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The convex hull of the all-different system  
with the inclusion property: a simple proof

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**The convex hull of the all-different system  
with the inclusion property: a simple proof**

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**Abstract**

An all-different constraint for a given family of discrete variables imposes the condition that no two variables in the family are allowed to take the same value. Magos et al. [Mathematical Programming, 132 (2012), pp. 209–260] gave a linear-inequality description of the convex hull of solutions to a system of all-different constraints, under a special assumption called inclusion property. The convex hull of solutions is in this case the intersection of the convex hulls of each of the all-different constraints of the system. We give a short and simple proof of this result, that in addition shows the total dual integrality of the linear system.

**Keywords:** all-different constraint, convex hull, integral polyhedron, total dual integrality.

**Mathematics Subject Classification:** 90C10, 90C27

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# 1 Introduction

In many combinatorial optimization problems one needs to impose one or more *all-different constraints*, i.e., conditions of the following type: for a given family of discrete variables, no two variables can be assigned the same value. All-different constraints arise, for instance, in problems related to timetabling, scheduling, manufacturing, and in several variants of the assignment problem (see, e.g., [WY01, MMA12] and the references therein).

Though all-different constraints are mainly studied in the context of Constraint Programming (see, e.g., [vH01]), when dealing with a problem that can be modeled as an integer linear program it is useful to have information on the polyhedral structure of the feasible solutions to a system of all-different constraints. For this reason, several authors studied linear-inequality formulations for the convex hull of solutions to a single all-different constraint or a system of all-different constraints [MMA12, WY01, Mag13, Lee02]. We remark that in some cases these descriptions are extended formulations, i.e., they make use of additional variables; however, here we are only interested in the description of the convex hull in the original space of variables.

If  $n$  variables  $x_1, \dots, x_n$  can take values in a finite domain  $D \subseteq \mathbb{R}$  and an all-different constraint is imposed on them, we will write (following the notation of [MMA12])

$$\{x_1, \dots, x_n\} \neq \quad (1)$$

$$x_1, \dots, x_n \in D. \quad (2)$$

Williams and Yan [WY01] proved that if  $D = \{1, \dots, d\}$  for some positive integer  $d$ , then the convex hull of the vectors that satisfy (1)–(2) is described by the linear system

$$\sum_{j \in S} x_j \geq f(S), \quad S \subseteq [n], \quad (3)$$

$$\sum_{j \in S} x_j \leq g(S), \quad S \subseteq [n], \quad (4)$$

where  $[n] = \{1, \dots, n\}$  and, for  $S \subseteq [n]$ , we define

$$f(S) = \frac{|S|(|S| + 1)}{2} \quad \text{and} \quad g(S) = |S|(d + 1) - f(S). \quad (5)$$

Note that  $f(S)$  is the sum of the  $|S|$  smallest positive integers, while  $g(S)$  is the sum of the  $|S|$  largest integers that do not exceed  $d$ , therefore inequalities

(3)–(4) are certainly valid for every vector  $x$  satisfying (1)–(2). The result extends to an arbitrary finite domain  $D \subseteq \mathbb{R}$  (with  $|D| \geq n$ ) by defining  $f(S)$  (resp.,  $g(S)$ ) as the sum of the smallest (resp., largest) elements in  $D$ , for every  $S \subseteq [n]$ . Note however that in the following we assume  $D = \{1, \dots, d\}$  for some positive integer  $d$ .

Williams and Yan [WY01] showed that if  $d > n$  then all of inequalities (3)–(4) are facet-defining, thus the convex hull of (1)–(2) needs an exponential number of inequalities to be described in the original space of variables  $x_1, \dots, x_n$ . However, they also gave polynomial-size extended formulations for the convex hull of (1)–(2).

When  $n = d$ , (1)–(2) is the set of permutations of the elements in  $[n]$ , and its convex hull is called permutahedron. In this case, the whole family of inequalities (4) can be dropped and replaced by the equation  $\sum_{i \in [n]} x_i = f([n])$ . The permutahedron admits an extended formulation with  $O(n \log n)$  constraints and variables [Goe09].

System (3)–(4) not only defines an integral polyhedron, but it also has the stronger property of being *totally dual integral*. We recall that a linear system of inequalities  $Ax \leq b$  is said to be totally dual integral if for every integer vector  $c$  such that the linear program  $\max\{cx : Ax \leq b\}$  has finite optimum, the dual linear program has an optimal solution with integer components. It is known that if  $Ax \leq b$  is totally dual integral and  $b$  is an integer vector, then the polyhedron defined by  $Ax \leq b$  is integral. The total dual integrality of system (3)–(4) follows from the fact that  $f$  (resp.,  $g$ ) is a supermodular (resp., submodular) function, along with a classical result on polymatroid intersection [Edm70] (see also [Sch03, Theorem 46.2]). A complete proof of the total dual integrality of (3)–(4) can be found in [Mag13].

In a more general setting, we might have  $m \geq 1$  all-different constraints, each enforced on a different subset of variables  $N_i \subseteq [n], i \in [m]$ . In this case, we have the system of conditions

$$\{x_j : j \in N_i\}_{\neq}, \quad i \in [m], \tag{6}$$

$$x_1, \dots, x_n \in D. \tag{7}$$

The following inequalities are of course valid for the convex hull of solutions to (6)–(7):

$$\sum_{j \in S} x_j \geq f(S), \quad S \subseteq N_i, i \in [m], \tag{8}$$

$$\sum_{j \in S} x_j \leq g(S), \quad S \subseteq N_i, i \in [m]. \tag{9}$$

However, the above inequalities do not give, in general, the convex hull of the vectors that satisfy (6)–(7). Furthermore, there are examples in which some integer solutions to system (8)–(9) do not lie in the convex hull of the points satisfying (6)–(7).

A special case, studied in [MMA12], in which constraints (8)–(9) do yield the convex hull of solutions to (6)–(7) is now described. Define  $N = [n]$  and assume that  $N = T \cup U$ , where  $T$  and  $U$  are disjoint nonempty subsets of  $N$ . Define  $T_i = N_i \cap T$  and  $U_i = N_i \cap U$  for  $i \in [m]$ . If the  $T_i$ 's form a monotone family of subsets ( $T_1 \supseteq T_2 \supseteq \dots \supseteq T_m$ ) and the  $U_i$ 's are pairwise disjoint, then Magos et al. [MMA12] say that the *inclusion property* holds. They showed that in this case inequalities (8)–(9) provide the convex hull of solutions to (6)–(7).

The proof of Magos et al. [MMA12] is rather lengthy and involved (overall, it consists of about 25 pages). The purpose of this note is to give a simple proof of their result. Indeed, we show something more: we prove that, under the inclusion property, system (8)–(9) is totally dual integral. Our proof is an extension of the classical proof of the total dual integrality of polymatroids (see, e.g., [Sch03, Chapter 44]). Specifically, in Section 2 we describe a greedy algorithm that solves linear optimization over (8)–(9), under the inclusion property. The correctness of the algorithm is shown in Section 3 by completing the solution returned by the algorithm with a dual solution such that the complementary slackness conditions are satisfied. The result of Section 3 also implies the total dual integrality of system (8)–(9), as the dual solution is integer whenever the primal objective function coefficients are all integers. We conclude in Section 4 with an extension of the result.

## 2 Primal algorithm

Assume that the inclusion property holds for an all-different system (6)–(7). Recall that:

- $N = [n] = T \cup U$ , with  $T$  and  $U$  disjoint and nonempty;
- $T_i = N_i \cap T$  and  $U_i = N_i \cap U$  for  $i \in [m]$ ;
- $T_1 \supseteq \dots \supseteq T_m$ ;
- $U_i \cap U_j = \emptyset$  for all distinct  $i, j \in [m]$ .

Wlog,  $N = N_1 \cup \dots \cup N_m$  and  $T = T_1 = [t]$  for some positive integer  $t$ . Also, recall that  $D = [d]$ . We assume that  $d \geq \max_{i \in [m]} |N_i|$ , otherwise

both (6)–(7) and (8)–(9) are infeasible. We use the notation  $t_i = |T_i|$  and  $u_i = |U_i|$  for  $i \in [m]$ .

In what follows, we will sometimes identify an index  $j \in N$  with the corresponding variable  $x_j$ ; e.g., we will indifferently say “the indices in  $T$ ” or “the variables in  $T$ ”.

Consider the problem of minimizing a linear objective function  $cx$  over the polytope defined by (8)–(9), where  $c$  is a row-vector in  $\mathbb{R}^n$ . If we define  $\mathcal{S} = \bigcup_{i \in [m]} \{S : S \subseteq N_i\}$ , the problem of minimizing  $cx$  over the polytope defined by (8)–(9) can be written as follows:

$$\min \quad cx \tag{10}$$

$$\text{s.t.} \quad \sum_{j \in S} x_j \geq f(S), \quad S \in \mathcal{S}, \tag{11}$$

$$-\sum_{j \in S} x_j \geq -g(S), \quad S \in \mathcal{S}. \tag{12}$$

We give a greedy algorithm that solves the above linear program for an arbitrary  $c \in \mathbb{R}^n$ . Since the solution returned by the algorithm will be a vector satisfying (6)–(7), this will prove that system (11)–(12) (i.e., system (8)–(9)) defines the convex hull of (6)–(7). The algorithm that we present can be seen as an extension of the greedy algorithm for polymatroids (see, e.g., [Sch03, Chapter 44]), and also as an extension of the algorithm given in [Mag13] for the case  $m = 1$ .

The procedure is shown in Algorithm 13 and is now illustrated. Throughout the algorithm, we maintain  $d$  clusters of variables  $V_1, \dots, V_d$ , i.e.,  $d$  (possibly empty) disjoint subsets of  $N$  gathering those variables that will be assigned the same value at the end of the algorithm. At the beginning (lines 2–3) we have  $t$  nonempty clusters  $V_1, \dots, V_t$ , where  $V_j = \{j\}$  for  $j \in [t]$ , while the other clusters  $V_{t+1}, \dots, V_d$  are empty. Thus every variable in  $T = [t]$  is assigned to a different cluster (as these variables are not allowed to take the same value), while the variables in  $U$  are not assigned to any cluster. During the execution of the algorithm, each variable in  $U$  will be assigned to a cluster, and no variable will be ever moved from a cluster to another.

Notation  $r(j)$  indicates the index of the cluster to which variable  $x_j$  is assigned. With each cluster  $V_j, j \in [d]$ , we associate a *pseudo-cost*  $\gamma_j$ , which is the sum of the costs of all variables in the cluster.

For  $i = 1, \dots, m$ , at the  $i$ th iteration of the algorithm we assign each variable in  $U_i$  to a different cluster (lines 4–10). Note that a variable in  $U_i$  cannot be assigned to a cluster  $V_j$  with  $j \in T_i$ , as  $V_j$  contains  $j$  for every  $j \in T_i$  and no variable in  $U_i$  is allowed to take the same value as a

variable in  $T_i$ . Thus only the clusters  $V_j$  with  $j \in [d] \setminus T_i$  are *feasible* for the variables in  $U_i$ . Lines 5–6 order the feasible clusters and the variables in  $U_i$  according to their pseudo-costs and costs, respectively (with ties broken arbitrarily). This is needed to assign the variables in  $U_i$  to the feasible clusters in a greedy fashion (line 8): among the variables with nonnegative cost, the one with the highest cost is assigned to the feasible cluster with the highest pseudo-cost (independently of the sign of the pseudo-cost), then the variable with the second highest cost is assigned to the feasible cluster with the second highest pseudo-cost, and so on; on the other hand, among the variables with negative cost, the one with the smallest cost is assigned to the feasible cluster with the smallest pseudo-cost, then the variable with the second smallest cost is assigned to the feasible cluster with the second smallest pseudo-cost, and so on. Lines 9 and 10 consequently update the clusters and the pseudo-costs.

At the end of the above procedure, we simply assign value 1 to the variables in the cluster with the highest pseudo-cost, value 2 to those in the cluster with second highest pseudo-cost, and so forth (lines 11–12).

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**Algorithm 1:** Greedy algorithm for linear optimization over an all-different system with the inclusion property.

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1 begin
2   for each  $j \in [t]$  do  $V_j := \{j\}$ ,  $\gamma_j := c_j$ ,  $r(j) := j$ ;
3   for each  $j \in [d] \setminus [t]$  do  $V_j := \emptyset$ ,  $\gamma_j := 0$ ;
4   for  $i = 1, \dots, m$  do
5     define a bijection  $\sigma : [d - t_i] \rightarrow [d] \setminus T_i$  such that
6      $\gamma_{\sigma(1)} \geq \dots \geq \gamma_{\sigma(d-t_i)}$ ;
7     define a bijection  $\pi : [u_i] \rightarrow U_i$  such that  $c_{\pi(1)} \geq \dots \geq c_{\pi(u_i)}$ ;
8     for each  $j \in [u_i]$  do
9       if  $c_{\pi(j)} \geq 0$  then  $r(\pi(j)) := \sigma(j)$  else
10         $r(\pi(j)) := \sigma(d - t_i - u_i + j)$ ;
11         $V_{r(\pi(j))} := V_{r(\pi(j))} \cup \{\pi(j)\}$ ;
12         $\gamma_{r(\pi(j))} := \gamma_{r(\pi(j))} + c_{\pi(j)}$ ;
13     define a bijection  $\sigma : [d] \rightarrow [d]$  such that  $\gamma_{\sigma(1)} \geq \dots \geq \gamma_{\sigma(d)}$ ;
14     for each  $j \in N$  do  $\bar{x}_j := \sigma^{-1}(r(j))$ ;
15   return  $\bar{x}$ 

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Note that if two variables belong to the same set  $N_i$  for some  $i \in [m]$ , then they are assigned to different clusters; therefore they receive different

values. This implies that the solution returned by the algorithm satisfies the given all-different system (6)–(7), and thus also (11)–(12). The optimality of the solution will follow from the existence of a dual solution satisfying the complementary slackness conditions, as we prove in the next section.

### 3 Dual solution and total dual integrality

**Theorem 3.1.** *Under the inclusion property, inequalities (8)–(9) define the convex hull of the vectors satisfying (6)–(7).*

*Proof.* We show that for every  $c \in \mathbb{R}^n$  the linear program (10)–(12) has an optimal solution that satisfies (6)–(7). For this purpose, fix  $c \in \mathbb{R}^n$  and run Algorithm 13. Let  $\bar{x}$  be the solution returned by the algorithm. Since  $\bar{x}$  satisfies (6)–(7), we only need to prove that  $\bar{x}$  is an optimal solution to (10)–(12).

Consider the dual problem of (10)–(12):

$$\max \sum_{S \in \mathcal{S}} (f(S)y_S - g(S)z_S) \quad (13)$$

$$\text{s.t. } \sum_{S \in \mathcal{S}: j \in S} (y_S - z_S) = c_j, \quad j \in N, \quad (14)$$

$$y_S, z_S \geq 0, \quad S \in \mathcal{S}. \quad (15)$$

We show that there exists a dual feasible solution  $(\bar{y}, \bar{z})$  such that  $\bar{x}$  and  $(\bar{y}, \bar{z})$  satisfy the complementary slackness conditions:

(a') for every  $S \in \mathcal{S}$ , if  $\bar{y}_S > 0$  then  $\sum_{j \in S} \bar{x}_j = f(S)$ ;

(b') for every  $S \in \mathcal{S}$ , if  $\bar{z}_S > 0$  then  $\sum_{j \in S} \bar{x}_j = g(S)$ .

Note that since  $\bar{x}$  satisfies (6)–(7),  $\sum_{j \in S} \bar{x}_j = f(S)$  if and only if  $\{\bar{x}_j : j \in S\} = \{1, \dots, |S|\}$ , and  $\sum_{j \in S} \bar{x}_j = g(S)$  if and only if  $\{\bar{x}_j : j \in S\} = \{d - |S| + 1, \dots, d\}$ . Then we can rewrite conditions (a') and (b') as follows:

(a) for every  $S \in \mathcal{S}$ , if  $\bar{y}_S > 0$  then  $\{\bar{x}_j : j \in S\} = \{1, \dots, |S|\}$ ;

(b) for every  $S \in \mathcal{S}$ , if  $\bar{z}_S > 0$  then  $\{\bar{x}_j : j \in S\} = \{d - |S| + 1, \dots, d\}$ .

Observe that if  $m = 1$  then Algorithm 13 reduces to the algorithm given in [Mag13] and thus returns an optimal solution. Therefore in this case there exists a dual solution  $(\bar{y}, \bar{z})$  satisfying the complementary slackness conditions.

Let  $p$  be the number of distinct pseudo-costs at the end of the algorithm, and  $q$  be the number of distinct nonzero costs of the variables in  $U_m$ :

$$p = |\{\gamma_j : j \in [d]\}|, \quad q = |\{c_j : c_j \neq 0, j \in U_m\}|.$$

Assume by contradiction that there is no dual solution satisfying conditions (a) and (b). Among all instances with this property, we choose an instance  $I$  such that the vector  $(m, p+q)$  is lexicographically minimum. As observed above,  $m \geq 2$ .

**Case 1** Suppose that  $c_j = 0$  for all  $j \in U_m$ . If we remove the  $m$ th all-different constraint and variables  $x_j$  with  $j \in U_m$ , we obtain a new instance  $I'$  with  $m-1$  constraints (note that  $m-1 \geq 1$ ). If we run the algorithm (with the same tie-breaking choices as we did for instance  $I$ ), we execute exactly the same operations as in the first  $m-1$  iterations of the algorithm applied to instance  $I$ . Then, since  $c_j = 0$  for all  $j \in U_m$ , the final pseudo-costs are the same for  $I$  and  $I'$ . Thus we obtain a solution  $\bar{x}'$  for  $I'$  which is identical to  $\bar{x}$ , except that  $\bar{x}'$  does not have the entries with index  $j \in U_m$ . Then, by the minimality of  $I$ , for  $I'$  there is a dual solution  $(\bar{y}, \bar{z})$  that satisfies conditions (a) and (b). One immediately checks that this dual solution is also feasible for the original instance  $I$ , and conditions (a) and (b) are still satisfied. This is a contradiction.

**Case 2** Assume that  $c_j \neq 0$  for some  $j \in U_m$ . Wlog,  $c_j > 0$  for some  $j \in U_m$ . Define  $c^* = \max\{c_j : j \in U_m\} > 0$  and  $C = \{j \in U_m : c_j = c^*\}$ .

Recall that, for  $j \in N$ ,  $r(j)$  denotes the index of the cluster containing  $j$ . We extend this notation to subsets: for  $J \subseteq N$ , we define  $r(J) = \{r(j) : j \in J\}$ .

Define  $A = r(C)$  and  $\gamma_0 = \min\{\gamma_j : j \in A\}$ . We claim that if  $\gamma_j \geq \gamma_0$  for some  $j \notin A$ , then  $j \in T_m$ . To see this, assume by contradiction that there is an index  $j \notin A \cup T_m$  such that  $\gamma_j \geq \gamma_0$ . Since  $j \notin A$ ,  $V_j$  was not assigned a variable in  $C$ ; and since  $j \notin T_m$ , cluster  $V_j$  was feasible at the  $m$ th iteration of the algorithm. This implies that before the execution of the  $m$ th iteration the pseudo-cost  $\gamma_j$  was at most as large as  $\gamma_k$  for every  $k \in A$ . But then the final pseudo-cost  $\gamma_j$  would be smaller than the final pseudo-cost  $\gamma_k$  for  $k \in A$  (because if  $V_j$  is assigned some variable at the  $m$ th iteration, the cost of this variable is by assumption smaller than  $c^*$ , while  $V_k$  is assigned a variable of cost  $c^*$ ). If we choose  $k$  to be an index in  $A$  such that  $\gamma_k = \gamma_0$ , we obtain a contradiction, as we assumed  $\gamma_j \geq \gamma_0 = \gamma_k$ .

Define  $B = \{j \in T_m : c_j \geq \gamma_0\}$ . Note that  $r(B) = B$  and  $\gamma_j = c_j$  for  $j \in T_m$ . By the above observation,  $\gamma_j \geq \gamma_0$  if and only if  $j \in A \cup B$ . Since  $r(B \cup C) = A \cup B$ , this implies that

$$\{\bar{x}_j : j \in B \cup C\} = \{1, \dots, |B \cup C|\}. \quad (16)$$

Let  $\hat{c} = \max\{c_j : c_j < c^*, j \in U_m\}$ , with  $\hat{c} = -\infty$  if  $c_j = c^*$  for all  $j \in U_m$ , and  $\hat{\gamma} = \max\{\gamma_j : \gamma_j < \gamma_0\}$ , with  $\hat{\gamma} = -\infty$  if  $\gamma_0$  is the minimum of all pseudo-costs. Define  $\delta = \min\{c^*, c^* - \hat{c}, \gamma_0 - \hat{\gamma}\} > 0$ .

Construct a new instance  $I'$  that is identical to  $I$ , except that the costs now are

$$c'_j = \begin{cases} c_j - \delta, & j \in B \cup C, \\ c_j, & j \notin B \cup C. \end{cases}$$

We claim that by applying the algorithm to this new instance (with the same tie-breaking choices as for instance  $I$ ) we obtain the same solution  $\bar{x}' = \bar{x}$ . To see this, observe that the first  $m - 1$  iterations of the algorithm are identical for  $I$  and  $I'$ , as we only changed the costs of some variables in  $N_m$ . After the  $(m - 1)$ th iteration, the ordering of the variables in  $U_m$  that we chose at line 6 when solving instance  $I$  is still non-increasing for the new instance, as  $c'_j = c^* - \delta \geq \hat{c}$  for  $j \in C$ . Furthermore, again after the  $(m - 1)$ th iteration, the pseudo-costs of the clusters  $V_j$  with  $j \notin T_m$  are the same as they were for instance  $I$ . Thus the assignment of the elements in  $U_m$  to feasible clusters is the same for the two instances. It follows that after the  $m$ th iteration the pseudo-costs for the new instance are

$$\gamma'_j = \begin{cases} \gamma_j - \delta, & j \in A \cup B, \\ \gamma_j, & j \notin A \cup B. \end{cases}$$

Since  $\gamma'_j = \gamma_j - \delta \geq \gamma_0 - \delta \geq \hat{\gamma}$  for all  $j \in A \cup B$ , the ordering of line 11 is still non-increasing. We then obtain the same solution as for instance  $I$ , as claimed.

By the choice of  $\delta$ , for  $I'$  either the number of distinct pseudo-costs is  $q - 1$  (this happens if  $\delta = \gamma_0 - \hat{\gamma}$ ), or the number of distinct nonzero costs of the variables in  $U_m$  is  $p - 1$  (this happens if  $\delta \in \{c^*, c^* - \hat{c}\}$ ). Then, by the minimality of instance  $I$ , there is a dual solution  $(\bar{y}, \bar{z})$  that satisfies conditions (a) and (b) for  $I'$ . By setting  $\bar{y}_{B \cup C} = \delta$ , we obtain a dual solution for the original instance  $I$ , with conditions (a) and (b) still satisfied because of (16). This is a contradiction.  $\blacksquare$

**Corollary 3.2.** *Under the inclusion property, system (8)–(9) is totally dual integral.*

*Proof.* The above proof shows that if  $c$  is an integer vector then there is an optimal dual solution with integer components. (The existence of such a solution when  $m = 1$ , which is needed in the base step of the proof, was shown in [Mag13].) ■

Note that the proof of Theorem 3.1 immediately yields an algorithm that constructs an optimal dual solution, given the output of Algorithm 13. The dual algorithm can be summarized as follows. When  $q > 0$ , the costs of the variables and the pseudo-costs are modified, and either  $p$  or  $q$  is decreased by one; then either an entry of  $\bar{y}$  or an entry of  $\bar{z}$  is set to some positive value  $\delta$ . When  $q = 0$ , the  $m$ th all-different constraint is removed and the procedure is iterated.

### 3.1 A remark

One might wonder whether Theorem 3.1 and Corollary 3.2 can be proved more directly through the theory of submodular functions.

A set function  $k : 2^N \rightarrow \mathbb{R}$  is *submodular* if

$$k(S_1) + k(S_2) \geq k(S_1 \cup S_2) + k(S_1 \cap S_2) \text{ for every } S_1, S_2 \subseteq N,$$

while a set function  $h : 2^N \rightarrow \mathbb{R}$  is *supermodular* if

$$h(S_1) + h(S_2) \leq h(S_1 \cup S_2) + h(S_1 \cap S_2) \text{ for every } S_1, S_2 \subseteq N. \quad (17)$$

It is known that if  $h$  (resp.,  $k$ ) is a supermodular (resp., submodular) function defined on  $2^N$ , then the polyhedron described by the inequalities

$$\sum_{j \in S} x_j \geq h(S), \quad S \subseteq N, \quad (18)$$

$$\sum_{j \in S} x_j \leq k(S), \quad S \subseteq N, \quad (19)$$

is totally dual integral: this is a classical result on polymatroids [Edm70] (see also [Sch03, Theorem 46.2]).

In our system (8)–(9),  $f$  and  $g$  are not defined for every  $S \subseteq N$ , but only for  $S \subseteq N_i$  with  $i \in [m]$ . Assume that, under the inclusion property,  $f$  (resp.,  $g$ ) can be extended to a supermodular function  $h$  (resp., submodular function  $k$ ) defined on  $2^N$  in such a way that the integer solutions to (18)–(19) are precisely the integer vectors in the convex hull of (6)–(7). Then Theorem 3.1 and Corollary 3.2 would follow immediately. However, we now show that in general such an extension does not exist.

Consider the all-different system with  $N = \{1, 2, 3\}$ ,  $m = 2$ ,  $N_1 = \{1, 2\}$ ,  $N_2 = \{2, 3\}$ . The inclusion property is clearly satisfied. We show that if  $h, k : 2^N \rightarrow \mathbb{R}$  are extensions of  $f, g$  such that the integer solutions to (18)–(19) are precisely the integer vectors in the convex hull of (6)–(7), then  $h$  violates inequality (17) for  $S_1 = N_1$  and  $S_2 = N_2$ . First, note that  $h(N_1) = f(N_1) = 3$ ,  $h(N_2) = f(N_2) = 3$ , and  $h(N_1 \cap N_2) = f(N_1 \cap N_2) = 1$ , while  $f$  is not defined for the set  $N_1 \cup N_2 = \{1, 2, 3\}$ . Since the vector  $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 2, 1)$  is a feasible solution, (18) holds only if  $h(N_1 \cup N_2) = h(\{1, 2, 3\}) \leq \bar{x}_1 + \bar{x}_2 + \bar{x}_3 = 4$ . Then  $h(N_1) + h(N_2) = 6$  and  $h(N_1 \cup N_2) + h(N_1 \cap N_2) \leq 5$ , and therefore  $h$  violates inequality (17).

## 4 An extension

We finally present an extension of Theorem 3.1 and Corollary 3.2. In what follows, we say that a function  $\phi : \mathbb{N} \rightarrow \mathbb{R}$  is convex (resp., concave) if the piecewise linear interpolation of  $\phi$  is convex (resp., concave).

Consider a system of the form

$$\sum_{j \in S} x_j \geq f(S), \quad S \subseteq N_i, i \in [m], \quad (20)$$

$$-\sum_{j \in S} x_j \geq -g(S), \quad S \subseteq N_i, i \in [m], \quad (21)$$

where  $f(S) = \alpha(|S|)$  for some convex function  $\alpha : \mathbb{N} \rightarrow \mathbb{R}$  and  $g(S) = \beta(|S|)$  for some concave function  $\beta : \mathbb{N} \rightarrow \mathbb{R}$ , with  $\alpha(0) = \beta(0) = 0$ . We assume that  $\alpha(k) \leq \beta(k)$  for all  $k \in \mathbb{N}$ , otherwise the system is infeasible. Note that system (8)–(9) is of this form.

When  $m = 1$ , system (20)–(21) is totally dual integral. This follows from the fact that  $f$  is a supermodular function and  $g$  is a submodular function (see [Lov83, Proposition 5.1]), along with the result on polymatroids mentioned in Section 3.1. However, in general the above system is not totally dual integral for  $m > 1$ . We now observe that we have total dual integrality if the inclusion property holds.

**Theorem 4.1.** *Assume that  $f(S) = \alpha(|S|)$  for some convex function  $\alpha : \mathbb{N} \rightarrow \mathbb{R}$  and  $g(S) = \beta(|S|)$  for some concave function  $\beta : \mathbb{N} \rightarrow \mathbb{R}$ , with  $\alpha(k) \leq \beta(k)$  for all  $k \in \mathbb{N}$  and  $\alpha(0) = \beta(0) = 0$ . Then, under the inclusion property, system (20)–(21) is totally dual integral. Thus, if  $\alpha$  and  $\beta$  are integer valued, the polyhedron defined by inequalities (20)–(21) is integral.*

*Proof.* We extend Algorithm 13 so that it solves linear optimization over (20)–(21). The only modification is at line 12: if variable  $x_j$  is in cluster  $V_k$  (where  $k = r(j)$ ) and  $\gamma_k \geq 0$ , we set  $\bar{x}_j = \alpha(k) - \alpha(k - 1)$ ; otherwise, if  $\gamma_k < 0$ , we set  $\bar{x}_j = \beta(k) - \beta(k - 1)$ . Note that when  $f$  and  $g$  are the functions defined in (5), this new version of the algorithm reduces to the original form of Algorithm 13.

In the following we show that the solution returned by the modified algorithm is feasible for (20)–(21), and then we observe that it can be completed with a dual solution satisfying the complementary slackness conditions.

We show that  $\bar{x}$  satisfies (20) for every  $S \subseteq N_i$ ,  $i \in [m]$ . First we observe that since  $\alpha$  is a convex function,

$$\alpha(k) - \alpha(k - 1) \leq \alpha(h) - \alpha(h - 1) \quad \text{for every } h \geq k \geq 1. \quad (22)$$

Now fix  $S \subseteq N_i$  for some  $i \in [m]$ . If we define  $S^+ = \{j \in S : \gamma_{r(j)} \geq 0\}$  and  $S^- = \{j \in S : \gamma_{r(j)} < 0\}$ , then

$$\begin{aligned} \sum_{j \in S} \bar{x}_j &= \sum_{j \in S^+} (\alpha(r(j)) - \alpha(r(j) - 1)) + \sum_{j \in S^-} (\beta(r(j)) - \beta(r(j) - 1)) \\ &\geq \sum_{j \in S} (\alpha(r(j)) - \alpha(r(j) - 1)) \\ &\geq \sum_{k=1}^{|S|} (\alpha(k) - \alpha(k - 1)) = \alpha(|S|) = f(S), \end{aligned} \quad (23)$$

where the first inequality holds because  $\alpha(k) \leq \beta(k)$  for all  $k \in \mathbb{N}$ , and the second inequality follows from (22) along with the fact that the indices  $r(j)$  for  $j \in S$  are pairwise distinct. This shows that  $\bar{x}$  satisfies (20); for inequalities (21), the proof is similar.

The rest of the proof is the same as the proof of Theorem 3.1, except that conditions (a) and (b) need to be adapted to this more general context. Note that the complementary slackness conditions take again the form (a')–(b') of the proof of Theorem 3.1. By (23), it follows that  $\sum_{j \in S} \bar{x}_j = f(S)$  if and only if  $\{\bar{x}_j : j \in S\} = \{\alpha(k) - \alpha(k - 1) : k = 1, \dots, |S|\}$ . Therefore, the complementary slackness conditions can be written in the following form:

- (a) for every  $S \in \mathcal{S}$ , if  $\bar{y}_S > 0$  then  $\{\bar{x}_j : j \in S\} = \{\alpha(k) - \alpha(k - 1) : k = 1, \dots, |S|\}$ ;
- (b) for every  $S \in \mathcal{S}$ , if  $\bar{z}_S > 0$  then  $\{\bar{x}_j : j \in S\} = \{\beta(k) - \beta(k - 1) : k = 1, \dots, |S|\}$ .

The proof now proceeds as for Theorem 3.1. ■

The above result implies in particular that Theorem 3.1 and Corollary 3.2 also hold if  $D$  is an arbitrary finite subset of  $\mathbb{R}$  (with  $|D| \geq n$ ), provided that  $f(S)$  (resp.,  $g(S)$ ) is defined as the sum of the  $|S|$  smallest (resp., largest) elements in  $D$  for every  $S \subseteq N$ .

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