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CORE

DISCUSSION PAPER

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Strongly polynomial bounds for multiobjective and parametric global minimum cuts in graphs and hypergraphs

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Abstract

We consider multiobjective and parametric versions of the global minimum cut problem in undirected graphs and bounded-rank hypergraphs with multiple edge cost functions. For a fixed number of edge cost functions, we show that the total number of supported non-dominated (SND) cuts is bounded by a polynomial in the numbers of nodes and edges, i.e., is strongly polynomial. This bound also applies to the combinatorial facet complexity of the problem, i.e., the maximum number of facets (linear pieces) of the parametric curve for the parametrized (linear combination) objective, over the set of all parameter vectors such that the parametrized edge costs are nonnegative and the parametrized cut costs are positive. We sharpen this bound in the case of two objectives (the bicriteria problem), for which we also derive a strongly polynomial upper bound on the total number of non-dominated (Pareto efficient) cuts. In particular, the bicriteria global minimum cut problem in an n-node graph admits $O(n^3 \log n)$ SND cuts and $O(n^5 \log n)$ non-dominated (Pareto efficient) cuts. These results significantly improve on earlier graph cut results by Mulmuley (1999) and Armon and Zwick (2006). They also imply that the parametric curve and all SND cuts, and, for the bicriteria problems, all Pareto efficient cuts, can be computed in strongly polynomial time when the number of objectives is fixed.

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1. Introduction

We consider the multicriteria version of the global minimum cut problem in undirected graphs. Global minimum cut is extensively studied in combinatorial optimization since many practical problems in, e.g., routing (subtour elimination), communications and electrical networks, contain it as a subproblem [2]. Let $G = (V, E)$ be an undirected graph, and $c^1, \dots, c^k : E \to \mathbb{R}$ be k cost functions, or *criteria*, defined on its edges. A *cut* X in G is a subset of nodes $X \subseteq V$ such that $\emptyset \neq X \neq V$, and it determines the set $\delta(X)$ of edges with exactly one end in X. The *cost* of cut X w.r.t. criterion j is $c^j(X) := c^j(\delta(X))$.

Many concepts and algorithms for global minimum cut generalize to hypergraphs.[1] We recall that a *hypergraph* is a finite set of vertices V, together with a family E of subsets of V. Thus each $e \in E$ is a vertex subset $e \subseteq V$, and is called a hyperedge, or simply, an edge. We say that hypergraph $G = (V, E)$ is *rank-ρ* if $|e| \leqslant \rho$ for all $e \in E$. Thus an undirected graph is a rank-2 hypergraph, and is rank-ρ for all $\rho \geqslant 2$. A *cut* X in hypergraph G is a non-trivial node subset, i.e., $\emptyset \neq X \subset V$. It cuts the set of edges $\delta(X) := \{e \in E : e \cap X \neq \emptyset \neq e \setminus X\}$. Notice that this matches the definition of a cut for a graph. A hypergraph is *connected* if all cuts X are non-empty, i.e., $\delta(X) \neq \emptyset$ for every cut X. Given edge costs $c^j(e)$ ($e \in E$, $j = 1, \dots, k$), the cost of a cut X w.r.t. c^j is the total cost $c^j(X) := c^j(\delta(X)) = \sum_{e \in \delta(X)} c^j(e)$ of all the edges crossed by cut X.

Ideally we would like a cut that simultaneously minimizes all criteria, but such a solution usually does not exist. Therefore, we focus on *Pareto optimal solutions*, i.e., solutions that cannot be improved upon in any criterion without degrading another criterion. Each cut X is associated with its criteria vector (or point) $c(X) := (c^1(X), \dots, c^k(X))$ in the criteria space $\mathbb{R}^k$. Let $\mathcal{C}$ denote the set of all cuts, and $Y := \{c(X) : X \in \mathcal{C}\}$ be the set of all criteria points associated with cuts (note that different cuts may give rise to the same criteria point). A point $c(X')$ *dominates* $c(X)$, and by extension, cut X' dominates X, if $c^j(X') \leqslant c^j(X)$ for all $j = 1, \dots, k$, and $c^j(X') < c^j(X)$ for at least one j. If there is no $c(X') \in Y$ that dominates $c(X)$, then the point $c(X)$, and by extension, the cut X itself, is *non-dominated*, or Pareto optimal. In order to derive strongly polynomial bounds and algorithms, when we study the *multicriteria minimum cut problem* we assume that all edge costs are nonnegative, as negative edge costs may give rise to NP-hard minimization problems; and that all cuts have positive costs, i.e., that $c^j(X) > 0$ for every objective j and cut X, for otherwise the hypergraph $(V, E^{>0,j})$ induced by the positive-cost edges (i.e., with edge set $E^{>0,j} = \{e \in E : c^j(e) > 0\}$) is not connected and may have an exponential number of minimum (zero-cost) cuts.

The computation of non-dominated points is related to the field of *parametric optimization*. For $j = 1, \dots, k-1$ we put a non-negative parameter μ_j on criterion j, and we consider the parametric optimization problem

$$c^*(\mu) := \min_{X \in \mathcal{C}} c_\mu(X) \quad \text{where} \quad c_\mu(X) := \sum_{j=1}^{k-1} \mu_j c^j(X) + c^k(X). \tag{1}$$

[1] A reader only interested in graph cuts may skip the rest of this paragraph and the whole of Section 2; and in the sequel replace all occurrences of "(bounded-rank) hypergraph(s)" with "graph(s)" and of the term "$B(\rho)$" in the exponents with "1", as well as ignore all mentions of rank $\rho \geqslant 3$.

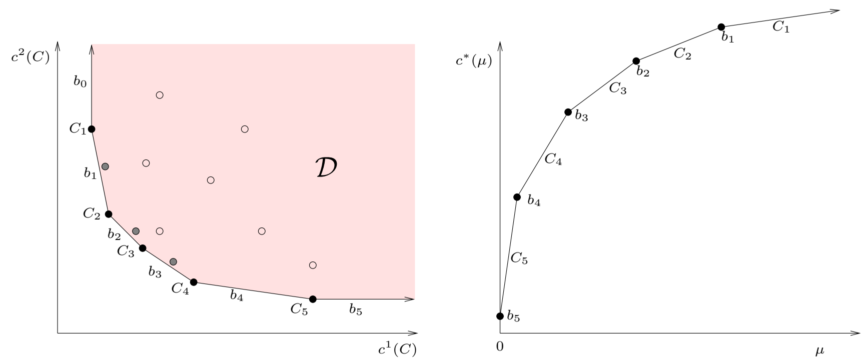

Figure 1. Chart (a) is an example of $\mathcal{D}$. Each point corresponds to a cut X with values $(c^1(X), c^2(X))$. SND points are black and are labeled C_i, UND points are gray, and dominated points are white. Chart (b) plots $c^*(\mu)$ for the same data. The extreme points and line segments of both graphs are labeled such that extreme point C_1 of chart (a) corresponds to line segment C_1 of the chart (b), extreme point b_2 of the chart (b) corresponds to line segment b_2 of chart (a), etc.

Notice that a multicriteria problem with k objectives corresponds to a parametric problem with a single objective and $k-1$ parameters. We call c_μ the *parametrized* objective. If there is some positive parameter vector $\mu \in \mathbb{R}^{k-1}_{>0}$ (i.e., with all components $\mu_j > 0$) such that cut X minimizes c_μ then $c(X)$ is non-dominated, we call it a *Supported Non-Dominated* (SND) point, and the cut X an *SND cut*. The non-dominated points that are not SND points are called *Unsupported Non-Dominated* (UND) points.

A natural object to define is the *dominant* $\mathcal{D}$ of the convex hull of Y, i.e., the set of all points $x \in \mathbb{R}^k$ that satisfy $x \geqslant y$ for some y in the convex hull of Y. The boundary of $\mathcal{D}$ (its lower convex hull) is the set of all points in $\mathcal{D}$ "accessible" as optimal solutions for parametrized objectives c_μ with $\mu \geqslant 0$. Thus all SND points must lie on the boundary of $\mathcal{D}$, and every vertex of $\mathcal{D}$ is actually an SND point. (When the costs are in "general position", the converse will also be true, namely, that every SND point is a vertex of $\mathcal{D}$. Otherwise, non-vertex SND points may arise as alternate optima for certain parameter vectors μ.) Function $c^* : \mathbb{R}^{k-1}_+ \mapsto \mathbb{R}$ is piecewise linear and concave, and its graph is in a sense the convex dual of the boundary of $\mathcal{D}$ (which is piecewise linear and convex) [10]. This duality interchanges vertices and facets, and so the fact that the vertices of $\mathcal{D}$ are SND points means that the facets of the graph of c^* also correspond to SND points. See Figure 1 for an example of this for $k = 2$.

In fact, our results apply to a slightly more general parametric minimum cut problem than arises from the multicriteria version, wherein we do not require either the objectives or the parameters to be nonnegative. However, in order to derive strongly polynomial bounds and algorithms, when we study the *parametric minimum cut problem* we make two assumptions that are similar to those for the multicriteria version: we assume that all parametrized edge costs $c_\mu(e) := \sum_{j=1}^{k-1} \mu_j c^j(e) + c^k(e)$ are nonnegative, and that all cuts have positive parametrized costs, i.e., that $c^*(\mu) > 0$. Thus we restrict attention to the following *relevant* *region* in the parameter space:

$$M = \left\{ \mu \in \mathbb{R}^{k-1} : c_\mu(e) \geqslant 0 \ \forall e \in E, \ \text{ and } c^*_\mu(X) > 0 \ \forall X \subset V, \ X \neq \emptyset \right\}.$$

Note that M is a convex polyhedral set, defined by a finite system of strict and non-strict linear inequalities. If, as in the multicriteria version, every objective c^j is nonnegative and its minimum is positive (i.e., its support graph is connected), then we may simply take as M the nonnegative orthant $\mathbb{R}^{k-1}_+$. Define $\mathcal{S}$ as the set of cuts $X \in \mathcal{C}$ which minimize $c_\mu(X)$ for some $\mu \in M$, which is precisely the set of SND cuts.

We define the *combinatorial facet complexity*[2] of the parametric minimum cut problem to be the maximum number of facets of the graph of c^*, which is equivalent to the maximum number of vertices of $\mathcal{D}$, and to the maximum number of SND points. Our main interest here is to study the combinatorial facet complexity of global minimum cut for hypergraphs, and thus also for graphs. We also derive sharper bounds on combinatorial facet complexity for the bicriterion case $k = 2$.

A natural subproblem of parametric minimum cut is solving single-criterion (ordinary) minimum cut, e.g., for some fixed value of μ. For graphs, the fastest deterministic algorithms for this problem run in $O(|E| \cdot |V| + |V|^2 \log |V|)$ time (Nagamochi and Ibaraki [26], and Stoer and Wagner [38]). The fastest randomized algorithm runs in $O(|E| \log^3 |V|)$ time (Karger [17]). These algorithms are faster than minimum s–t-cut algorithms that are based on network flows. See [27] for a detailed treatment of graph connectivity problems. For hypergraphs there exist polynomial time algorithms for finding a minimum cost cut in a hypergraph with nonnegative edge costs, see [20, 22, 33].

The multicriteria versions of several combinatorial optimization problems have been extensively studied, see Ehrgott [10] for a comprehensive survey. These problems are often *intractable* in the sense that the size of the set of (supported) non-dominated points grows exponentially in the input size. Furthermore, it is often hard even to verify if a given point is non-dominated. For example, Carstensen [3] shows that the combinatorial facet complexity of the minimum s-t-cut problem in a graph (namely, to find a minimum cost cut X that *separates* two given vertices s and t, in the sense that $|X \cap \{s,t\}| = 1$) is exponential, even for a single parameter. Mulmuley [25] gives a simpler proof of this result. In addition, it follows from Papadimitriou and Yannakakis [31, Theorem 6] that the *decision version* of the bicriteria minimum s-t-cut problem in a graph (namely, given a cut, is it non-dominated?) is strongly NP-hard.

Multicriteria *global* minimum cut in graphs are an exception to such intractability results. Mulmuley [25, Theorem 3.10] considers the combinatorial facet complexity of the parametric minimum cut problem with two objectives that may be negative,

[2]This is also called *parametric complexity* by some authors, such as Mulmuley [25] (see also [24, 21]) for the bicriterion case. We use a different terminology to avoid conflict with a different concept of parametric (or "parametrized", or "parameterized") complexity, e.g., [9], which deals with finding efficient algorithms for problems in which certain input or output "parameters" (or properties) are fixed. On the other hand, Fernández-Baca and Venkatachalam [13] use the term *combinatorial complexity* to refer to the total number of *faces* of all dimensions (here, 0 to $k-1$) of the graph of a parametric function (such as c^* here), whereas Schrijver [35] uses *facet complexity* to refer to the maximum input size of a rational linear inequality in a system that defines a polyhedron (such as $\mathcal{D}$ here).

but where all parametric edge costs are positive (a slight restriction, for the bicriteria case, of our more general parametric framework above). His Theorem 3.10 implies that the combinatorial facet complexity is $O(|V|^{19} \log |V| \log c_{\max})$, where $c_{\max} := \max_e \max\{c^1(e), c^2(e)\}$ is the maximum edge cost. This is a weakly polynomial bound, but Mulmuley also claims (without detailed proof) that the bit sizes of the costs can be assumed to have polynomial size without affecting the combinatorial facet complexity, and this yields a strongly polynomial bound. In addition, Armon and Zwick [2] show that, when the number k of criteria is fixed, the decision version of the multicriteria minimum cut problem in graphs can be solved in strongly polynomial time. They also point out the existence, when k is fixed, of a pseudo-polynomial time algorithm to find all the non-dominated points, and of a fully-polynomial time approximation scheme (FPTAS) for finding an *approximate Pareto set*, a notion defined by Papadimitriou and Yannakakis [31]. All these results may be surprising since the single-objective global minimum cut problem may be solved by solving $|V|-1$ minimum s-t-cut problems (e.g., by fixing s and letting t vary over the other nodes). Thus, for example, the parametric function c^* is the pointwise minimum of $|V|-1$ parametric minimum s-t cut functions, each of which possibly has an exponential number of breakpoints.

All these positive results rely on the deep and far-reaching fact that the single-criterion global minimum cut problem on graphs has at most a strongly polynomial number of near-optimal solutions. Indeed, given $\alpha \geqslant 1$, call a cut α-*approximate* if its cost is less than α times the minimum. For global minimum cut in graphs, Dinitz et al. [6] showed that there are at most $\binom{|V|}{2} = \theta(|V|^2)$ minimum (i.e., 1-approximate) cuts, and that this bound is tight. Nagamochi et al. [29] and Henzinger and Williamson [14] show that this bound also applies to the number of α-approximate cuts for all $\alpha < 4/3$ and all $\alpha < 3/2$, respectively; the latter authors also show that $3/2$ is the largest approximation factor for which this result holds. More generally, Karger [16] states (see [19] for a detailed proof) that for every $\alpha > 1$ the number of α-approximate cuts is $O(|V|^{2\alpha})$. All these results are subsumed by Karger's result [17] that for every $\alpha \geqslant 1$ the number of α-approximate cuts in a graph is $O(|V|^{\lfloor 2\alpha \rfloor})$.

In this paper, we give a strongly polynomial upper bound on the combinatorial facet complexity of global minimum cut on rank-ρ hypergraphs. Our bound does depend on ρ, and we show that this dependence on ρ is unavoidable. For the case of two nonnegative objectives (with positive minimum cut values), we also derive sharper bounds that are much smaller, and follow from a simpler proof, than in [25].

As pointed out above, our bounds of the combinatorial facet complexity of global minimum cut depend on a strongly polynomial upper bound on the number of approximate optimal solutions to global minimum cut. Hence our first step, in Section 2, is to generalize the results on approximate global minimum cut from graphs to hypergraphs. Given this, we derive in Section 3 a strongly polynomial bound on the combinatorial facet complexity of global minimum cut on hypergraphs. Then Section 4 studies in detail the bicriteria case, i.e., when $k = 2$. Finally, Section 5 shows how to adapt existing parametric algorithms for computing the graph of the parametric function c^*, the boundary of $\mathcal{D}$ (SND points and cuts), and for $k = 2$, the set of all non-dominated cuts. This paper derives from the conference paper [1], though it has been substantially extended, re-written, and corrected.

2. Approximate minimum cuts for a single objective

In this section we consider a hypergraph $G = (V, E)$ with a single, nonnegative edge cost function c that has positive minimum cut value, i.e., $c^*(G) := \min\{c(X) : \emptyset \neq X \subset V\} > 0$. (Without loss of generality we can discard all zero cost edges, and then this condition $c^*(G) > 0$ is equivalent to assuming that the hypergraph is connected.) We may also assume that distinct edges in E span different sets of vertices, and thus that $|E| = O(|V|^\rho)$, for if G contains "parallel" edges that span the same set of vertices, we may replace them by a single edge with cost equal to the sum of the costs of the parallel edges.

We extend Karger's bound [16] on the number of approximate cuts to bounded-rank hypergraphs. Our approach is based on a standard reduction (e.g., Ihler et al. [15]) to the minimum cut problem on a complete graph by replacing each hyperedge by a clique. This leads to a bound expressed in terms of the following function of ρ:

$$B(\rho) = \begin{cases} 1 & \text{if } \rho \leqslant 3, \text{ and} \\ \frac{\rho}{4} + \frac{1}{3} & \text{if } \rho \geqslant 4. \end{cases}$$

Theorem 1. *For every fixed integer $\rho \geqslant 2$, fixed scalar $\alpha \geqslant 1$, and rank-ρ hypergraph $G = (V, E)$ with nonnegative edge costs c and positive minimum cut cost, the number of α-approximate cuts is $O\Big(|V|^{\lfloor 2B(\rho)\alpha \rfloor}\Big)$.*

Proof. W.l.o.g., assume that every edge $e \in E$ has cardinality $|e| \geqslant 2$ (since edges e with $|e| \leqslant 1$ are not crossed by any cut). We prove the result by approximating the minimum cut problem in hypergraph $G = (V, E)$ with edge costs c, by the minimum cut problem in the complete graph $K(V) = (V, E_{K(V)})$ with edge costs $\tilde{c}$, whereby each edge $e \in E$ is replaced by a clique on the nodes in e and each edge in the clique has cost $c_e/(|e| - 1)$, i.e., by letting

$$\tilde{c}_{i,j} = \sum_{e \in E : \{i,j\} \subseteq e} \frac{c_e}{|e| - 1} \quad \text{for all } \{i, j\} \in E_{K(V)}. \tag{2}$$

In particular, every cardinality-2 edge $e = \{i, j\} \in E$ contributes its full cost c_e to $\tilde{c}_{i,j}$, and thus to the cost $\tilde{c}(X)$ of every cut X in $K(V)$ that crosses it. Note also that every cardinality-3 edge $e \in E$ that is crossed by cut X has two of its nodes on one side of the cut and the other node on the other side, and thus also contributes its exact cost $2(c_e/2) = c_e$ to $\tilde{c}(X)$. Therefore (as it is well known, e.g., Ihler et al. [15]), when $\rho \leqslant 3$ this transformation is exact, i.e., $\tilde{c}(X)) = c(X)$ for every cut X. The result then follows from Karger's bound when $\rho \leqslant 3$.

Now assume that $\rho \geqslant 4$. A cut X that crosses an edge $e \in E$ with cardinality $|e| \geqslant 4$ crosses at least $|e| - 1$ edges in the clique $K(e)$ (when exactly one node of e is on one side of the cut), and at most $|e|^2/4$ such edges (when half the nodes of e are on either side). Thus every edge $e \in E$ crossed by X contributes at least c_e and at most $(|e|^2/4)\frac{c_e}{|e|-1} \leqslant \frac{\rho^2/4}{\rho-1}\, c_e$ to the cost $\tilde{c}(X)$. Therefore, $c(X) \leqslant \tilde{c}(X)$ and

$$\tilde{c}(X) \leqslant \frac{\rho^2/4}{\rho - 1}\, c(X) = \left(\frac{\rho}{4} + \frac{1}{4}\left(1 + \frac{1}{\rho - 1}\right)\right) c(X) \leqslant B(\rho)\, c(X),$$

where the last inequality follows from $\rho \geqslant 4$. Let X^* denote a minimum cut for $(K(V), \tilde{c})$. If X is an α-optimal cut for (G, c), i.e., $c(X) \leqslant \alpha\, c^*(G)$, we have

$$\begin{aligned} \tilde{c}(X) &\leq B(\rho)\, c(X) \leqslant B(\rho)\, \alpha\, c^*(G) = B(\rho)\, \alpha\, c(X^*) \\ &\leq B(\rho)\, \alpha\, \tilde{c}(X^*), \end{aligned} \tag{3}$$

implying that X is a $\big(B(\rho)\, \alpha\big)$-optimal cut for $(K(V), \tilde{c})$. The result then again follows from Karger's bound when $\rho \geqslant 4$. □

The reduction to complete graphs used in the proof above is rather simple. We tried other approaches, such as generalizing Karger's randomized contraction algorithm [16, 19] to hypergraphs (as mentioned in [5] and [34]), but these alternatives did not lead to better bounds. It is possible that some other approach could improve on Theorem 2. To allow for this possibility, all our later results are expressed in terms of a generic function $B(\rho)$. The next example shows that the dependence of the number of minimum cuts on the rank ρ is unavoidable, by showing that there is no polynomial upper bound on this number when ρ grows faster than $\log |V|$.

Example: Consider any nondecreasing function $r : \mathbb{N} \mapsto \mathbb{N}$ such that $1 < r(n) < n$ for all $n \geqslant 3$, and $\lim_{n \to +\infty} r(n)/\log(n) = +\infty$. For every $n \geqslant 3$ let $m = \lfloor (n-1)/(r(n)-1) \rfloor$. Define hypergraph $G_{n,r} = (V_{n,r}\,,\, E_{n,r})$ with node set $V_{n,r} = \{1, \ldots, n\}$ and edge set $E_{n,r}$ consisting of the m edges $e_i = \{v \in V : 1+(i-1)(r(n)-1) \leqslant v \leqslant 1 + i(r(n)-1)\}$ for all $i = 1 \ldots, m$, and $e_{m+1} = \{v \in V : 1 + m(\rho - 1) \leqslant v \leqslant n\}$. Thus $G_{n,r}$ is an n-node, rank-$r(n)$ connected hypergraph with $m+1$ edges that form a "chain", in the sense that every edge e_i only intersects edges e_{i-1} (if $i \geqslant 2$) and e_{i+1} (if $i \leqslant m$). With unit costs $c(e_i) = 1$ for all $i = 1, \ldots, m+1$, the minimum cut cost is $c(G_{n,r}) = 1$ and the following $(2^{r(n)-1} - 1)m$ cuts are minimum: for every $i = 1, \ldots, m$, take every nonempty subset X of $e_i^0 := e_i \setminus \{1 + (i-1)(r(n)-1), 1+i(r(n)-1)\}$, and every subset $W = Z \cup \{1\} \cup \bigcup_{j<i} e_j$ such that $Z \subseteq e_i^0$. Indeed, each of these cuts X and W only crosses edge e_i. Thus the number of minimum cuts is at least $(2^{r(n)-1} - 1)m \geqslant 2^{r(n)-2} = n^{\Omega(r(n)/\log n)} = \Omega(n^p)$ for every integer p. □

3. Minimum cuts with multiple objectives

We are now given a hypergraph $G = (V, E)$, and k edge cost functions $c^1, \ldots, c^k : E \to \mathbb{R}$, where $k \geqslant 2$ is fixed. Before considering the multiobjective version of the problem, we start with its parametric version. Recall that for $\mu \in \mathbb{R}^{k-1}$ the parametrized cut cost is $c_\mu(X) = \sum_{e \in \delta(X)} c_\mu(e)$ for all cuts X in G.

The strategy for the following theorem is to express the parameter space M first as a union of regions defined by edges, then to cut each region into subregions via a hyperplane dissection. The region for edge a is the part of M where some minimum cut has a as a maximum cost edge w.r.t. c_μ. The hyperplanes come from $c_\mu(a)$ and, for each other edge e, a family of relative width β hyperplanes coming from $c_\mu(e)$ for some appropriate $\beta > 1$. The resulting subregions are small enough that we can apply Theorem 1 to get a small number of minimum cuts per region, and then we sum up across all subregions to get our bound.

Theorem 2. *Given a fixed integer $\rho \geqslant 2$, a rank-ρ hypergraph $G = (V, E)$, and k edge cost functions $c^1, \ldots, c^k$, the total number of cuts X minimizing $c_\mu(X)$ for μ in M is $O\Big(|E|^k |V|^{\lfloor 2B(\rho) \rfloor} \log^{k-1} |V|\Big)$.*

Proof. For every cut X in G let

$$M(X) = \left\{\mu \in M : c_\mu(X) = \min_W\{c_\mu(W) : \emptyset \neq W \subset V\}\right\}$$

denote the subset of parameter vectors in M for which X is a minimum cut. For every edge $a \in E$ let

$$\mathcal{S}_a = \left\{X \in \mathcal{S} : a \in \arg\max_{e \in X} c_\mu(e) \text{ for some } \mu \in M(X)\right\}$$

denote the set of all cuts X such that, for some μ for which X is a minimum cut, a is an edge in X with largest cost $c_\mu(a)$. It suffices to show that $|\mathcal{S}_a| = O\Big(|E|^{k-1}|V|^{\lfloor 2B(\rho)\rfloor}\log^{k-1}|V|\Big)$ and the result will follow.

Thus, in the rest of this proof, fix edge $a \in E$ such that $\mathcal{S}_a \neq \emptyset$. For every $X \in \mathcal{S}_a$ let

$$M_a(X) = \left\{\mu \in M(X) : a \in \arg\max_{e \in X} c_\mu(e)\right\}$$

so that $M_a(X) \neq \emptyset$, and let

$$M_a = \bigcup_{X \in \mathcal{S}_a} M_a(X)$$

denote the set of parameter vectors for which edge a has maximum cost in a minimum cut. We will cover the parameter region M_a with $L = O\left(|E|^{k-1}\log^{k-1}|V|\right)$ regions R_l $(l = 1, \ldots, L)$. We will then derive, for each region R_l, an $O\Big(|V|^{\lfloor 2B(\rho)\rfloor}\Big)$ bound on the total number of cuts W in $\mathcal{S}_a$ that are minimum for some $\mu \in R_l \cap M_a$, i.e., such that $R_l \cap M_a(W) \neq \emptyset$. The covering condition $M_a = \bigcup_{l=1}^L R_l$ implies $\mathcal{S}_a = \bigcup_{l=1}^L \{W \in S_a : R_l \cap M_a(W) \neq \emptyset\}$, and the result will follow.

First, note that for every $\mu \in M_a$ and cut W such that $a \in \arg\max_{e \in W} c_\mu(e)$ we have

$$c_\mu(a) \leqslant c_\mu(W) \leqslant |E| c_\mu(a). \tag{4}$$

Since $c_\mu(W) > 0$ for every $W \in \mathcal{S}_a \neq \emptyset$, this implies $c_\mu(a) > 0$.

Now fix a constant $\beta > 1$ whose value will be specified later, and let $\alpha = \frac{\beta-1}{|E|} > 0$. Compute $p = 1 + \lceil \log \frac{|E|}{\alpha} / \log\beta \rceil$ so that $\alpha\beta^{p-1} > |E|$, and observe that $p = O(\log|V|)$ (since $p = O(\log|E|)$, and $|E| = O(|V|^\rho)$ due to no parallel edges, and ρ being constant).

Notice that $c_\mu(a)$, considered as a function on $\mu \in \mathbb{R}^{k-1}$, is an affine function, and so $(\mu, c_\mu(a))$ is a hyperplane in $\mathbb{R}^k$. Define the functions $g_i : \mathbb{R}^{k-1} \to \mathbb{R}$ by $g_0(\mu) = 0$ and

$$g_i(\mu) = \alpha\,\beta^{i-1} c_\mu(a) \quad \text{for } i = 1, \ldots, p, \quad \text{and}$$

$g_{p+1}(\mu) = +\infty$. Thus for $1 \leqslant i \leqslant p$ we have that $(\mu, g_i(\mu))$ is a set of hyperplanes in $\mathbb{R}^k$ all parallel to $(\mu, c_\mu(a))$, and at exponentially increasing distances from each other.

Inequality (4) implies that, for every $\mu \in M_a$ and every cut W such that $a \in \arg\max_{e \in W} c_\mu(e)$,

$$c_\mu(e) \leqslant c_\mu(W) \leqslant |E| c_\mu(a) < g_p(\mu). \tag{5}$$

Since $c_\mu(a) > 0$,

$$0 = g_0(\mu) < g_1(\mu) < \cdots < g_p(\mu) < g_{p+1}(\mu) = +\infty \tag{6}$$

for all $\mu \in M_a$.

For each $i = 0, 1, \ldots, p$ such that the difference $g_i(\mu) - c_\mu(e)$ is not constant (for $i > 0$ this is equivalent to $(\mu, c_\mu(a))$ not being parallel to $(\mu, c_\mu(e))$), consider the hyperplane $H_{i,e} = \{\mu \in \mathbb{R}^{k-1} : g_i(\mu) = c_\mu(e)\}$. Note that $H_{i,e}$ is the intersection of the hyperplanes $(\mu, g_i(\mu))$ and $(\mu, c_\mu(e))$ in $\mathbb{R}^k$ projected down to $\mathbb{R}^{k-1}$, the space of the μ's. Now $H_{i,e}$ divides $\mathbb{R}^{k-1}$ into two regions, one where $g_i(\mu) \leqslant c_\mu(e)$, and the other where $g_i(\mu) \geqslant c_\mu(e)$.

Recall that, for fixed dimension d, N hyperplanes divide the d-dimensional space $\mathbb{R}^d$ into $O(N^d)$ regions ([37], see also [39] for further references). Thus the $O(|E| \log(|V|))$ hyperplanes $H_{i,e}$ define regions $R_1, \ldots, R_L$ that cover the set $M_a \subset \mathbb{R}^{k-1}$, and $L = O\left(|E|^{k-1} \log^{k-1} |V|\right)$. By (6), for each $l \in \{1, \ldots, L\}$ and each $e \in E$ there exists an index $i(e,l) \in \{0, 1, \ldots, p\}$ such that

$$R_l \cap M_a = \left\{\mu \in M_a : g_{i(e,l)}(\mu) \leqslant c_\mu(e) \leqslant g_{i(e,l)+1}(\mu)\ \forall e \in E\right\}.$$

Consider any such region R_l and any cut $W \in \mathcal{S}_a$. Note that (5) implies that $i(e,l) \leqslant p-1$ for all $e \in \delta(W)$. Let

$$R_l(W) = \left\{\mu \in R_l : a \in \arg\max_{e \in W} c_\mu(e)\right\},$$

the subset of R_l such that our fixed arc a has the maximum $c_\mu(a)$ value among all edges in cut W. Then for every $\mu \in R_l(W)$, the definition of $i(e,l)$ implies that

$$\begin{aligned} c_\mu(W) &\geqslant c_\mu(a) + \sum_{e \in \delta(W) \setminus \{a\} \,:\, 1 \leqslant i(e,l) \leqslant p-1} c_\mu(e) \\ &\geqslant c_\mu(a) \left(1 + \sum_{e \in \delta(W) \setminus \{a\} \,:\, 1 \leqslant i(e,l) \leqslant p-1} \alpha\, \beta^{i(e,l)-1}\right). \end{aligned}$$

If we define

$$\gamma_l(W) = 1 + \sum_{e \in \delta(W) \setminus \{a\} \,:\, 1 \leqslant i(e,l) \leqslant p-1} \alpha\, \beta^{i(e,l)-1}$$

then we can re-write this as

$$c_\mu(W) \geqslant c_\mu(a) + \sum_{e \in \delta(W) \setminus \{a\} \,:\, 1 \leqslant i(e,l) \leqslant p-1} c_\mu(e) \geqslant c_\mu(a)\, \gamma_l(W) \tag{7}$$

for every $\mu \in R_l(W)$.

The definition of α also implies that, for every $\mu \in R_l(W)$,

$$\begin{aligned} c_\mu(W) &= c_\mu(a) + \sum_{e \in \delta(W) \setminus \{a\} \,:\, i(e,l)=0} c_\mu(e) + \sum_{e \in \delta(W) \setminus \{a\} \,:\, 1 \leqslant i(e,l) \leqslant p-1} c_\mu(e) \\ &\leqslant c_\mu(a) + |E|\, \alpha\, c_\mu(a) + c_\mu(a)\, \beta\, (\gamma_l(W) - 1) \\ &= c_\mu(a)\, \beta\, \gamma_l(W). \qquad (8) \end{aligned}$$

Now fix a cut $X \in \mathcal{S}_a$ such that there exist $\mu \in R_l \cap M_a(X)$. Thus X is a minimum cut for the cost c_μ. For every cut $W \in \mathcal{S}_a$ such that there exists

$\nu \in R_l \cap M_a(W)$, we have

$$\begin{aligned}
c_\mu(W) &\leqslant c_\mu(a)\,\beta\,\gamma_l(W) && \text{(by (8))}\\
&= \frac{c_\mu(a)}{c_\nu(a)}\,\beta\,c_\nu(a)\,\gamma_l(W)\\
&\leqslant \frac{c_\mu(a)}{c_\nu(a)}\,\beta\,c_\nu(W) && \text{(by (7))}\\
&\leqslant \frac{c_\mu(a)}{c_\nu(a)}\,\beta\,c_\nu(X) && \text{(by optimality of } W \text{ for } \nu)\\
&\leqslant \frac{c_\mu(a)}{c_\nu(a)}\,\beta^2\,c_\nu(a)\,\gamma_l(X) && \text{(by (8))}\\
&= \beta^2\,c_\mu(a)\,\gamma_l(X)\\
&\leqslant \beta^2\,c_\mu(X) && \text{(by (7))}.
\end{aligned}$$

Therefore, every cut W such that $R_l \cap M_a(W) \neq \emptyset$ is a β^2-approximate cut for the parametrized cost c_μ.

If $\rho \leqslant 3$ then choose any β such that $1 < \beta < \sqrt{3/2}$, so $\lfloor 2B(\rho)\beta^2 \rfloor = 2 = \lfloor 2B(\rho) \rfloor$. Otherwise $\rho \geqslant 4$ and $2B(\rho)$ is not integer, and so there exists $\beta > 1$ such that $\lfloor 2B(\rho)\beta^2 \rfloor = \lfloor 2B(\rho) \rfloor$. In either case, by Theorem 1, the total number of such cuts W is $O\Big(|V|^{\lfloor 2B(\rho)\beta^2 \rfloor}\Big) = O\Big(|V|^{\lfloor 2B(\rho) \rfloor}\Big)$, and the proof is complete. □

Recall that the combinatorial facet complexity of the parametrized minimum cut problem over the relevant set M of parameters is the maximum number of facets of the graph of $c^*(\mu)$ when $\mu \in M$; equivalently, of vertices of the dominant $\mathcal{D}$, and also of SND points. Thus Theorem 2 implies:

Corollary 3. *Given a fixed integer $\rho \geqslant 2$, a rank-ρ hypergraph $G = (V, E)$, and a fixed number k of edge cost functions $c^1, \ldots, c^k$, the combinatorial facet complexity of the parametrized minimum cut problem over set M is $O\Big(|E|^k\, |V|^{\lfloor 2B(\rho) \rfloor} \log^{k-1} |V|\Big)$.*

We now briefly comment on the multicriteria minimum cut problem. As explained earlier, we now assume that all edge costs are nonnegative, i.e., all $c^j \geqslant 0$, and that every cut cost $c^j(X) > 0$ ($j = 1, \ldots, k$, $\emptyset \neq X \subset V$) (and therefore that G is connected). Under these assumptions, the relevant region in the parameter space is the nonnegative orthant, $M = \mathbb{R}_+^{k-1}$ and, by Theorem 2, the number of supported non-dominated (SND) points is $O\Big(|E|^k\, |V|^{\lfloor 2B(\rho) \rfloor} \log^{k-1} |V|\Big)$. In Theorem 6 in the next Section, we derive a strongly polynomial bound on the total number of (supported and unsupported) non-dominated points when $k = 2$. However, for $k \geqslant 3$ we leave the following:

Open Problem: *In a graph or a bounded-rank hypergraph, with a fixed number $k \geqslant 3$ of nonnegative edge cost functions such that each cut cost is positive, is the number of unsupported non-dominated (UND) points polynomially bounded?*

4. Tighter bounds for minimum cuts with two nonnegative objectives

When the bound on the number of SND cuts from Theorem 2 is specialized to $k = 2$ objectives, it becomes $O\big(|E|^2 |V|^{\lfloor 2B(\rho) \rfloor} \log |V|\big)$. This section sharpens this bound to $O\big(|V|^{1+\lfloor 2B(\rho)\rfloor}\log|V|\big)$, an improvement by a factor $|E|^2/|V|$. For graphs (and rank-3 hypergraphs), this improvement is from $O(|E|^2|V|^2\log|V|)$ to $O\left(|V|^3\log|V|\right)$. We also derive a strongly polynomial bound on the total number of (supported and unsupported) nondominated points. Our proof strategy will rely on partitioning the criteria space, instead of the parameter space as in Section 3. For this reason, we now assume that both edge cost functions c^1 and c^2 are nonnegative.

For general optimization problems with several objectives, Papadimitriou and Yannakakis [31] use a partitioning of the criteria space into rectangular regions of exponentially increasing sizes and obtain an approximate Pareto curve of weakly polynomially bounded size. Here we partition the criteria space according to value of criterion c^1, and we use two properties of global minimum cuts: First (in the proofs of Lemma 4 and Lemma 5), the polynomial bound on the number of approximate cuts from Theorem 1; and second (in the proof of Theorem 6), properties of edge contractions. We derive *strongly* polynomial bounds on the total number of *all exact* SND cuts.

The main argument used in the proof of Lemma 4 below is that every SND cut in a c^1-cost interval of relative width $\beta > 1$ is a β-approximation for the objective c_μ defined by a particular facet of the dominant polyhedron $\mathcal{D}$. Then, by choosing a small enough constant β, the number of SND cuts is shown to be $O\left(|V|^{\lfloor 2B(\rho)\rfloor}\right)$ per interval.

Lemma 4. *Given a fixed integer $\rho \geqslant 2$, a rank-ρ hypergraph $G = (V, E)$, two nonnegative edge cost functions c^1 and c^2 such that all cut costs are positive, and two reals $b > a > 0$, the total number of SND cuts X with c^1-cost satisfying $a \leqslant c^1(X) \leqslant b$ is $O\left(|V|^{\lfloor 2B(\rho)\rfloor}\ \log(b/a)\right)$.*

Proof. First, suppose that $\rho \geqslant 4$. If we choose $\beta > 1$ such that $2B(\rho)\beta = (\frac{\rho}{2}+\frac{2}{3})\beta < \frac{\rho}{2}+1$, then $\lfloor 2B(\rho)\beta\rfloor = \lfloor 2B(\rho)\rfloor$. Since $2B(\rho)\beta = (\frac{\rho}{2}+\frac{2}{3})\beta < \frac{\rho}{2}+1$ iff $\beta < \frac{3\rho+6}{3\rho+4}$, we can choose $\beta \in (1, \frac{3\rho+6}{3\rho+4})$ to ensure that $\beta > 1$ and $\lfloor 2B(\rho)\beta\rfloor = \lfloor 2B(\rho)\rfloor$. If instead $\rho \in \{2, 3\}$, we can similarly choose $\beta \in (1, \frac{3}{2})$ to again ensure that $\beta > 1$ and $\lfloor 2B(\rho)\beta\rfloor = \lfloor 2B(\rho)\rfloor$.

Now choose $n = \lfloor \log(b/a)/\log\beta\rfloor + 1 = O\left(\log(b/a)\right)$ to ensure that $\beta^{n-1}a \leqslant b < \beta^n a$. Thus we can cover the "target" interval $[a, b]$ with half-open intervals $I_j = \left[\beta^{j-1}a,\ \beta^j a\right)$ for $j = 1, \ldots, n$, such that $b \in I_n$.

Let $\mathcal{X}_j$ denote the set of SND cuts X such that $c^1(X) \in I_j$. If $\mathcal{X}_j \neq \emptyset$ then let W be a cut in $\mathcal{X}_j$ with largest c^2-cost. Let $\mu > 0$ be a parameter value for which W maximizes the parametrized objective $c_\mu = \mu c^1 + c^2$. Since we only have two criteria, for every SND cut $X \in \mathcal{X}_j$ such that $\left(c^1(X), c^2(X)\right) \neq \left(c^1(W), c^2(W)\right)$, and for every non-dominated cut X such that $c^1(W) < c^1(X) < \beta^j a$, we have $c^2(X) < c^2(W)$ and $\beta^{j-1}a \leqslant c^1(W) < c^1(X) < \beta^j a$. Thus

$$\begin{aligned} c_\mu(X) &< \mu\, c^1(X) + c^2(W) < \mu\beta^j a + c^2(W) \\ &< \beta\left(\mu\beta^{j-1}a + c^2(W)\right) \leqslant \beta\left(\mu\, c^1(W) + c^2(W)\right) \\ &= \beta\, c_\mu(W). \end{aligned} \tag{9}$$

Therefore, every cut in $\mathcal{X}_j$ is a β-approximate cut for the parametrized objective c_μ. By Theorem 1, the total number $|\mathcal{X}_j|$ of such cuts is $O\big(|V|^{\lfloor 2B(\rho)\beta\rfloor}\big) = O\big(|V|^{\lfloor 2B(\rho)\rfloor}\big)$, and the total number of SND cuts X with $a \leqslant c^1(X) \leqslant b$ is at most $\sum_{j=1}^n |\mathcal{X}_j| = O\big(|V|^{\lfloor 2B(\rho)\rfloor}\log(b/a)\big)$, as claimed. ☐

The proof of the next lemma uses the fact that there are two types of non-dominated cuts with c^1 cost in a given width-β interval: Those that follow the first SND cut in that interval (if any), which are again β-approximations; and those that precede it, which we show are 2-approximations for a suitable c_μ objective.

Lemma 5. *Given a fixed integer $\rho \geqslant 2$, a rank-ρ hypergraph $G = (V, E)$, two nonnegative edge cost functions c^1 and c^2 such that all cut costs are positive, and two reals $b > a > 0$, the total number of non-dominated cuts X with c^1-cost satisfying $a \leqslant c^1(X) \leqslant b$ is $O\left(|V|^{\lfloor 4B(\rho) \rfloor} \log(b/a)\right) = O\left(|V|^{\max\{4,\rho+1\}} \log(b/a)\right)$.*

Proof. Using the same notation as in the proof of Lemma 4, let $\mathcal{N}[a, b]$ denote the set of all non-dominated cuts W with $c^1(W) \in [a, b]$, so we need to bound $|\mathcal{N}[a, b]|$. Our proof considers a number of cases and subcases. Cases 1 to 1.1.1 comprise the bulk of the proof; the other cases and subcases yield to simpler treatment.

: Case 1. There is at least one SND cut Z with $c^1(Z) \in [a, b]$: We will need an interval $[a', b'] \subseteq [a, b]$ such that there is an SND cut Z_1 with $c^1(Z_1) \leqslant a'$, an SND cut Z_2 with $c^1(Z_2) \geqslant b'$, and $\mathcal{N}[a', b'] = \mathcal{N}[a, b]$. To this end, define $a'' = \min\{c^1(X) : X \in \mathcal{S}\}$, so $a'' \leqslant c^1(Z)$ and there is no non-dominated cut W with $c^1(W) < a''$. Similarly, define $b'' = \max\{c^1(X) : X \in \mathcal{S}\}$, so $b'' \geqslant c^1(Z)$ and there is no non-dominated cut W with $c^1(W) > b''$. Thus we set $a' = \max\{a, a''\}$ and $b' = \min\{b, b''\}$ to get the property we want, and we restrict attention to the interval $[a', b']$.

: Case 1.1 $a' < b'$: Let $x_1 < x_2 < \cdots < x_N$ denote the distinct values of $c^1(X)$ for all SND cuts X with $a' < c^1(X) < b'$. It follows from Lemma 4 that $N = O\big(|V|^{\lfloor 2B(\rho) \rfloor} \log(b'/a')\big)$. Let $X_1, \ldots, X_N$ be corresponding SND cuts, i.e., with $c^1(X_i) = x_i$. Define $x_0 = \max\{c^1(X) : X \in \mathcal{S} \text{ with } c^1(X) \leqslant a'\}$ with corresponding SND cut X_0, and $x_{N+1} = \min\{c^1(X) : X \in \mathcal{S} \text{ with } c^1(X) \geqslant b'\}$ with corresponding SND cut X_{N+1}. For $i = 0, 1, \ldots, N$ define $\nu(i)$ as the value of μ such that $c_\mu(X_i) = c_\mu(X_{i+1}) = \min\{c_\mu(X) : X \in \mathcal{C}\}$. Thus, $\nu(0) > \nu(1) > \cdots > \nu(N)$ are the successive breakpoints of the parametric curve $c^*(\mu)$ over the interval of μ values for which SND cuts X that minimize c_μ have $c^1(X) \in [a', b']$.

- For example, suppose that in Figure 1 (a) we have $c^1(C_2) < a < c^1(C_3)$ and $b > c^1(C_5)$. Then $a' = a$ and $b' = c^1(C_5)$; $N = 2$; $x_0 = c^1(C_2)$, $x_1 = c^1(C_3)$, $x_2 = c^1(C_4)$ and $x_3 = c^1(C_5)$; $\nu(0) = b_2$, $\nu(1) = b_3$ and $\nu(2) = b_4$.

Every non-dominated cut $W \in \mathcal{N}[a', b']$ has $c^1(W) \in [x_i,\ x_{i+1}]$ for some $i \in \{0, 1, \ldots, N\}$. Since W is non-dominated, its c^2 cost satisfies $c^2(W) \in [c^2(X_{i+1}),\, c^2(X_i)]$. Since X_i and X_{i+1} are both minimum cuts for $\mu = \nu(i)$, we have $c_{\nu(i)}(X_i) = c_{\nu(i)}(X_{i+1})$ and so

$$
\begin{aligned}
c_{\nu(i)}(W) &\leqslant \nu(i)\, c^1(X_{i+1}) + c^2(X_i) \\
&< \nu(i)\, c^1(X_{i+1}) + c^2(X_{i+1}) + \nu(i)\, c^1(X_i) + c^2(X_i) \\
&= 2\, c_{\nu(i)}(X_i). \qquad (10)
\end{aligned}
$$

That is, W is a 2-approximate cut for the parametrized objective $c_{\nu(i)}$. Now fix one of the intervals $I'_j = \left[\beta^{j-1} a',\ \beta^j a'\right)$ that cover $[a', b']$ and let $\mathcal{N}(I'_j) := \{W \in \mathcal{N}[a', b'] : c^1(W) \in I'_j\}$.

: **Case 1.1.1. There is an SND cut with c^1 cost in I'_j:** Let X_i be the SND cut with the least c^1 cost among all SND cuts with c^1 cost in I'_j. Every $W \in \mathcal{N}(I'_j)$ with $c^1(W) < c^1(X_i)$ satisfies $c^1(W) \in [x_{i-1}, x_i]$ and so by (10) is a 2-approximate cut for the parametrized objective $c_{\nu(i-1)}$; there are again $O\left(|V|^{\lfloor 4B(\rho) \rfloor}\right)$ such non-dominated cuts. Every other $W \in \mathcal{N}(I'_j)$ satisfies $c^1(X_i) \leqslant c^1(W) < \beta^j a'$ and, by (9), is a β-approximate cut for the parametrized objective $c_{\nu(i)}$; thus, for β chosen as in the proof of Lemma 4, there are $O\big(|V|^{\lfloor 2B(\rho) \rfloor}\big)$ such cuts, and so a total of $O\big(|V|^{\lfloor 4B(\rho) \rfloor}\big)$ non-dominated cuts in I'_j. Since $|\mathcal{N}(I'_j)| = O\big(|V|^{\lfloor 4B(\rho) \rfloor}\big)$ for each of the $O(\log(b'/a')$ intervals I'_j, we get $|\mathcal{N}[a,b]| = |\mathcal{N}[a',b']| = O\left(|V|^{\lfloor 4B(\rho) \rfloor} \log(b'/a')\right) = O\left(|V|^{\lfloor 4B(\rho) \rfloor} \log(b/a)\right)$, as claimed.

: **Case 1.1.2. I'_j does not contain the c^1 cost of any SND cut:** Then every $W \in \mathcal{N}(I'_j)$ has $c^1(W) \in [x_i,\, x_{i+1}]$ for some i such that $x_i < \beta^{j-1} a'$ and $\beta^j a' \leqslant x_{i+1}$, hence is a 2-approximate cut for the parametrized objective $c_{\nu(i)}$. By Theorem 1 there are $O\left(|V|^{\lfloor 4B(\rho) \rfloor}\right)$ such cuts.

: **Case 1.2 $a' = b'$:** Then $\mathcal{N}[a', b']$ is just the set of all SND cuts X with $c^1(X) = a'$, and the proof of Lemma 4 shows that $|\mathcal{N}[a', a']| = O\big(|V|^{\lfloor 2B(\rho) \rfloor}\big) = O\left(|V|^{\lfloor 4B(\rho) \rfloor} \log(b/a)\right)$.

Case 2. There is no SND cut Z with $c^1(Z) \in [a, b]$: Then either (i) there is no SND cut Z at all with $c^1(Z) < a$, and thus $\mathcal{N}[a,b] = \emptyset$; or else (ii) let as above X_0 be an SND cut with largest cost $c^1(X_0) < a$ and $\nu(0)$ the corresponding parameter value, then (10) implies that every $W \in \mathcal{N}[a,b]$ is a 2-approximate cut for the parametrized objective $c_{\nu(0)}$, and thus $|\mathcal{N}[a,b]| = O\big(|V|^{\lfloor 4B(\rho) \rfloor}\big) = O\left(|V|^{\lfloor 4B(\rho) \rfloor} \log(b/a)\right)$, as claimed.

The proof is complete. □

The proof of the next theorem uses the following observation: every cut either has its c^1 cost in the interval $\left[\frac{c^1(E)}{|E|}, c^1(E)\right]$ between the average and total c^1 edge costs, that has relative width $|E| = O(|V|^\rho)$ (hence $\log |E| = O(\log |V|)$) and to which we apply the two preceding lemmas; or else the cut cannot contain any edge e with cost $c^1(e) \geqslant \frac{c^1(E)}{|E|}$, in which case we can contract all such costlier-than-average edges and recurse. Since there is at least one such edge to contract, the number of vertices decreases by at least one in each iteration and we are done after fewer than $|V|$ rounds of edge contractions.

Theorem 6. *Given a fixed integer $\rho \geqslant 2$, a rank-ρ hypergraph $G = (V, E)$, and two nonnegative edge cost functions such that all cut costs are positive, the total number of SND cuts is $O\Big(|V|^{1+\lfloor 2B(\rho) \rfloor} \log |V|\Big)$ and the total number of non-dominated cuts is $O\left(|V|^{1+\lfloor 4B(\rho) \rfloor} \log |V|\right) = O\left(|V|^{\max\{5, \rho+2\}} \log |V|\right)$.*

Proof. Our strategy is to repeatedly apply Lemma 4 to a sequence of hypergraphs obtained by contracting edges. As in [20, 22], by *contracting* an edge e in hypergraph $G = (V, E)$ with edge costs c^j to get $G' = (V', E')$ we mean replacing all nodes in e by a single "contracted node" v_e (i.e., $V' = (V \setminus e) \cup \{v_e\}$), and every edge $f \in E$ by an edge f' of same costs, $c^j(f') = c^j(f)$, wherein all nodes in $e \cap f$, if any, are replaced by the single node v_e (i.e., $f' = (f \setminus e) \cup \{v_e\}$ if $f \cap e \neq \emptyset$, and $f' = f$ otherwise). We then remove from E all edges f with $|f| = 1$ (loops, which no cut in G' can cross), and all nodes that do not belong to any edge in E' (isolated nodes). Conversely, every cut in the contracted hypergraph corresponds, after expanding back the contracted node v_e, to a cut with the same cost in the original hypergraph. Note that this generalizes edge contraction in graphs, e.g., [27].

Suppose that X is a cut in G, and we contract e to get G'. If e is not in $\delta(X)$ ("X does not cross e"), then $X' := X \setminus \{v_e\}$ is a cut in G' where f' is in $\delta(X')$ if and only if f is in $\delta(X)$. Thus $c^j(X) = c^j(X')$ for $j = 1, 2$. On the other hand, if X is a cut in G with $e \in \delta(X)$ ("X crosses e"), then there is no cut in G' corresponding to X.

We construct a sequence of m rank-ρ hypergraphs, $i = 1, \ldots, m$. Hypergraph i is $G_i = (V_i, E_i)$ with edge costs $c_i = (c_i^1, c_i^2)$, such that $(G_0, c_0) = (G, (c^1, c^2))$. Hypergraph (G_i, c_i) is derived from (G_{i-1}, c_{i-1}) by contracting all edges in $e \in E_{i-1}$ with cost $c_{i-1}^1(e) \geqslant a_{i-1} := \frac{c_{i-1}^1(E_{i-1})}{|E_{i-1}|}$, so long as $|V_{i-1}| > \rho$. Since there is at least one edge $e \in E_{i-1}$ with c^1-cost no less than the average a_{i-1}, there will be at least one edge to contract. Thus $|V_i| < |V_{i-1}|$ and so $m \leqslant |V| - \rho$.

We associate with each hypergraph G_i in the sequence an interval $[a_i, b_i]$ where, as defined above, $a_i := \frac{c_i^1(E_i)}{|E_i|}$ is the average edge c_i^1-cost in G_i, and $b_i = c_i^1(E_i) = |E_i| a_i$ is the total c_i^1-cost. Since $c_i^1 \geqslant 0$, the c_i^1-cost of every cut X in G_i satisfies $c_i^1(X) \leqslant c_i^1(E_i) = b_i$. Every cut X in G_i either has cost $c_i^1(X) \geqslant a_i$, and therefore $c_i^1(X) \in [a_i, b_i]$, or else it cannot cross any edge e with cost $c_i^1(e) \geqslant a_i$. In the latter case, if $i < m$ then X does not cross any edge which is contracted when defining G_{i+1}. Thus there is a cut X' in G_{i+1} corresponding to X, and its cost $c_i^1(\delta_i(X)) = c_{i+1}^1(\delta_{i+1}(X')) \leqslant b_{i+1}$.

Let s_i (resp. n_i) denote the number of SND (resp. nondominated) cuts in G_i with cost $c_i^1(X)$ in $[a_i, b_i]$. It follows that s_i (resp. n_i) is also the number of SND (resp. nondominated) cuts in the original hypergraph G with cost $c^1(X) \in [a_i, b_i]$. Then the total number of SND cuts in G is at most $\sum_{i=1}^m s_i$, and it suffices to prove that each $s_i = O\Big(|V|^{\lfloor 2B(\rho) \rfloor} \log |V|\Big)$. Similarly, it suffices to prove that each $n_i = O\Big(|V|^{\lfloor 4B(\rho) \rfloor} \log |V|\Big)$. These are certainly true for $i = m$ since $|V_m| \leqslant \rho$ and thus $s_m \leqslant n_m < 2^\rho = O(1)$. For $i < m$, $s_i = O\left(|V_i|^{\lfloor 2B(\rho) \rfloor} \log |E_i|\right)$ by Lemma 4, and $n_i = O\left(|V_i|^{\lfloor 4B(\rho) \rfloor} \log |E_i|\right)$ by Lemma 5. Since $|E_i| = O(|V_i|^\rho)$ and ρ is fixed, $\log |E_i| = O(\log |V_i|) = O(\log |V|)$. □

Since distinct facets of the graph of c^* are associated with distinct SND cuts, we have:

Corollary 7. *Under the assumptions of Theorem 6, the combinatorial facet complexity of the bi-objective parametrized minimum cut problem is* $O\Big(|V|^{1+\lfloor 2B(\rho) \rfloor} \log |V|\Big)$.

Notice that in the case of graphs (where $\rho = 2$) the bounds in Theorem 6 specialize to there being $O(|V|^3 \log |V|)$ SND cuts and $O(|V|^5 \log |V|)$ UND cuts. The conference version of this paper [1] claimed a slightly better bound of $O(|V|^3)$ SND cuts, but there was an error in its analysis of its Algorithm 2 in the proof of its Claim 3. On the other hand, the present $O(|V|^5 \log |V|)$ bound on the number of UND cuts is better than the $O(|V|^7)$ bound of [1].

5. Algorithms

In this section we show how to combine the preceding results with several existing algorithms, and derive strongly polynomial time algorithms for computing the optimum cost function f; the sets of supported and, for $k = 2$, unsupported non-dominated points; and the sets of all corresponding minimum cuts. For this we use three types of existing algorithms:

(1) Algorithms for solving the ordinary (single-objective) minimum cut problems, i.e., to determine the value $c^*(\mu)$ and a corresponding minimum cut for any given $\mu \in M$, see the Introduction. Let $MC = MC(|V|, |E|)$ denote the running time of such an algorithm for finding a minimum cut in a hypergraph with $|V|$ vertices and $|E|$ edges. Recall that $MC = O(|E| \cdot |V| + |V|^2 \log |V|)$ ([26, 38] for graphs; [20, 22, 33] for hypergraphs).

(2) Algorithms for construction of the optimum cost function f for μ in a convex parameter region, see [12, Section 3.2] or [13, Section 30.4.2]. Theorem 30.2 in the latter reference implies that if the upper envelope of a piecewise linear concave function f of d parameters has Φ facets and Ψ vertices then, using the hyperplane probing methods of Dobkin et al. [7, 8], function f may be obtained by evaluating $c^*(\mu)$ at $O(\Phi + d\ \Psi)$ "probe" points μ, and then constructing the full upper convex hull derived from the resulting hyperplanes. The latter task is dual, and computationally equivalent, to the well-studied question of constructing the convex hull of a given set of points, e.g., [32, Section 3.4], [23, Section 7.3]. Chan's algorithm [4], the fastest convex hull algorithm known today, achieves this in $O\Big(\Phi \log \Phi + (\Psi\Phi)^{1+1/(\lfloor d/2 \rfloor+1)} \log^{O(1)} \Phi\Big)$ time. For $k = 2$ (hence $d = 1$), the whole graph can be constructed in $O(\Phi\, T)$ total time [11], where T is the time needed for an evaluation of $c^*(\mu)$ (i.e., MC for our problem).

(3) Algorithms for producing, for a single objective, *all* minimum cuts and all 2-approximate cuts. Let $MC1$ and $MC2$ denote the total time to produce all minimum cuts and all 2-approximate cuts, respectively. For the case where G is a graph, there is a well-known "cactus representation" of all minimum cuts [6], which is a of tree of cycles such that cut X is minimum if and only if it cuts exactly two arcs in one cycle. Nagamochi et al. [28] use a cactus representation to show that $MC1 = O(|E|^2|V| + |V|^2|E|)$, while it follows from [27] that $MC2 = O(|E|\ |V|^4)$; see also [18] and [30, Chapter 2] for randomized graph cut algorithms. If G is a rank-3 hypergraph then, as shown in the proof of Theorem 1, the cost of a cut X is the same in graph G with edge cost function c as in the complete graph $K(V)$ with edge cost function $\tilde{c}$ defined in (2). Therefore one may apply the method of [28] to $(K(V), \tilde{c})$ and produce in $MC1 = O(|E_{K(V)}|^2|V| + |V|^2|E_{K(V)}|) = O(|V|^5)$ time all minimum cuts for (G, c). If G is a rank-ρ hypergraph with $\rho \geqslant 4$, then, by (3) with $\alpha = 1$, it suffices to produce all $B(\rho)$-approximate cuts in $(K(V), \tilde{c})$ and then only retain those that are minimum for (G, c). It follows from [27] that this can be done in $MC1 = O(|E_{K(V)}|\ |V|^{2B(\rho)}) = O(|V|^{2B(\rho)+2})$ time.

Combining these algorithms and our earlier results we obtain:

Theorem 8. *Given a hypergraph $G = (V, E)$, and a fixed number k of nonnegative edge cost functions $c^1, \ldots, c^k$ such that $c^j(X)$ is positive for all cuts X in G:*

(i) *If $k = 2$, there exists an algorithm that constructs the optimum cost function in $O\left(|V|^{1+\lfloor 2B(\rho)\rfloor} \log |V|\ MC\right)$ time, and all cuts defining SND and non-dominated points in $O\left(|V|^{1+\lfloor 2B(\rho)\rfloor} \log |V|\ MC1\right)$ and $O\left(|V|^{1+\lfloor 2B(\rho)\rfloor} \log |V|\ MC2\right)$ time, respectively.*

(ii) *If $k \geqslant 3$, the time needed to construct the optimum cost function and all cuts defining SND points is*

$$O\left(|E|^{k\left\lfloor \frac{k-1}{2}\right\rfloor} \quad |V|^{\lfloor 2B(\rho)\rfloor \left\lfloor \frac{k-1}{2}\right\rfloor} \quad \log^{(k-1)\left\lfloor \frac{k-1}{2}\right\rfloor + O(1)} |V|\right) \tag{11}$$

Proof. *(i)* follows from the use of the Eisner and Severance method [11] with $\Phi = O\left(|V|^{1+\lfloor 2B(\rho)\rfloor} \log |V|\right)$ from Theorem 6, and from the fact, established in the proof of Lemma 5, that every non-dominated point is defined by a 2-approximate cut for a facet-inducing parametrized objective.

(ii) follows from Theorem 30.2 in [13] using $d = k - 1$, $\Phi = O(\widehat{\Phi})$ where $\widehat{\Phi} = |E|^k |V|^{\lfloor 2B(\rho)\rfloor} \log^{k-1} |V|$ by Theorem 2, $\Psi = O(\widehat{\Psi})$ where $\widehat{\Psi} = \widehat{\Phi}^{\left\lfloor \frac{k-1}{2}\right\rfloor}$ by McMullen's Upper Bound Theorem (e.g., [23, Section 7.3]; see also [36]). Indeed, using Chan's convex hull algorithm [4], the time needed for constructing the lower convex hull from the Φ hyperplanes is bounded by (11). Since the time MC for each "probe" (evaluation of $c^*(\mu)$ for a given $\mu \in M$) satisfies $MC = O(\widehat{\Phi})$ when $k \geqslant 3$, the lower hull construction time (11) dominates the total time for constructing the optimum cost function. Furthermore, for each of the Φ SND points one may produce in $MC1$ time all corresponding cuts. Since $MC1 = O(\widehat{\Phi})$ when $k \geqslant 3$, the lower hull construction time (11) again dominates the total computation time. □

6. Conclusion

We have extended Karger's bound [16] on the number of approximate global minimum cuts from graphs to hypergraphs. This then allowed us to derive strongly polynomial bounds on the number of Supported (and for the case of two objectives, Unsupported) Non-Dominated cuts w.r.t. multiple objectives (or multiple parameters) in both hypergraphs and graphs. These bounds, when combined with existing algorithms, lead to strongly polynomial algorithms for computing all such cuts.

In addition to the open problem at the end of Section 3 (for a fixed number $k \geqslant 3$ of criteria and fixed rank $\rho \geqslant 2$, is there a strongly polynomial bound on the total number of non-dominated points?) another open problem is to find a non-trivial *lower bound* on the complexity of parametric global minimum cut. It is easy to produce classes of instances with $\Theta(|V|)$ different minimum cuts in the graph of f, but so far we do not have any classes of instances with superlinear complexity. It would be interesting to close this gap; or even for graphs and just two criteria, to reduce the gap between the present $\Omega(|V|)$ and $O(|V|^3 \log |V|)$ bounds.

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