N}$. Conventionally, for a profile $(X_i)_{i\in N}$ of sets, we write $X := \prod_{i\in N} X_i$; for $i \in N$, $X_{-i} := \prod_{k\neq i} X_k$. We write $x := (x_i)_{i\in N} \in X$; for $i \in N$, $x_{-i} := (x_k)_{k\neq i} \in X_{-i}$. For $T \subset N$, we write $X_T := \prod_{i\in T} X_i$ and $x_T := (x_i)_{i\in T} \in X_T$. A set of probability distributions on a set X is denoted by $\Delta(X)$.

An action distribution $\mu \in \Delta(A)$ is a *correlated equilibrium* of $\mathbf{g}$ if, for each $i \in N$ and each $a_i \in A_i$ with $\mu_i(a_i) > 0$,

$$\sum_{a_{-i}\in A_{-i}} \mu(a_{-i}|a_i) g_i(a_i, a_{-i}) \geq \sum_{a_{-i}\in A_{-i}} \mu(a_{-i}|a_i) g_i(a_i', a_{-i})$$

for each $a_i' \in A_i$, where $\mu_i(a_i)$ is the marginal probability of a_i and $\mu(a_{-i}|a_i)$ is the conditional probability of a_{-i} given a_i. A profile $(\mu_i)_{i\in N} \in \prod_{i\in N} \Delta(A_i)$ is a *Nash equilibrium* of $\mathbf{g}$ if $\mu \in \Delta(A)$ with $\mu(a) = \prod_{i\in N} \mu_i(a_i)$ for all $a \in A$ is a correlated equilibrium of $\mathbf{g}$.

Consider an incomplete information game with the set N of players and the set A of action profiles (same as in $\mathbf{g}$). Let Θ_i be a countable set of types of a player $i \in N$, and let P be the common prior probability distribution on Θ with $P_i(\theta_i) := \sum_{\theta_{-i}\in\Theta_{-i}} P(\theta_i, \theta_{-i}) > 0$. A payoff function of a player $i \in N$ is a bounded function $u_i : A \times \Theta \to \mathbb{R}$. Since we also fix the set Θ of type profiles, we denote an incomplete information game by $(\mathbf{u}, P)$, where $\mathbf{u} := (u_i)_{i\in N}$.

For a player $i \in N$ a strategy is a function $\sigma_i : \Theta_i \to \Delta(A_i)$. Let Σ_i be a set of i's strategies. We write $\sigma_i(a_i|\theta_i)$ for the probability that an action $a_i \in A_i$ is chosen given a type $\theta_i \in \Theta_i$ under a strategy $\sigma_i \in \Sigma_i$. For a subset T of N, a probability of an action profile $a_T \in A_T$ given a type profile $\theta_T \in \Theta_T$ under a strategy profile $\sigma_T \in \Sigma_T$ is denoted by $\sigma_T(a_T|\theta_T) := \prod_{i\in T} \sigma_i(a_i|\theta_i)$.

A strategy profile $\sigma \in \Sigma$ is a *Bayesian Nash equilibrium* of $(\mathbf{u}, P)$ if, for each $i \in N$ and $\theta_i \in \Theta_i$,

$$\sum_{\theta_{-i}\in\Theta_{-i}} P(\theta_{-i}|\theta_i) \sum_{a\in A} \sigma(a|\theta) u_i(a, \theta) \geq \sum_{\theta_{-i}\in\Theta_{-i}} P(\theta_{-i}|\theta_i) \sum_{a\in A} \sigma_i'(a_i|\theta_i)\sigma_{-i}(a_{-i}|\theta_{-i}) u_i(a, \theta)$$

for all $\sigma_i' \in \Sigma_i$, where $P(\theta_{-i}|\theta_i) = P(\theta_i, \theta_{-i})/P_i(\theta_i)$.

Given a complete information game $\mathbf{g}$ and an incomplete information game $(\mathbf{u}, P)$, for each $i \in N$, consider the subset $\bar{\Theta}_i$ of Θ_i such that, if $\theta_i \in \bar{\Theta}_i$ is realized, i's payoffs are given by g_i

and he knows his payoffs:

$$\bar{\Theta}_i = \{\theta_i \in \Theta_i | u_i(a, (\theta_i, \theta_{-i})) = g_i(a) \text{ for all } a \in A, \theta_{-i} \in \Theta_{-i} \text{ with } P(\theta_i, \theta_{-i}) > 0\}.$$

An incomplete information game $(\mathbf{u}, P)$ is an ε-*elaboration* of $\mathbf{g}$ if $P(\bar{\Theta}) = 1 - \varepsilon$, where $\varepsilon \in [0, 1]$. Kajii and Morris [5] prove following useful lemma.

Lemma 1. *Let $\{(\mathbf{u}^m, P^m)\}$ be a sequence of ε-elaborations with $\varepsilon^m \to 0$ and let σ^m be a Bayesian Nash equilibrium of $(\mathbf{u}^m, P^m)$. Then there exist a subsequence $\{\sigma^l\}$ of $\{\sigma^m\}$ and a correlated equilibrium $\mu \in \Delta(A)$ of $\mathbf{g}$ such that $\sum_{\theta \in \Theta} P^l(\theta)\sigma^l(a|\theta) \to \mu(a)$ for each $a \in A$.*

A type $\theta_i \in \Theta_i \backslash \bar{\Theta}_i$ of a player $i \in N$ is *committed* if it has a strictly dominant action $a_i^{\theta_i} \in A_i$:

$$u_i\left((a_i^{\theta_i}, a_{-i}), (\theta_i, \theta_{-i})\right) > u_i\left((a_i, a_{-i}), (\theta_i, \theta_{-i})\right)$$

for each $a_i \in A_i \backslash \{a_i^{\theta_i}\}$, $a_{-i} \in A_{-i}$ and all $\theta_{-i} \in \Theta_{-i}$ with $P(\theta_i, \theta_{-i}) > 0$. An ε-elaboration of $\mathbf{g}$ is *canonical* if, for each $i \in N$, each $\theta_i \in \Theta_i \backslash \bar{\Theta}_i$ is a committed type.

Morris and Ui [8] study the sets of correlated equilibria robust to canonical elaborations.

Definition 1. A set $\mathcal{E} \subseteq \Delta(A)$ of correlated equilibria is *robust to canonical elaborations* in $\mathbf{g}$ if, for each $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon < \bar{\varepsilon}$, each canonical ε-elaboration of $\mathbf{g}$ has a Bayesian Nash equilibrium $\sigma \in \Sigma$ with $\max_{a \in A} |\mu(a) - \sum_{\theta \in \Theta} P(\theta)\sigma(a|\theta)| \leq \delta$ for some $\mu \in \mathcal{E}$.

Notice that if a set $\mathcal{E}$ is a singleton, then the correlated equilibrium in the set is a Nash equilibrium distribution robust to canonical elaborations in a sense of Kajii and Morris [4].

The reason to consider a set-valued notion is that robust equilibrium may not exist, as shown in Kajii and Morris [5].[3] On the contrary, robust set always exists, though maybe trivially as a set of all correlated equilibria.

Originally, the stronger notion of robustness to *all* elaborations was proposed by Kajii and Morris [5]. To get a corresponding set-valued notion one allows for all, not only canonical ε-elaborations in Definition 1. It is clear that a set of correlated equilibria robust to all elaborations is also robust to canonical elaborations.[4]

3 Saddle points and robust sets of equilibria

In a complete information game fix a player j in N and the set T of the other players. A saddle function is a real-valued function f on the set of action profiles such that, for each member i of T,

[3]In fact, Haimanko and Kajii [3] show that even two-person zero-sum game may have no robust equilibrium.
[4]Whether the converse holds is an open question.

every best-response against his belief over the other players' actions in a game where i's payoff is given by f is a best-response against the same belief in the original game; and j's best-response against her belief over the other players' actions in a game where j's payoff is given by $-f$ is a best-response against the same belief in the original game as well.[5]

Definition 2. Let $j \in N$ and $T := N\backslash\{j\}$. A function $f : A \to \mathbb{R}$ is a j-*saddle function* of $\mathbf{g}$ if, for each $i \in T$,

$$\arg\max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) f(a) \subseteq \arg\max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a)$$

for all $\lambda_i \in \Delta(A_{-i})$; and for j,

$$\arg\min_{a_j \in A_j} \sum_{a_T \in A_T} \lambda_j(a_T) f(a) \subseteq \arg\max_{a_j \in A_j} \sum_{a_T \in A_T} \lambda_j(a_T) g_j(a)$$

for all $\lambda_j \in \Delta(A_T)$. A *value of* f is $v^* := \max_{\mu_T \in \Delta(A_T)} \min_{\mu_j \in \Delta(A_j)} \sum_{a \in A} \mu_T(a_T) \mu_j(a_j) f(a)$. A Nash equilibrium $(\mu_i^*)_{i \in N} \in \prod_{i \in N} \Delta(A_i)$ is a *saddle point of* f if $\sum_{a \in A} \left(\prod_{i \in N} \mu_i^*(a_i)\right) f(a) = v^*$.

In what follows we fix $\mathbf{g}$, a player j in N and a j-saddle function f of $\mathbf{g}$ with a saddle point $(\mu_i^*)_{i \in N} \in \prod_{i \in N} \Delta(A_i)$. Therefore, abusing notation, we simply say that f is a *saddle function* and omit reference to f when discussing a value and saddle points.

A game with a saddle function does not have a saddle point if the set of maximinimizers of f does not contain a product distribution. On the other hand, it is clear that, if a product distribution is a maximinimizer of a saddle function, then it is generated by a saddle point.

Games with a saddle function generalize best-response potential games introduced in Morris and Ui [8]: add a dummy player j with a singleton action set to a best-response potential game, now a best-response potential function is a saddle function and a best-response potential maximizer, a saddle point.

Let $\mathcal{E}$ be a set of correlated equilibria of $\mathbf{g}$ inducing an expectation of f equal to v^*:

$$\mathcal{E} := \{\mu \in \Delta(A) | \mu \text{ is a correlated equilibrium of } \mathbf{g} \text{ and } \textstyle\sum_{a \in A} \mu(a) f(a) = v^*\}.$$

We are ready to state the main result of the paper.

Theorem 1. *If* $\mathbf{g}$ *has a saddle function with a saddle point, then the corresponding set* $\mathcal{E}$ *is robust to canonical elaborations in* $\mathbf{g}$.

Generally, a set $\mathcal{E}$ is not a singleton. If $\mathcal{E}$ is a singleton and there exists a saddle point, then the unique Nash equilibrium distribution in the set is robust to canonical elaborations.

[5] Note that games with a j-saddle function is a special class of multi-potential games introduced by Monderer [6], or games with a partition $\{\{j\}, T\}$-potential introduced by Uno [13].

4 Proof of the main result

Let ($\mathbf{u}$,P) be a canonical ε-elaboration of $\mathbf{g}$. Define a function $V : \Sigma \to \mathbb{R}$ by

$$V(\sigma) := \sum_{\theta\in\Theta}\sum_{a\in A} P(\theta)\sigma(a|\theta)f(a).$$

For each $i \in N$, let $\bar{\Sigma}_i := \{\sigma_i \in \Sigma_i | \sigma_i(a_i^{\theta_i}|\theta_i) = 1 \text{ for all } \theta_i \in \Theta_i \setminus \bar{\Theta}_i\}$ be a set of strategies such that all committed types choose their dominant actions. A Bayesian Nash equilibrium $(\sigma_T^*, \sigma_j^*) \in \Sigma$ of an ε-elaboration ($\mathbf{u}$,P) is a *quasi-saddle point* of ($\mathbf{u}$,P) if

$$\sigma_T^* \in \arg\max_{\sigma_T\in\bar{\Sigma}_T}\min_{\sigma_j\in\bar{\Sigma}_j} V(\sigma_T, \sigma_j).$$

A saddle point of $\mathbf{g}$ is a quasi-saddle point of a degenerate 0-elaboration of $\mathbf{g}$ where Θ_i is a singleton for each i in N. The key observation is that for canonical ε-elaborations with ε close to zero, quasi-saddle points induce a value of V close to v^*.

The following generalization of von Stengel and Koller's [14] result on team vs. adversary games guarantees the existence of a quasi-saddle point in every canonical ε-elaboration of $\mathbf{g}$.[6]

Lemma 2. *Let $j \in N$ and $T := N\backslash\{j\}$. Consider a game $(N, (S_i)_{i\in N}, (g_i)_{i\in N})$ such that, for each $i \in T$, $g_i = f$ and $g_j = -f$, where $f : S \to \mathbb{R}$ is a continuous multilinear function. For each $i \in N$, suppose that S_i is a compact convex subset of a locally convex Hausdorff space. Then,* $\arg\max_{s_T\in S_T}\min_{s_j\in S_j} f(s_T, s_j)$ *is nonempty. Furthermore, for each $s_T^* \in$* $\arg\max_{s_T\in S_T}\min_{s_j\in S_j} f(s_T, s_j)$*, there exists $s_j^* \in S_j$ such that (s_T^*, s_j^*) is a Nash equilibrium.*[7]

If we let a game in Lemma 2 have $S_i = \bar{\Sigma}_i$ for each $i \in N$ and $f = V$, then we can show that (s_T^*, s_j^*) is a quasi-saddle point of a canonical ε-elaboration ($\mathbf{u}$,P).

Lemma 3. *Every canonical ε-elaboration of* $\mathbf{g}$ *has a quasi-saddle point.*

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Suppose that $(\mu_i^*)_{i\in N} \in \prod_{i\in N}\Delta(A_i)$ is a saddle point of $\mathbf{g}$. Fix a canonical ε-elaboration $(\mathbf{u}, P)$ of $\mathbf{g}$. By Lemma 3 there exists a quasi-saddle point $\sigma \in \Sigma$ of $(\mathbf{u}, P)$. First, we shall find a lower and an upper bounds on $V(\sigma)$. For each $i \in T$ and each $\theta_i \in \bar{\Theta}_i$, let $\sigma_T^* \in \bar{\Sigma}_T$ be such that $\sigma_i^*(a_i|\theta_i) = \mu_i^*(a_i)$ for each $a_i \in A_i$. Since $\sigma_T \in \arg\max_{\sigma'_T\in\bar{\Sigma}_T}\min_{\sigma'_j\in\bar{\Sigma}_j} V(\sigma'_T, \sigma'_j)$, we have $V(\sigma_T, \sigma_j) \geq \min_{\sigma'_j\in\bar{\Sigma}_j} V(\sigma_T^*, \sigma'_j)$.

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[6]The proofs omitted in the text are referred to Appendix A.

[7]Note that Sion's minimax theorem is not applicable in this case since f is not quasi-concave in s_T. Note also that in a finite game S_i is a set of mixed strategies, i.e. a finite dimensional simplex.

Let $\varepsilon_T \geq 0$ be a marginal probability that there exists a player in T of committed type, i.e., $\varepsilon_T := P\left((\Theta_T \setminus \bar{\Theta}_T) \times \Theta_j\right) \leq \varepsilon$. By definition of V for each $\sigma'_j \in \bar{\Sigma}_j$ we have

$$\begin{aligned} V(\sigma^*_T, \sigma'_j) &= \sum_{\theta \in (\Theta_T \setminus \bar{\Theta}_T) \times \Theta_j} P(\theta) \sum_{a \in A} \sigma^*_T(a_T|\theta_T)\sigma'_j(a_j|\theta_j) f(a) \\ &\quad + \sum_{\theta \in \bar{\Theta}_T \times \Theta_j} P(\theta) \sum_{a \in A} \left(\prod_{i \in T} \mu^*_i(a_i) \right) \sigma'_j(a_j|\theta_j) f(a) \\ &\geq \varepsilon_T f_{min} + \sum_{\theta \in \bar{\Theta}_T \times \Theta_j} P(\theta) \sum_{a \in A} \left(\prod_{i \in T} \mu^*_i(a_i) \right) \sigma'_j(a_j|\theta_j) f(a), \end{aligned}$$

where $f_{min} := \min_{a \in A} f(a)$. Observe that $\sum_{a \in A} \left(\prod_{i \in T} \mu^*_i(a_i)\right) \mu_j(a_j) f(a) \geq v^*$ for all $\mu_j \in \Delta(A_j)$, since $(\mu^*_i)_{i \in N} \in \prod_i \Delta(A_i)$ is a saddle point. It follows that

$$\varepsilon_T f_{min} + (1 - \varepsilon_T) v^* \leq \varepsilon_T f_{min} + \sum_{\theta \in \bar{\Theta}_T \times \Theta_j} P(\theta) \sum_{a \in A} \left(\prod_{i \in T} \mu^*_i(a_i) \right) \sigma_j(a_j|\theta_j) f(a).$$

Therefore $\varepsilon_T f_{min} + (1 - \varepsilon_T) v^* \leq V(\sigma)$. By the symmetric argument for player j we get $V(\sigma) \leq \varepsilon_j f_{max} + (1 - \varepsilon_j) v^*$, where $f_{max} := \max_{a \in A} f(a)$ and $\varepsilon_j := P\left(\Theta_T \times (\Theta_j \setminus \bar{\Theta}_j)\right) \leq \varepsilon$. Combining lower and upper bounds, we obtain

$$\varepsilon_T(f_{min} - v^*) \leq V(\sigma) - v^* \leq \varepsilon_j(f_{max} - v^*). \tag{1}$$

To complete the proof we show that for each $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \leq \bar{\varepsilon}$, each canonical ε-elaboration of $\mathbf{g}$ has a Bayesian Nash equilibrium $\sigma \in \Sigma$ with $\max_{a \in A} |\mu(a) - \sum_{\theta \in \Theta} P(\theta)\sigma(a|\theta)| \leq \delta$ for some $\mu \in \mathcal{E}$.

To get a contradiction suppose that for some $\delta > 0$ there exists a sequence $\{(\mathbf{u}^m, P^m)\}$ of canonical ε-elaboration of $\mathbf{g}$ with $\varepsilon^m \to 0$ such that $\max_{a \in A} |\sum_{\theta \in \Theta} P^m(\theta)\sigma^m(a|\theta) - \mu(a)| > \delta$ for all $\mu \in \mathcal{E}$, where $\sigma^m \in \Sigma$ is a quasi-saddle point of $(\mathbf{u}^m, P^m)$. By Lemma 1 there exist a subsequence $\{\sigma^l\}$ of $\{\sigma^m\}$ and a correlated equilibrium $\nu \in \Delta(A)$ of $\mathbf{g}$ such that $\sum_{\theta \in \Theta} P^l(\theta)\sigma^l(a|\theta) \to \nu(a)$ for all $a \in A$. Since $\sigma^l \in \Sigma$ is a quasi-saddle point of $(\mathbf{u}^l, P^l)$, by (1) we have $\sum_{a \in A} \nu(a) f(a) = v^*$. Therefore $\nu \in \mathcal{E}$. The contradiction completes the proof. □

5 Discussion

5.1 Generalized potentials

Morris and Ui [8] use the generalized potentials to find robust sets of equilibria. Generalized potentials are real-valued functions defined on the domain $\mathcal{A} = \prod_{i \in N} \mathcal{A}_i$ where for each $i \in N$, a set $\mathcal{A}_i \subseteq 2^{A_i} \backslash \{\emptyset\}$ is a covering of A_i. That is $\mathcal{A}_i$ is a collection of nonempty subsets of A_i such that $\bigcup_{X_i \in \mathcal{A}_i} X_i = A_i$. Given a nonempty subset $S \subseteq N$ and $\Lambda_S \in \Delta(\mathcal{A}_S)$ let

$$\Delta_{\Lambda_S}(A_S) := \{\lambda \in \Delta(A_S) | \lambda(a_S) = \sum_{X_S \in \mathcal{A}} \Lambda_S(X_S)\lambda^{X_S}(a_S) \text{ for each } a_S \in A_S \text{ and}$$
$$\lambda^{X_S} \in \Delta(A_S) \text{ with } \sum_{a_S \in X_S} \lambda^{X_S}(a_S) = 1 \text{ for each } X_S \in \mathcal{A}_{\mathcal{S}}\}$$

be the set of distributions over A_S induced by Λ_S.

A function $F : \mathcal{A} \to \mathbb{R}$ is a *generalized potential function* of $\mathbf{g}$ if, for each $i \in N$, all $Q_i \in \Delta(\mathcal{A}_{-i})$ and all $q_i \in \Delta_{Q_i}(A_{-i})$,

$$X_i \cap \arg\max_{a_i' \in A_i} \sum_{a_{-i} \in A_{-i}} q_i(a_{-i}) g_i(a_i', a_{-i}) \neq \emptyset$$
$$\text{for every } X_i \in \arg\max_{X_i' \in \mathcal{A}_i} \sum_{X_{-i} \in \mathcal{A}_{-i}} Q_i(X_{-i}) F(X_i', X_{-i})$$

such that X_i is maximal in the argmax set ordered by the set inclusion relation. An action subspace X^* is a *generalized potential maximizer (GP-maximizer)* if $F(X^*) > F(X)$ for each $X \in \mathcal{A} \setminus \{X^*\}$. Consider a set

$$\mathcal{E}_{X^*} := \{\mu \in \Delta(A) | \mu \text{ is a correlated equilibrium of } \mathbf{g} \text{ such that } \textstyle\sum_{a \in X^*} \mu(a) = 1\}.$$

Morris and Ui [8] show the following result.

Theorem 2. *If* $\mathbf{g}$ *has a generalized potential function with a GP-maximizer* X^**, then* $\mathcal{E}_{X^*}$ *is robust to canonical elaborations in* $\mathbf{g}$.

Theorem 2 is the most general known sufficient condition for robustness of sets of equilibria. However, in the example below we demonstrate that Theorem 1 guarantees robustness of strictly smaller sets than Theorem 2 of Morris and Ui [8]. Therefore Theorem 1 establishes robustness results for the class of games uncovered by the existing literature.

Example 1. Consider a three-player game above. Row, Column and Matrix are choosing a row, a column and a matrix respectively. The payoff function is a saddle function. It is easy

	L	R
U	$3, \ 3, -3$	$-1, -1, \ 1$
D	$-1, -1, \ 1$	$-1, -1, \ 1$

TL

	L	R
U	$-1, -1, \ 1$	$-1, -1, \ 1$
D	$-1, -1, \ 1$	$3, \ 3, -3$

TR

	L	R
U	$-1, -1, \ 1$	$-1, -1, \ 1$
D	$3, \ 3, -3$	$-1, -1, \ 1$

BL

	L	R
U	$-1, -1, \ 1$	$3, \ 3, -3$
D	$-1, -1, \ 1$	$-1, -1, \ 1$

BR

to verify that the set $\mathcal{E}$ consists from all convex combinations of distributions generated by strategy profiles where Row and Column randomize with probabilities $(\frac{1}{2}, \frac{1}{2})$ over their actions and Matrix randomizes with Prob(TL) = Prob$(TR) = x$ and Prob(BL) = Prob$(BR) = y$ where $2(x + y) = 1$ and $x, y \geq 0$. These distributions are Nash equilibrium distributions of the game. But, a GP-maximizer must include some inferior pure strategy equilibrium for Row and Column like (D, R, TL) with their payoff equal to -1.

A best-response potential function introduced by Morris and Ui [8] is a special case of a generalized potential when the domain is simply $\mathcal{A}_i = \{\{a_i\} | a_i \in A\}$ for each i in N. One might wonder whether a saddle function has a similar generalization. Indeed, we can define a "generalized" saddle function with a domain being a covering of an action set for each player. Under appropriate assumptions we obtain the result analogous to the robustness of GP-maximizer. We refer the construction to Appendix B.

5.2 Team vs. adversary games

A special case of games with a saddle function are games of a team vs. adversary studied by von Stengel and Koller [14]. In $\mathbf{g}$ let $j \in N$ and $T := N \backslash \{j\}$. We call j an adversary and T a team. A game is a team vs. adversary game if $g_i = f$ for each $i \in T$ and $g_j = -f$, where $f : A \to \mathbb{R}$. A set of mixed strategies of $i \in N$ is $S_i := \Delta(A_i)$. A *team-maximin* strategy profile is $s^*_T \in S_T$ such that $s^*_T \in \arg\max_{s_T \in S_T} \min_{s_j \in S_j} \sum_{a \in A} \left(\prod s_i(a_i)\right) f(a)$. A *team-maxmin equilibrium* is a Nash equilibrium $s \in S$ such that $s_T \in S_T$ is a team-maximin strategy profile. The following result of von Stengel and Koller [14] is a special case of Lemma 2.

Theorem 3. *Any team-maximin strategy profile is a part of a team-maximin equilibrium.*

A team players' payoff function is a saddle function of a team vs. adversary game. A saddle point, if it exists, is a team-maximin equilibrium, but converse does not hold. Von Stengel and Koller [14] suggest that a team-maxmin equilibrium is the most reasonable solution to a team vs. adversary games and therefore is an appropriate method of equilibrium selection. However, in the following example we show that a unique team-maxmin equilibrium may not be robust to canonical elaborations.

Example 2. A three-player game below has a unique team-maximin equilibrium (U, L, B).

T:

	L	R
U	$1, \ 1, -1$	$-1, -1, 1$
D	$-1, -1, \ 1$	$-1, -1, 1$

B:

	L	R
U	$0, \ 0, \ 0$	$-1, -1, \ 1$
D	$-1, -1, \ 1$	$1, \ 1, -1$

Notice that this game does not have a saddle point. We shall show that the team-maximin equilibrium (U, L, B) is not robust to canonical elaborations. Consider the following ε-elaboration of the game above. A Row and a Column players' sets of types are $\Theta_R = \{0, 1, 2, \dots\}$ and $\Theta_C = \{1, 2, \dots\}$ respectively; the adversary has a single type θ_A. We represent the probability distribution on type profiles in a table below, where rows and columns are Row and Column' types respectively:

	1	2	3	$\cdots$	m	$m+1$	$\cdots$
0	ε	0	0	$\cdots$			
1	$\varepsilon(1-\varepsilon)$	$\varepsilon(1-\varepsilon)^2$	0	$\cdots$			
2	0	$\varepsilon(1-\varepsilon)^3$	$\varepsilon(1-\varepsilon)^4$				
$\vdots$	$\vdots$	$\vdots$		$\ddots$			
m					$\varepsilon(1-\varepsilon)^{2m-1}$	$\varepsilon(1-\varepsilon)^{2m}$	$\cdots$
$m+1$					0	$\varepsilon(1-\varepsilon)^{2m+1}$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\ddots$

Formally the probability distribution on type profiles is given by

$$P(\theta_R, \theta_C, \theta_A) = \begin{cases} \varepsilon(1-\varepsilon)^{2m-1} & \text{if } \theta_R = \theta_C = m, \\ \varepsilon(1-\varepsilon)^{2m} & \text{if } \theta_R = m, \ \theta_C = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\theta_R = 0$ be a committed type of Row with the strictly dominant action D. All other types of Row as well as all types of Column and the adversary have payoffs as in the complete information game. Observe that $\frac{\varepsilon(1-\varepsilon)^{2m-1}}{\varepsilon(1-\varepsilon)^{2m}+\varepsilon(1-\varepsilon)^{2m}} > \frac{1}{2}$ for all $\varepsilon > 0$. Hence, by induction, when the adversary chooses B with probability greater or equal to $\frac{2}{3}$, for all types of Row and Column dominant actions are respectively D and R. Thus, we can find a sequence of canonical ε-elaborations with all equilibrium distributions bounded away from (U, L, B).

In case of team vs. adversary games condition of Theorem 1 has an intuitive interpretation. Call the difference between the best correlated equilibrium payoff and the best Nash equilibrium payoff of the team, the value of correlation within a team. When the value of correlation within a team is zero, then a team-maxmin equilibrium is guaranteed to be robust.

A Appendix: proofs omitted in the text

Proof of Lemma 2. Following von Stengel and Koller [14] we call T *a team* and j *the adversary*. A profile $s^*_T \in S_T$ is a *team-maximin profile* if $s^*_T \in \arg\max_{s_T \in S_T} \min_{s_j \in S_j} f(s_T, s_j)$. A profile (s^*_T, s^*_j) is a *team-maximin equilibrium* if it is a Nash equilibrium such that s^*_T is a team-maximin profile. We write $v^* := \max_{s_T \in S_T} \min_{s_j \in S_j} f(s_T, s_j)$.

The existence of a team-maximin profile is guaranteed by compactness and continuity assumptions.

We shall prove the existence of a team-maximin equilibrium. Let s^*_T be a team-maximin profile. We want to find a best response of the adversary such that no team member $i \in T$ has an incentive to deviate from his team-maximin strategy s^*_i when the other team members play their team-maximin strategies $s^*_{-i,j}$. For each $i \in T$ and $s_i \in S_i$, let $H_i(s_i) := \{s_j \in S_j | f(s_i, s^*_{-i,j}, s_j) \leq v^*\}$ be a set of strategies of the adversary such that given $s_j \in H_i(s_i)$, player i's payoff from s_i when others play team-maximin profile $s^*_{-i,j}$ is not higher than v^*. For each $i \in T$ and $s_i \in S_i$, it is clear that $H_i(s_i)$ is nonempty, convex and compact (since f is continuous). We construct a correspondence which fixed point is a desired best response of the adversary.

Define maps $\psi_i : S \rightrightarrows S_T$ for each $i \in T$, $\psi_j : S \rightrightarrows S_j$ and $\psi : S \rightrightarrows S$ by

$$\begin{aligned} \psi_i(s) &:= \arg\max_{s_i \in S_i} f(s^*_{-i,j}, s_j) \text{ for each } i \in T, \\ \psi_j(s) &:= \bigcap_{i \in T} H_i(s_i), \\ \psi(s) &:= \prod_{i \in N} \psi_i(s). \end{aligned}$$

Suppose that $(\tilde{s}_T, s^*_j)$ is a fixed point of ψ. We shall show that (s^*_T, s^*_j) is a Nash equilibrium. First, we want to show that, for each $i \in T$, $s^*_i \in \arg\max_{s_i \in S_i} f(s_i, s^*_{-i,j}, s^*_j)$. Observe that $f(\tilde{s}_i, s^*_{-i,j}, s^*_j) = f(s^*_T, s^*_j) = v^*$ for each $i \in T$. Indeed, if there exists $i \in T$ such that $\tilde{s}_i \neq s^*_i$ and $f(\tilde{s}_i, s^*_{-i,j}, s^*_j) < v^*$, then $i \in T$ prefers s^*_i to $\tilde{s}_i$ given s^*_j and $s^*_{-i,j}$, a contradiction. And, if there exists $i \in T$ such that $\tilde{s}_i \neq s^*_i$ and $f(\tilde{s}_i, s^*_{-i,j}, s^*_j) > v^*$ then $s^*_j \notin H_i(\tilde{s}_i)$, a contradiction. Thus, for each $i \in T$ we have $s^*_i \in \arg\max_{s_i \in S_i} f(s_i, s^*_{-i,j}, s^*_j)$. Next, since s^*_T is a team-maximin profile with value v^* and $f(s^*_T, s^*_j) = v^*$ we have $s^*_j \in \arg\min_{s_j \in S_j} f(s^*_T, s_j)$. Hence, (s^*_T, s^*_j) is a team-maximin equilibrium.

It remains to show that ψ has a fixed point. It is clear that ψ is upper hemicontinuous since

ψ is a product of upper hemicontinuous best reply correspondences and a correspondence ψ_j which is an intersection of upper hemicontinuous correspondences $H_i : S_i \rightrightarrows S_j$. Moreover, it has convex values as a product of convex valued correspondences. The set S is a nonempty compact convex subset of a locally convex Hausdorff space. Then, if ψ has nonempty convex compact values, it has a closed graph and thus ψ has a fixed point by Kakutani-Fan-Glicksberg Theorem.

So, to guarantee the existence of a fixed point of ψ it suffices to show that ψ has nonempty values. For each $i \in T$ the set $\psi_i(s)$ is nonempty for all $s \in S$ by Weierstrass' Theorem. We assert now that $\bigcap_{i\in T} H_i(s_i) \neq \emptyset$ for all $s_T \in S_T$. Suppose that there exists $\bar{s}_T \in S_T$ such that $\bigcap_{i\in T} H_i(\bar{s}_i) = \emptyset$. For all $s_j \in S_j$, define a vector $\mathbf{f}(s_j) := (f(\bar{s}_i, s^*_{-i,j}, s_j))_{i\in T} \in \mathbb{R}^{n-1}$. Let $K := \{\mathbf{f}(\bar{s}_T, s_j) \in \mathbb{R}^{n-1} | s_j \in S_j\}$ and $D := \{\mathbf{y} \in \mathbb{R}^{n-1} | y_i \leq v^*\}$. Note that K is convex and compact subset of $\mathbb{R}^{n-1}$ since S_j is compact and convex and f is linear in $s_j \in S_j$. Obviously, D is a convex and closed subset of $\mathbb{R}^{n-1}$. Moreover, since there does not exist $s_j \in H_i(\bar{s}_i)$ for each $i \in T$, we have $K \cap D = \emptyset$. By Separating Hyperplane Theorem there exists a linear functional $\pi := (\pi_i)_{i\in T}$ on $\mathbb{R}^{n-1}$ strongly separating K and D, which clearly can be taken to satisfy $\sum \pi_i = 1$ and $\pi_i \geq 0$. Define $\hat{v} := \min_{\mathbf{y}\in K} \pi\mathbf{y} > v^*$.

For $\delta > 0$, define $s^\delta_T := [(1 - \delta\pi_i)s^*_i + \delta\pi_i\bar{s}_i]_{i\in T}$. We shall show that if $\delta > 0$ is sufficiently small, then $v^* < f(s^\delta_T, s_j)$ for all $s_j \in S_j$. Let $\hat{S}_T := \{s_T \in S_T |\text{there exist } i, k \in T \text{ such that } s_i \neq s^*_i \text{ and } s_k \neq s^*_k\}$. By multi-linearity of f we can write

$$\begin{aligned}
f(s^\delta_T, s_j) &= \left(\prod_{i\in T}(1-\delta\pi_i)\right) f(s^*_T, s_j) + \delta^2 \sum_{s_T\in\hat{S}_T} \frac{\lambda(s_T,\delta)}{\delta^2} f(s_T, s_j) + \dots \\
&\quad + \sum_{i\in T}\left(\delta\pi_i \prod_{k\neq i}(1-\delta\pi_k)\right) f(\bar{s}_i, s^*_{-i,j}, s_j) \\
&= \left(\prod_{i\in T}(1-\delta\pi_i)\right) f(s^*_T, s_j) + \delta^2 \sum_{s_T\in\hat{S}_T} \frac{q(s_T,\delta)}{\delta^2} f(s_T, s_j) + \dots \\
&\quad + \left(\sum_{i\in T}\delta\pi_i \prod_{k\neq i}(1-\delta\pi_k)\right) \sum_{i\in T}\left(\frac{\pi_i\prod_{k\neq i}(1-\delta\pi_k)}{\sum_{i\in T}\pi_i\prod_{k\neq i}(1-\delta\pi_k)}\right) f(\bar{s}_i, s^*_{-i,j}, s_j).
\end{aligned}$$

where $q(\cdot)$ is a coefficient.[8] Observe that $\left(\frac{\pi_i\prod_{k\neq i}(1-\delta\pi_k)}{\sum_{i\in T}\pi_i\prod_{k\neq i}(1-\delta\pi_k)}\right)_{i\in T} \to \pi$ as $\delta \to 0$. So, there exists $\delta' > 0$ such that, for all $\delta < \delta'$,

$$\sum_{i\in T}\left(\frac{\pi_i\prod_{k\neq i}(1-\delta\pi_k)}{\sum_{i\in T}\pi_i\prod_{k\neq i}(1-\delta\pi_k)}\right) f(\bar{s}_i, s^*_{-i,j}, s_j) \geq \frac{\hat{v}-v^*}{2}.$$

[8]See Supplement for a detailed explanation.

Thus for all $\delta < \delta'$ we obtain an inequality

$$f(s_T^\delta, s_j) \geq v^* \prod_{i \in T}(1 - \delta\pi_i) + \check{v}\delta^2 \sum_{s_T \in \hat{S}_T} \frac{q(s_T, \delta)}{\delta^2} + \left(\frac{\hat{v} - v^*}{2}\right) \delta \sum_{i \in T} \left(\pi_i \prod_{k \neq i}(1 - \delta\pi_k)\right)$$

for all $s_j \in S_j$, where $\check{v} := \min_{s \in S} f(s)$ and so $\check{v} \leq v^* < \frac{\hat{v}-v^*}{2}$. Since $q(\cdot)$ is a bounded function of δ, it follows that $\frac{\delta \sum_{i \in T}\left(\pi_i \prod_{k \neq i}(1-\delta\pi_k)\right)}{\delta^2 \sum_{s_T \in \hat{S}_T} \frac{q(s_T,\delta)}{\delta^2}} \to \infty$ as $\delta \to 0$. Therefore, there exists $\bar{\delta} > 0$ such that, for all $\delta < \min\{\bar{\delta}, \delta'\}$, we have $f(s_T^\delta, s_j) > v^*$ for all $s_j \in S_j$, which contradicts to s_T^* being a team-maximin profile. Thus $\bigcap_{i \in T} H_i(s_i) \neq \emptyset$ for each $s_T \in S_T$. □

Proof of Lemma 3. Fix a canonical ε-elaboration $(\mathbf{u}, P)$ of $\mathbf{g}$. Consider a complete information game $\mathbf{V} := (N, (\bar{\Sigma}_i)_{i \in N}, ((V)_{i \in T}, -V))$. For each $i \in N$, the strategy set $\bar{\Sigma}_i$ is a convex subset of locally convex Hausdorff space and is compact in a product topology by Tyhonoff's theorem. The payoff function V is continuous and linear in a strategy of each player. By Lemma **??** there exists $\sigma_T^* \in \arg\max_{\sigma_T \in \bar{\Sigma}_T} \min_{\sigma_j \in \bar{\Sigma}_j} V(\sigma_T, \sigma_j)$ and $\sigma_j^* \in \Xi_j$ such that (σ_T^*, σ_j^*) is a Nash equilibrium of $\mathbf{V}$. We will show that σ^* is also a quasi-saddle point of $(\mathbf{u}, P)$. It suffices to show that σ^* is a Bayesian Nash equilibrium of $(\mathbf{u}, P)$.

First, for each $i \in N$ and all $\theta_i \in \Theta_i \setminus \bar{\Theta}_i$, we have $\sigma_i^*(\theta_i) \in \arg\max u_i(\sigma_{-i}^*, \theta_i)$ by definition of $\bar{\Sigma}_i$. Next, we need to show that, for each $i \in N$,

$$\sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) \sum_{a \in A} \sigma_i^*(a_i|\theta_i)\sigma_{-i}^*(a_{-i}|\theta_{-i})u_i(a, \theta) \geq \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) \sum_{a \in A} \sigma_i(a_i|\theta_i)\sigma_{-i}^*(a_{-i}|\theta_{-i})u_i(a, \theta), \quad (2)$$

for all $\theta_i \in \bar{\Theta}_i$ and $\sigma_i(\theta_i) \in \Delta(A_i)$. Fix $i \in T$. We have $V(\sigma_i^*, \sigma_{-i}^*) \geq V(\sigma_i, \sigma_{-i}^*)$ for all $\sigma_i \in \bar{\Sigma}_i$. We can rewrite it as

$$\sum_{\theta_i \in \bar{\Theta}_i} P_i(\theta_i) \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) \sum_{a \in A} \sigma_i^*(a_i|\theta_i)\sigma_{-i}^*(a_{-i}|\theta_{-i})f(a) \geq \sum_{\theta_i \in \bar{\Theta}_i} P_i(\theta_i) \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) \sum_{a \in A} \sigma_i(a_i|\theta_i)\sigma_{-i}^*(a_{-i}|\theta_{-i})f(a)$$

for all $\sigma_i \in \bar{\Sigma}_i$. Inequality (2) follows for each $i \in T$ since f is a saddle function of $\mathbf{g}$. A symmetric argument for j concludes the proof. □

B Appendix: generalized saddle functions

First we describe the domain of a saddle function. For each $i \in N$, let $\mathcal{A}_i \subseteq 2^{A_i} \backslash \emptyset$ be a covering of A_i, i.e. $\mathcal{A}_i$ is a collection of nonempty subsets of A_i such that $\bigcup_{X_i \in \mathcal{A}_i} X_i = A_i$.

Given a nonempty subset $S \subseteq N$ and $\Lambda_S \in \Delta(\mathcal{A}_S)$ we write

$$\Delta_{\Lambda_S}(A_S) := \{\lambda \in \Delta(A_S) | \lambda(a_S) = \sum_{X_S \in \mathcal{A}} \Lambda_S(X_S)\lambda^{X_S}(a_S) \text{ for each } a_S \in A_S \text{ and}$$
$$\lambda^{X_S} \in \Delta(A_S) \text{ with } \sum_{a_S \in X_S} \lambda^{X_S}(a_S) = 1 \text{ for each } X_S \in \mathcal{A}_S\}$$

for the set of distributions over A_S induced by Λ_S.

Definition 3. Let $j \in N$ and $T := N \backslash \{j\}$. A function $F : \mathcal{A} \to \mathbb{R}$ is a *generalized saddle function (GS-function)* of $\mathbf{g}$ if for each $i \in T$, all $Q_i \in \Delta(\mathcal{A}_{-i})$ and all $q_i \in \Delta_{Q_i}(A_{-i})$,

$$X_i \cap \arg\max_{a'_i \in A_i} \sum_{a_{-i} \in A_{-i}} q_i(a_{-i}) g_i(a'_i, a_{-i}) \neq \emptyset$$
$$\text{for every } X_i \in \arg\max_{X'_i \in \mathcal{A}_i} \sum_{X_{-i} \in \mathcal{A}_{-i}} Q_i(X_{-i}) F(X'_i, X_{-i})$$

such that X_i is maximal in the argmax set ordered by the set inclusion relation; and for j and all $Q_j \in \Delta(\mathcal{A}_T)$ and all $q_j \in \Delta_{Q_j}(A_T)$,

$$X_j \cap \arg\max_{a'_j \in A_j} \sum_{a_T \in A_T} q_j(a_T) g_j(a'_j, a_T) \neq \emptyset$$
$$\text{for every } X_j \in \arg\min_{X'_j \in \mathcal{A}_j} \sum_{X_T \in \mathcal{A}_T} Q_j(X_T) F(X'_j, X_T)$$

such that X_j is maximal in the argmin set ordered by the set inclusion relation.

A *generalized value of* F is $v^* := \max_{\Lambda_T \in \Delta(\mathcal{A}_T)} \min_{\Lambda_j \in \Delta(\mathcal{A}_j)} \sum_{X \in \mathcal{A}} \Lambda_T(X_T)\Lambda_j(X_j)F(X)$. A profile $(\Lambda^*_i)_{i \in N} \in \prod_{i \in N} \Delta(\mathcal{A}_i)$ is a *generalized saddle point (GS-point)* of F if

$$v^* = \sum_{X \in \mathcal{A}} \left(\prod_{i \in N} \Lambda^*_i(X_i) \right) F(X),$$
$$\Lambda^*_j \in \arg\min_{\Lambda_j \in \Delta(\mathcal{A}_j)} \sum_{X \in \mathcal{A}} \left(\prod_{i \in T} \Lambda^*_i(X_i) \right) \Lambda_j(X_j) F(X),$$
$$\Lambda^*_i \in \arg\max_{\Lambda_i \in \Delta(\mathcal{A}_i)} \sum_{X \in \mathcal{A}} \Lambda_i(X_i) \left(\prod_{i \neq k} \Lambda_k(X_k) \right) F(X) \text{ for each } i \in T.$$

In what follows we fix a player j in N and a saddle function F. Therefore, abusing notation, we omit reference to F when discussing GS-points, the value and other notions.

We restrict attention to a special class of GS-functions such that, for each $i \in N$, each element $X_i \in \mathcal{A}_i$ is maximal in $\mathcal{A}_i$ ordered by the set inclusion relation, i.e., for each $X_i \in \mathcal{A}_i$, we have $X_i \nsubseteq X_i'$ for all $X_i' \in \mathcal{A}_i$ with $X_i' \neq X_i$. Suppose $\mathbf{g}$ has such a GS-function $F : \mathcal{A} \to \mathbb{R}$ and GS-point $(\Lambda_i^*)_{i \in N}$. Consider a set of correlated equilibria

$$\begin{aligned}\mathcal{E} : \;=\; & \{\mu \in \Delta(A) | \mu \text{ is a correlated equilibrium of } \mathbf{g} \text{ and } \; \mu \in \Delta_\Lambda \\ & \text{for } \Lambda \in \Delta(\mathcal{A}) \text{ such that } \sum_{X \in \mathcal{A}} \Lambda(X) F(X) = v^*\}.\end{aligned}$$

We obtain the result analogous to Theorem 1.

Theorem 4. *If* $\mathbf{g}$ *has a GS-function* $F : \mathcal{A} \to \mathbb{R}$ *such that for each* $X_i \in \mathcal{A}_i$*, we have* $X_i \nsubseteq X_i'$ *for each* $X_i' \in \mathcal{A}_i$ with $X_i' \neq X_i$ *and GS-point, then* $\mathcal{E}$ *is robust to canonical elaborations in* $\mathbf{g}$*.*

In the reminder of the section we prove Theorem 4. Let $\mathbf{g}$ be a game with GS-function F and $(\mathbf{u},P)$ be a canonical ε-elaboration of $\mathbf{g}$. For each $i \in N$, consider a collection of mappings

$$\begin{aligned}\Xi_i = \{ & \xi_i : \Theta_i \to \Delta(\mathcal{A}_i) | \text{for all } \theta_i \in \Theta_i \setminus \bar{\Theta}_i, \\ & \text{if } \xi_i(X_i|\theta_i) > 0, \text{ then } X_i \text{ contains every undominated action of type } \theta_i\}.\end{aligned}$$

Define a function $V : \Xi \to \mathbb{R}$ by

$$V(\xi) := \sum_{\theta \in \Theta} \sum_{X \in \mathcal{A}} P(\theta) \xi(X|\theta) F(X).$$

A profile $\xi \in \Xi$ is a *generalized quasi-saddle point* (GQS-point) of an ε-elaboration $(\mathbf{u},P)$ of $\mathbf{g}$ if

$$\begin{aligned}& \xi_T \in \arg\max_{\xi_T' \in \Xi_T} \min_{\xi_j' \in \Xi_j} V(\xi_T', \xi_j'), \\ & \xi_j \in \arg\min_{\xi_j' \in \Xi_j} V(\xi_T, \xi_j'), \\ & \xi_i \in \arg\max_{\xi_i' \in \Xi_i} V(\xi_i', \xi_{-i}) \text{ for all } i \in T.\end{aligned}$$

Note that GS-point of $\mathbf{g}$ is GQS-point of a degenerate 0-elaboration of $\mathbf{g}$ where Θ_i is a singleton for each $i \in N$. The proof of the existence of GQS-point is analogous to the proof of Lemma 22.

Lemma 4. *Every canonical* ε*-elaboration of* $\mathbf{g}$ *has GQS-point.*

For $i \in N$ let $\Sigma_i(\xi_i) := \prod_{\theta_i \in \Theta_i} \Delta_{\xi_i(\theta_i)}(A_i)$ be a set of strategies in $(\mathbf{u}, P)$ induced by $\xi_i \in \Xi_i$. Let $\Sigma(\xi) := \prod_{i \in N} \Sigma_i(\xi_i)$ for all $\xi \in \Xi$. Next we show that, for each GQS-point $\xi^* \in \Xi$ there exists a Bayesian Nash equilibrium of an ε-elaboration $(\mathbf{u}, P)$ in $\Sigma(\xi^*)$. This result is analogous to Lemma 6 in Morris and Ui [8].

Lemma 5. *If $\xi^* \in \Xi$ is a GQS-point of a canonical ε-elaboration of $\mathbf{g}$, then there exists a Bayesian Nash equilibrium in $\Sigma(\xi^*)$.*

Proof. Suppose that $\xi^* \in \Xi$ is a GQS-point of a canonical ε-elaboration $(\mathbf{u}, P)$ of $\mathbf{g}$. For each $i \in N$, let a correspondence $\beta_i : \Sigma_{-i}(\xi^*_{-i}) \rightrightarrows \Sigma_i(\xi^*_i)$ be such that $\beta_i(\sigma_{-i}) = \arg\max_{\sigma_{i\in\Sigma_i}} u_i(\sigma_i, \sigma_{-i}) \cap \Sigma_i(\xi^*_i)$, for all $\sigma_{-i} \in \Sigma_{-i}(\xi^*_{-i})$; and let $\beta : \Sigma(\xi^*) \rightrightarrows \Sigma(\xi^*)$ be such that $\beta(\sigma) = \prod_{i\in N} \beta_i(\sigma_{-i})$ for all $\sigma \in \Sigma$. Thus β is a best response correspondence restricted to $\Sigma(\xi^*)$. We show that there exists a fixed point $\sigma^* \in \Sigma(\xi^*)$ of β, which must be a Bayesian Nash equilibrium of $(\mathbf{u}, P)$. Notice that for each $i \in N$ a set $\Sigma_i(\xi^*_i) \subseteq \Sigma_i$ is compact and convex. Moreover it is easy to show that β has closed graph and convex values. Below we show that β has nonempty values.

Fix $i \in T$ and $\sigma_{-i} \in \Sigma_{-i}(\xi^*_{-i})$. First we show that, for each $\theta_i \in \Theta_i$ and each $X_i \in \mathcal{A}_i$ such that $\xi^*_i(X_i|\theta_i) > 0$ we have

$$X_i \cap \arg\max_{a_i\in A_i} \sum_{\theta_{-i}\in\Theta_{-i}} \sum_{a_{-i}\in A_{-i}} P(\theta_{-i}|\theta_i)\sigma_{-i}(a_{-i}|\theta_{-i})u_i(a_i, a_{-i}, \theta) \neq \emptyset. \tag{3}$$

Suppose that $\theta_i \in \Theta_i \setminus \bar{\Theta}_i$. Then (1) holds since each $X_i \in \mathcal{A}_i$ such that $\xi^*_i(X_i|\theta_i) > 0$ contains every undominated action of $\theta_i \in \Theta_i \setminus \bar{\Theta}_i$.

Suppose that $\theta_i \in \bar{\Theta}$. By definition of a GQS-point, we have

$$\xi^*_i \in \arg\max_{\xi_i\in\Xi_i} \sum_{\theta\in\Theta} \sum_{X\in\mathcal{A}} P(\theta_i, \theta_{-i})\xi_i(X_i|\theta_i)\xi^*_{-i}(X_{-i}|\theta_{-i})F(X),$$

which implies that

$$\xi^*_i(\theta_i) \in \arg \max_{\xi_i(\theta_i)\in\Delta(\mathcal{A}_i)} \sum_{\theta_{-i}\in\Theta_{-i}} \sum_{X\in\mathcal{A}} P(\theta_{-i}|\theta_i)\xi_i(X_i|\theta_i)\xi^*_{-i}(X_{-i}|\theta_{-i})F(X).$$

Therefore, for each $X_i \in \mathcal{A}_i$ such that $\xi^*_i(X_i|\theta_i) > 0$, we must have

$$\begin{aligned} X_i &\in \arg\max_{X'_i\in\mathcal{A}_i} \sum_{X_{-i}\in\mathcal{A}_{-i}} \left(\sum_{\theta_{-i}\in\Theta_{-i}} P(\theta_{-i}|\theta_i)\xi^*_{-i}(X_{-i}|\theta_{-i}) \right) F(X'_i, X_{-i}) \\ &= \arg\max_{X'_i\in\mathcal{A}_i} \sum_{X_{-i}\in\mathcal{A}_{-i}} Q_i^{\theta_i}(X_{-i})F(X'_i, X_{-i}), \end{aligned}$$

where $Q_i^{\theta_i}(X_{-i}) := \sum_{\theta_{-i}\in\Theta_{-i}} P(\theta_{-i}|\theta_i)\xi^*_{-i}(X_{-i}|\theta_{-i})$ for all $X_{-i} \in \mathcal{A}_{-i}$.

By restriction on the domain $\mathcal{A}$ it follows that, for each $X_i \in \mathcal{A}_i$ with $\xi^*_i(X_i|\theta_i) > 0$, all

$\theta_{-i} \in \Theta_{-i}$, and all $q_i^{\theta_i} \in \Delta_{Q_i^{\theta_i}}(A_{-i})$, by definition of a GS-function,

$$
\begin{aligned}
X_i \cap \arg\max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} q_i^{\theta_i}(a_{-i}) u_i(a_i, a_{-i}, \theta) &= X_i \cap \arg\max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} q_i^{\theta_i}(a_{-i}) g_i(a_i, a_{-i}) \\
&\neq \emptyset
\end{aligned}
$$

It remains to show that $\sigma_{-i} \in \Sigma_{-i}(\xi_{-i}^*)$ induces a distribution in the set $\Delta_{Q_i^{\theta_i}}(A_{-i})$, i.e. if $q_i(a_{-i}) = \sum_{\theta_{-i}} P(\theta_{-i}|\theta_i)\sigma_{-i}(a_{-i}|\theta_{-i})$ for all $a_{-i} \in A_{-i}$, then $q_i \in \Delta_{Q_i^{\theta_i}}(A_{-i})$. To see this, consider a distribution $q_i^{X_{-i}} \in \Delta(A_{-i})$ such that

$$
q_i^{X_{-i}}(a_{-i}) := \begin{cases} \frac{\sum_{\theta_{-i}} P(\theta_{-i}|\theta_i)\sigma_{-i}(a_{-i}|\theta_{-i})}{Q_i^{\theta_i}(X_{-i})} & \text{if } Q_i^{\theta_i}(X_{-i}) \neq 0, \\ \frac{1}{|X_{-i}|} & \text{if } Q_i^{\theta_i}(X_{-i}) = 0 \text{ and } a_{-i} \in X_{-i}, \\ 0 & \text{if } Q_i^{\theta_i}(X_{-i}) = 0 \text{ and } a_{-i} \notin X_{-i} \end{cases}
$$

for all $a_{-i} \in A_{-i}$. Now we can write

$$
q_i(a_{-i}) = \sum_{X_{-i}} Q_i^{\theta_i}(X_{-i}) q_i^{X_{-i}}(a_{-i}).
$$

Thus, $q_i \in \Delta_{Q_i^{\theta_i}}(A_{-i})$ by definition of $\Delta_{Q_i^{\theta_i}}(A_{-i})$. It follows that

$$
X_i \cap \arg\max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} q_i(a_{-i}) u_i(a_i, a_{-i}, \theta) \neq \emptyset,
$$

hence

$$
X_i \cap \arg\max_{a_i \in A_i} \sum_{\theta_{-i} \in \Theta_{-i}} \sum_{a_{-i} \in A_{-i}} P(\theta_{-i}|\theta_i)\sigma_{-i}(a_{-i}|\theta_{-i}) u_i(a_i, a_{-i}, \theta) \neq \emptyset.
$$

Now we show that, for each $i \in T$, for all $\sigma_{-i} \in \Sigma_{-i}(\xi_{-i}^*)$, there exists a best response in the set $\Sigma_i(\xi_i^*)$. Since $\Sigma_i(\xi_i) := \prod_{\theta_i \in \Theta_i} \Delta_{\xi_i(\theta_i)}(A_i)$, for each $\theta_i \in \Theta_i$ we have

$$
\Delta_{\xi_i^*(\theta_i)}(A_i) \cap \arg\max_{a_i \in A_i} \sum_{\theta_{-i} \in \Theta_{-i}} \sum_{a_{-i} \in A_{-i}} P(\theta_{-i}|\theta_i)\sigma_{-i}(a_{-i}|\theta_{-i}) u_i(a_i, a_{-i}, \theta) \neq \emptyset.
$$

Fix $i \in T$ and $\sigma_{-i} \in \Sigma_{-i}(\xi_{-i}^*)$. For each $\theta_i \in \Theta_i$ and $X_i \in \mathcal{A}_i$ such that $\xi_i^*(X_i|\theta_i) > 0$ consider a nonempty set

$$
\bar{X}_i := X_i \cap \arg\max_{a_i \in A_i} \sum_{\theta_{-i} \in \Theta_{-i}} \sum_{a_{-i} \in A_{-i}} P(\theta_{-i}|\theta_i)\sigma_{-i}(a_{-i}|\theta_{-i}) u_i(a_i, a_{-i}, \theta).
$$

And a set of distributions for each $X_i \in \mathcal{A}_i$

$$M(X_i,\theta_i) := \begin{cases} \{\lambda_i^{X_i} \in \Delta(A_i) | \sum_{a_i \in \bar{X}_i} \lambda_i^{X_i}(a_i) = 1\} & \text{if } \xi_i^*(X_i|\theta_i) \neq 0, \\ \{\lambda_i^{X_i} \in \Delta(A_i) | \sum_{a_i \in X_i} \lambda_i^{X_i}(a_i) = 1\} & \text{if } \xi_i^*(X_i|\theta_i) = 0. \end{cases}$$

The strategy profile σ_i such that for each θ_i and a_i,

$$\sigma_i(a_i|\theta_i) := \sum_{X_i} \xi_i^*(X_i|\theta_i)\lambda_i^{X_i}(a_i|\theta_i)$$

where $\lambda_i^{X_i}(a_i|\theta_i) \in M(X_i,\theta_i)$, belongs to $\Sigma_i(\xi_i^*)$. It is also a best response to σ_{-i} since each $a_i \in A_i$ such that $\sigma_i(a_i|\theta_i) > 0$ belongs to some $\bar{X}_i$. Hence, for each $i \in T$ we established that β_i has nonempty values. We can use a similar argument to show that β_j has nonempty values. Thus by Kakutani-Fan-Glicksberg fixed point theorem there exists a Bayesian Nash equilibrium in $\Sigma(\xi^*)$. □

We need the characterization of $\Delta_\Lambda(A)$ given in Lemma 3 in Morris and Ui [8].

Lemma 6. *For all $\Lambda \in \Delta(\mathcal{A})$ we have $\lambda \in \Delta_\Lambda(A)$ if and only if*

$$\sum_{a \in B} \lambda(a) \quad \geq \quad \sum_{\substack{X \in \mathcal{A} \\ X \subseteq B}} \Lambda(X)$$

for every $B \in 2^A$.

Now we are ready to prove Theorem 4.

Proof of Theorem 4. Suppose that $(\Lambda_i^*)_{i \in N} \in \prod_{i \in N} \Delta(\mathcal{A}_i)$ is a saddle point of $\mathbf{g}$. Fix a canonical ε-elaboration $(\mathbf{u}, P)$ of $\mathbf{g}$. By Lemma 4 there exists a GQS-point $\xi \in \Xi$ of $(\mathbf{u}, P)$. First, we shall find a lower bound and an upper bounds on $V(\xi)$. For each $i \in T$ and each $\theta_i \in \bar{\Theta}_i$, let $\xi_T^* \in \Xi_T$ be such that $\xi_i^*(X_i|\theta_i) = \Lambda_i^*(X_i)$ for each $X_i \in \mathcal{A}_i$. Since $\xi_T \in \arg\max_{\xi'_T \in \Xi_T} \min_{\xi'_j \in \Xi_j} V(\xi'_T, \xi'_j)$, we have $V(\xi_T, \xi_j) \geq \min_{\xi'_j \in \Xi_j} V(\xi_T^*, \xi'_j)$.

Let $\varepsilon_T \geq 0$ be a marginal probability that one of the players in T is of committed type, i.e.,

$\varepsilon_T := P((\Theta_T \setminus \bar{\Theta}_T) \times \Theta_j)$. By definition of V for all $\xi'_j \in \Xi_j$ we have

$$\begin{aligned} V(\xi^*_T, \xi'_j) = & \sum_{\theta \in (\Theta_T \setminus \bar{\Theta}_T) \times \Theta_j} P(\theta) \sum_{X \in \mathcal{A}} \xi^*_T(X_T|\theta_T) \xi'_j(X_j|\theta_j) F(X) \\ & + \sum_{\theta \in \bar{\Theta}_T \times \Theta_j} P(\theta) \sum_{X \in \mathcal{A}} \left(\prod_{i \in T} \Lambda^*_i(X_i) \right) \xi'_j(X_j|\theta_j) F(X) \\ & \geq \varepsilon_T F_{min} + \sum_{\theta \in \bar{\Theta}_T \times \Theta_j} P(\theta) \sum_{X \in \mathcal{A}} \left(\prod_{i \in T} \Lambda^*_i(X_i) \right) \xi'_j(X_j|\theta_j) F(X), \end{aligned}$$

where $F_{min} := \min_{X \in \mathcal{A}} F(X)$. Observe that $\sum_{X \in \mathcal{A}} \left(\prod_{i \in T} \Lambda^*_i(X_i) \right) \Lambda_j(X_j) F(X) \geq v^*$ for all $\Lambda_j \in \Delta(\mathcal{A}_j)$, since $(\Lambda^*_i)_{i \in N} \in \prod_{i \in N} \Delta(\mathcal{A}_i)$ is a GS-point. It follows that

$$\varepsilon_T F_{min} + (1 - \varepsilon_T) v^* \leq \varepsilon_T F_{min} + \sum_{\theta \in \bar{\Theta}_T \times \Theta_j} P(\theta) \sum_{X \in \mathcal{A}} \left(\prod_{i \in T} \Lambda^*_i(X_i) \right) \xi_j(X_j|\theta_j) F(X).$$

Therefore $\varepsilon_T F_{min} + (1 - \varepsilon_T) v^* \leq V(\xi)$. By the symmetric argument for player j we get $V(\xi) \leq \varepsilon_j F_{max} + (1 - \varepsilon_j) v^*$, where $F_{max} := \max_{X \in \mathcal{A}} F(X)$ and $\varepsilon_j := P(\Theta_T \times (\Theta_j \setminus \bar{\Theta}_j))$. Combining lower and upper bounds we obtain

$$\varepsilon_T (F_{min} - v^*) \leq V(\xi) - v^* \leq \varepsilon_j (F_{max} - v^*). \tag{4}$$

To complete the proof we show that for each $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \leq \bar{\varepsilon}$, each canonical ε-elaboration of $\mathbf{g}$ has a Bayesian Nash equilibrium $\sigma \in \Sigma$ with $\max_{a \in A} |\mu(a) - \sum_{\theta \in \Theta} P(\theta) \sigma(a|\theta)| \leq \delta$ for some $\mu \in \mathcal{E}$.

To get a contradiction suppose that for some $\delta > 0$ there exists a sequence $\{(\mathbf{u}^m, P^m)\}$ of canonical ε-elaboration of $\mathbf{g}$ with $\varepsilon^m \to 0$ such that $\max_{a \in A} | \sum_{\theta \in \Theta} P^m(\theta) \sigma^m(a|\theta) - \mu(a)| > \delta$ for all $\mu \in \mathcal{E}$, where $\sigma^m \in \Sigma(\xi^m)$ is a Bayesian Nash equilibrium of $(\mathbf{u}^m, P^m)$ and $\xi \in \Xi$ is a GQS-point of $(\mathbf{u}^m, P^m)$. A Bayesian Nash equilibrium $\sigma^m \in \Sigma(\xi^m)$ of $(\mathbf{u}^m, P^m)$ exists by Lemma 5. By (4) and compactness of $\Delta(\mathcal{A})$ there exist a subsequence $\{\xi^k\}$ in Ξ and $\Lambda \in \Delta(\mathcal{A})$ such that $\sum_{\theta \in \Theta} P^k(\theta) \xi^k(X|\theta) \to \Lambda(X)$ for each $X \in \mathcal{A}$ and $\sum_{X \in \mathcal{A}} \Lambda(X) F(X) = v^*$. By Lemma 1 there exist a subsequence $\{\sigma^l\}$ of $\{\sigma^k\}$ and a correlated equilibrium $\nu \in \Delta(A)$ of $\mathbf{g}$ such that $\sum_{\theta \in \Theta} P^l(\theta) \sigma^l(a|\theta) \to \nu(a)$ for all $a \in A$.

Now we demonstrate that $\nu \in \Delta_\Lambda$. By Lemma 6 it's enough to show that

$$\sum_{a \in B} \nu(a) \geq \sum_{\substack{X \in \mathcal{A} \\ X \subseteq B}} \Lambda(X)$$

for each $B \in 2^A$. But it follows immediately since, for each $l = 1, 2, \ldots$ and each $B \in 2^A$,

$$\sum_{a \in B} \nu^l(a) \geq \sum_{\substack{X \in \mathcal{A} \\ X \subseteq B}} \sum_{\theta \in \Theta} P^l(\theta) \xi^l(X|\theta).$$

Therefore $\nu \in \Delta_\Lambda$ and so $\nu \in \mathcal{E}$. The contradiction completes the proof. □

C Supplement for the proof of Lemma 2

By linearity of f in s_i we can write

$$\begin{aligned} f(s_T^\delta, s_j) &= f((1-\delta\pi_i)s_i^* + \delta\pi_i\bar{s}_i, s_{-i,j}^\delta, s_j) \\ &= (1-\delta\pi_i) f(s_i^*, s_{-i,j}^\delta, s_j) + \delta\pi_i f(\bar{s}_i, s_{-i,j}^\delta, s_j). \end{aligned}$$

By linearity of f in s_m we can further write

$$\begin{aligned} f(s_T^\delta, s_j) &= (1-\delta\pi_i) f(s_i^*, (1-\delta\pi_m)s_m^* + \delta\pi_m\bar{s}_m, s_{-i,j,m}^\delta, s_j) + \\ &\quad + \delta\pi_i f(\bar{s}_i, (1-\delta\pi_m)s_m^* + \delta\pi_m\bar{s}_m, s_{-i,j,m}^\delta, s_j) \\ &= (1-\delta\pi_i)(1-\delta\pi_m) f(s_i^*, s_m^*, s_{-i,j,m}^\delta, s_j) + (1-\delta\pi_i)\delta\pi_m f(s_i^*, \bar{s}_m, s_{-i,j,m}^\delta, s_j) + \\ &\quad + \delta\pi_i(1-\delta\pi_m) f(\bar{s}_i^*, s_m^*, s_{-i,j,m}^\delta, s_j) + \delta^2\pi_i\pi_m f(\bar{s}_i^*, \bar{s}_m, s_{-i,j,m}^\delta, s_j). \end{aligned}$$

We can proceed with expansion for all $n-1$ members of T. Therefore we get 2^{n-1} terms in total. For $s_T \in S_T$, let $I(s_T) \subseteq T$ be the set of players who deviate from s_T^* in a profile $s_T \in S_T$. The coefficient of $s_T \in S_T$ is given by

$$q(s_T) = \prod_{i \in I(s_T)} \delta\pi_i \prod_{i \in T \setminus I(s_T)} (1-\delta\pi_i). \tag{5}$$

In proof we defined a set of players where more than one player deviates by $\hat{S}_T := \{s_T \in S_T | \exists i, k \in T \text{ such that } s_i \neq s_i^*, s_k \neq s_k^*\}$. From (B.1) it's obvious that the coefficients are well defined and are bounded functions of $\delta \in [0, 1]$.

D Appendix: examples of games with saddle functions

Example 3 (Exchange economies with quasi-linear utility functions)**.** Consider an exchange economy with quasi-linear utility functions. There are traders $1, ..., n$ and commodities $1, ..., L, L+1$. The consumption space of the first L commodities is $\mathbb{R}_+^L$ and that of the last commodity $L+1$, called the numeraire, is $\mathbb{R}$. Each trader i has a strictly increasing utility function $u_i : \mathbb{R}^L_+ \times \mathbb{R} \rightarrow \mathbb{R}$ such that $u_i(x_i, m_i) = v_i(x_i) + m_i$, where $x_i \in \mathbb{R}^L_+$ is i's consumption vector of L commodities and $m_i \in \mathbb{R}$ is that of the composite commodity. Each trader i endowed with $(e^1_i, ..., e^L_i, \bar{m}_i)$, where $\bar{m}_i$ is assumed to be sufficiently large it is not binding. Assuming that the composite commodity is the numeraire, the price vector of commodities in units of the composite commodity is denoted by $p \in \mathbb{R}^L$. We write $x_i = (x^l_i)^L_{l=1} \in \mathbb{R}^L_+$ and $e_i = (e^l_i)^L_{l=1} \in \mathbb{R}^L$, and $x = (x_i)^n_{i=1} \in (\mathbb{R}^L_+)^n$ and $m = (m^i)^n_{i=1} \in \mathbb{R}^n$.

The abstract economy *a la* Arrow and Debreu (1954) consists of the player set $N = \{0, 1, ..., n\}$, where player 0 is the auctioneer and player $i = 1, ..., n$ is a trader; a set $\mathbb{R}^L$ of auctioneer's actions and, for each trader $i = 1, ..., n$, a set $\mathbb{R}^L_+ \times \mathbb{R}$ of i's actions; a set $Z_0((x, m)) = \mathbb{R}^L_+$ of auctioneer's feasible actions and, for each trader $i = 1, ..., n$, a set $Z_i((x_{-i}, m_{-i}), p) = \{(x_i, m_i) \in \mathbb{R}^L_+ \times \mathbb{R} | p \cdot x_i + m_i = p \cdot e_i + \bar{m}_i\}$ of i's feasible actions; and auctioneer's payoff function $g_0(x, m, p) = \sum^n_{i=1} p \cdot (x_i - e_i) + \sum^n_{i=1}(m_i - \bar{m}_i)$ and i's payoff function $g_i(x, m, p) = v_i(x_i) + m_i$ for $i = 1, ..., n$.[9]

The abstract economy degenerates to a complete information game consisting of the player set N ; a set $\mathbb{R}^L$ of auctioneer's actions and, for trader $i = 1, ..., n$, a set $\mathbb{R}^L_+$ of i's actions; and the payoff functions

$$\begin{aligned} \hat{g}_0(p, x) &= \sum^n_{i=1} p \cdot (x_i - e_i) \\ \hat{g}_i(p, x) &= v_i(x_i) + p \cdot (e_i - x_i) + \bar{m}_i \end{aligned}$$

for $i = 1, ..., n$. Define a function $f : \mathbb{R}^L \times (\mathbb{R}^L_+)^n$ such that

$$f(p, x) = \sum^n_{i=1} \hat{g}_i(p, x).$$

For the auctioneer, for each $x \in (\mathbb{R}^L_+)^n$,

$$\hat{g}_0(p, x) - \hat{g}_0(p', x) = -[f(p, x) - f(p', x)]$$

for all $p, p' \in \mathbb{R}^L$, and for each trader $i = 1, ..., n$, for each $p \in \mathbb{R}^L$ and for each $x_{-i} \in (\mathbb{R}^L_+)^{n-1}$,

$$\hat{g}_i(p, x_i, x_{-i}) - \hat{g}_i(p, {x'}_i, x_{-i}) = f(p, x_i, x_{-i}) - f(p, {x'}_i, x_{-i})$$

for all $x_i, x'_i \in \mathbb{R}^L$. Thus, we can show that the exchange economy with quasi-linear utility

[9] We assume the consumers' budget constraints are not inequalities but equalities.

functions has a saddle function f.

Example 4 (Congestion games with an attacker)**.** Consider a congestion game with an attacker.[10] There are a finite set E of facilities and $n+1$ players. Each player $i = 1, .., n$, which we call a commuter, chooses a subset a_i of facilities, where $a_i \in A_i \subseteq 2^E$ and A_i is a set of i's actions. We write $T = \{1, \dots, n\}$. Player 0, which we call an attacker, chooses one facility a_0, where $a_0 \in A_0 = E$ and A_0 is a set of 0's actions. When m commuters uses a facility e and the attacker chooses $a_0 \in A_0$, the cost $c_e(m, a_0)$ for each user by using the facility is

$$c_e(m, a_0) = \begin{cases} d_e(m) & \text{if } a_0 \neq e \\ t + d_e(m), & \text{if } a_0 = e \end{cases},$$

where $d_e(m)$ is a facility cost for each user when the number of users of facility e is m, and $t \in \mathbb{R}$ is the amount of money robbed by the attacker when a commuter meets the attacker. For each action profile $a_T \in A_T$, let $n_e(a_T)$ be a number of users of facility e under a:

$$n_e(a_T) = \#\{i \in T | e \in a_i\}.$$

Then, for each $i \in T$, i's payoff function g_i is the sum of the costs of facilities he selects times minus one:

$$g_i(a_T, a_0) = -\sum_{e \in a_i} c_e(n_e(a_T), a_0).$$

The attacker's payoff is the sum of levies collected:

$$g_0(a, a_0) = n_{a_0}(a)t.$$

Define a function $f : A_T \times A_0 \to \mathbb{R}$ such that

$$f(a_T, a_0) = -[\sum_{e \in \cup_{i=1}^n a_i} \sum_{m=1}^{n_e(a_T)} d_e(m) + n_{a_0}(a_T)t].$$

[10] A related class of games is the congestion games with "malicious" player studied by Babaioff et al. [1].

Then, for each $i \in T$, for all $a_i, a'_i \in A_0$, all $a_{-i} \in A_{-i}$,

$$\begin{aligned}
f(a_i, a_{-i}) - f(a'_i, a_{-i}) &= -[\sum_{e \in a_i} \sum_{m=1}^{n_e(a_T)} d_e(m) + n_{a_0}(a_T)t] \\
&\quad +[\sum_{e \in a'_i} \sum_{m=1}^{n_e(a'_i, a_{T\setminus\{i\}})} d_e(m) + n_{a_0}(a'_i, a_{T\setminus\{i\}})t] \\
&= -[\sum_{e \in a_i} c_e(n_e(a_T), a_0) - \sum_{e \in a'_i} c_e(n_e(a'_i, a_{T\setminus\{i\}}), a_0)] \\
&= g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i}).
\end{aligned}$$

We also have

$$\begin{aligned}
f(a_T, a_0) - f(a_T, a'_0) &= -[n_{a_0}(a_T)t - n_{a'_0}(a_T)t] \\
&= -[g_0(a_T, a_0) - g_0(a_T, a'_0)]
\end{aligned}$$

for all $a_0, a'_0 \in A_0$ and $a_T \in A_T$. Thus, we can show that f is a saddle function of the game.

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