Electing a Parliament

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Abstract

We present a model where a society elects candidates belonging to two parties to a national parliament. The electoral rule determines the seats distribution between the two parties. The policy outcome is a function of the number of seats the two parties win in the election. We analyze two electoral rules, multidistrict majority and single district proportional. We prove that under both systems there is a unique pure strategy perfect equilibrium outcome. We compare the outcomes under the two systems.

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1 Introduction

In parliamentary democracies, policies are the outcome of a legislative debate between political parties in power and at the opposition. The number of seats each party has in parliament determines its ability to influence such debate and shape policies. Hence,

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the strength of the parties in the next parliament is of concern to policy motivated voters when electing their representatives. This paper presents a tractable theoretical model of a parliamentary election where the composition of parliament, through its effect on the policy outcome, matters for the behavior of voters.

Specifically, we consider a model of a parliamentary election where the electorate is composed of policy motivated citizens with single-peaked preferences over one policy issue, each of whom votes strategically for two political parties, with given policy positions - one leftist, the other rightist, to elect a number of representatives to the national parliament. The electoral result determines the number of seats for the two parties in parliament. In turn, the policy outcome is determined by the number of seats the two parties obtain in the election. In particular, we assume that the policy outcome is more leftist, the higher the number of seats in parliament obtained by the left, which captures the idea of a parliamentary compromise between the two parties with different strengths. We consider two electoral systems which are widely used in real-world elections: the multidistrict majority system and the single district proportional system.

First, we study the multidistrict majority system whereby the electorate of the country is divided into several districts and each district elects one member of parliament by majority rule. We analyze this voting model with the tools of non-cooperative game theory. As soon as we start thinking about the appropriate solution concept for our model, an obstacle comes up: too many situations survive as Nash equilibria. In fact, any strategy combination whereby no voter is decisive is a Nash equilibrium, since a non-decisive voter cannot change the outcome by changing her vote. Some of these equilibria may be rather unpalatable, since voters may happen to use (weakly) dominated strategies. One may be tempted to rush to the conclusion that the appropriate solution concept involves restricting attention to undominated strategies. In political scenarios of the all-or-nothing type, whereby the party that wins the election is free to implement its preferred policy while the losing party has no say on policy, the undominated equilibrium is, indeed, a powerful tool that leads voters to vote sincerely for their preferred party and generates as the equilibrium outcome the one favored by the median voter of the population.[1] Unfortunately, here matters are more complicated, since different policy outcomes obtain for different compositions of the parliament.

Although sincere voting per se does not help, a "conditional" version of it, which we call district sincerity, turns out to be the appropriate solution concept. Intuitively, a strategy combination is district sincere if each voter who strictly prefers one of the two parties to win in her district - taking as given the strategies of the voters in the other districts- votes for such a party. We prove that our voting game has a unique district sincere outcome in pure strategy. Such an outcome is characterized by a number of seats for the left equal to the number of districts, where their median voters prefer the policy implemented should the left win exactly that number of seats in parliament to the policy implemented should the left win one less seat. Since the policy outcome is more leftist the higher is the number of districts carried by the left party, and the districts can be ordered according to how leftist their median voter is, the number of seats for the left - and, thus, the allocation of seats for the two parties- in parliament in a pure strategy district sincere equilibrium is pinned down uniquely as the one favored by the median voter of a critical district.

District sincerity is attractive since it provides a direct and intuitive way to arrive

[1] Incidentally, this purely majoritarian scenario could be subsumed under our analysis. We have chosen not to dwell upon it, since it is well understood in the literature.

at the solution. However, the reader may legitimately wonder whether we have forced such a solution through the choice of an arbitrary solution concept. This is not so. Indeed, district sincerity is closely related to trembling-hand perfection, a standard refinement of Nash equilibrium. As we have seen, the Nash solution concept may include some unreasonable situations as equilibria, since it allows players to use weakly dominated strategies when they are not decisive. The idea of perfection consists in trying to eliminate unreasonable Nash Equilibria making sure that players are decisive with positive probability, assuming that players "tremble" and make mistakes with some probability when playing. Trembling-hand perfect equilibria, as defined by Selten (1975), are the limiting equilibria when the probability with which the players tremble and make mistakes vanishes. We show, first, that every perfect equilibrium is district sincere. Intuitively, consider a scenario with a small, but non-negligible chance for a voter to be decisive in her district, as a result of errors by the other voters, but a negligible chance to be decisive overall. Since district sincerity requires voters to behave as if they are decisive in their district, such a voter would want to change her strategy, should her trembling-hand perfect equilibrium strategy violate district sincerity. Next, we prove the existence of a unique pure strategies perfect equilibrium outcome, which is, then, the unique district sincere outcome in pure strategies.

We move on to show that the logic of our result goes through in a single district proportional election as well. In this case, a single, national district is populated by voters who elect their representatives from the two parties[2] to a national parliament

[2]It may seem, at first sight, unrealistic to have only two parties with a proportional system, in the light of "Duverger's Hypothesis", which claims that proportional representation favors a multiparty political system. However, political systems such as those of Australia, Austria, Germany and Ireland cast serious doubts on the validity of Duverger's Hypothesis (see Riker (1982)). In a theoretical paper, De Sinopoli and Iannantuoni (2007) show that proportional representation leads

in a proportional election. We choose a general mechanism to transform votes into seats, requiring a minimum number of votes to obtain a certain number of seats in parliament for the left and allowing for any majority premium. Again, the policy outcome is more leftist the higher is the number of seats won by the left. With only one district, district sincerity does not have bite. Trembling-hand perfection, though, comes to the rescue, since the chance for a voter to be decisive, as a result of errors by the other voters, between the election of a critical number of leftist representatives or one less, when the number of votes obtained by the left is equal to the minimum number of votes required to obtain precisely such a critical number of seats, is infinitely higher than in all other situations. As in the multidistrict majority case, we can prove that there exists a unique pure strategy perfect equilibrium outcome.

In sum, under both a majoritarian and a proportional system, the pure perfect equilibrium outcome is unique.[3] We exploit the uniqueness of the outcome to compare equilibrium policies in the two electoral systems. This is intended to address a general presumption in the literature, namely that the outcomes of proportional systems should be more moderate than those of majoritarian systems. Our framework suggests that this may be true for some - but, interestingly, not for all- distributions of voters in the districts. We start our comparison with two leading cases, with full voters' homogeneity across districts and with maximal voters' dishomogeneity across districts. In the former case, we find that the outcome may differ depending on which electoral system is adopted. A single district proportional system favors a more moderate outcome, since it protects minorities dispersed in different districts more than a multidistrict majority system. In the latter, the outcomes are, instead, the same

to a two-party system, when the electorate votes strategically.

[3] Such outcome cannot be eliminated by sophisticated voting.

independently of the electoral system. Indeed, in a multidistrict majority system, with a higher concentration of like-minded voters, votes are wasted on a candidate who would win anyway. In situations falling in between the previous two polar cases, where a number of districts are populated by very homogeneous voters, while some other districts are extremely polarized, the single district proportional system still leads to outcomes which are as moderate as - or possibly more than- those induced by the multidistrict majoritarian system.

We can show, however, that a general result, whereby in all circumstances the outcome under a proportional system is at least as moderate as the outcome under a majoritarian system, cannot be hoped for. Indeed, it may be the case that a multi-district majority system could lead to a more moderate outcome than a single district proportional system. This would happen, for instance, with an overwhelmingly, say, leftist electorate which is highly concentrated in some districts, where the left wins by a landslide, while the rightist minority is sufficiently well distributed elsewhere for the right to carry several districts by a small margin. This would moderate the outcome with respect to a single district proportional election where an overwhelmingly leftist electorate would lead to a clearer parliamentary majority for the left. In these circumstances, though, the moderation of the outcome and the protection of minorities achieved by the majoritarian system, would seem to fly in the face of democratic representation.

The paper is related to the theoretical literature in political economy that analyzes the strategic voting behavior of individual citizens in various institutional settings and to the game-theoretical literature that tries to identify the appropriate solution concept for voting models. As regards the political economy literature, our district sincerity concept is related to the conditional sincerity concept adopted in Ingberman and Rosenthal (1997). In their work, given the policy positions of two parties, strategic voters vote in different contests, each one under majority rule. The authors show how a form of divided government, namely a situation in which the majority in one contest is in favor of the party which loses the other contest, may occur in equilibrium when the voters vote in a conditional sincere manner, i.e. the voters' strategies are sincere contest by contest. Our assumption that the policy outcome depends on the *number of seats* the two parties obtain in parliament, constitutes a novelty with respect to the literature on parliamentary elections - e.g. Alesina and Rosenthal (1995, 1996) and Persson and Tabellini (2000)- where policies depend on the *share of votes* the two parties obtain in the election. Alesina and Rosenthal (1995, 1996) analyze the strategic behavior of voters in an institutional setting with two branches of government, the executive, elected by plurality rule, and the legislature, elected by proportional rule, with the policy outcome being the result of a compromise between them, where the compromise is captured by the linear combination of the position of the elected President with the share of votes his party obtains in the legislative election. They rely on coalition-proof Nash equilibrium as a solution concept[4] to circumvent the difficulties of dealing with a continuum of voters. The main implication of their model is that divided government can be explained by the behavior of moderate voters, who counterpoise a president of one party voting in favor of the other party in the congressional election. Persson and Tabellini (1999, 2000) look at two-party competitions in two different scenarios defined by the electoral rule: proportional versus majoritarian elections. Proportional rule is characterized by a single national district. Majoritarian elections are defined under plurality rule

[4]More precisely they use a refinement of coalition-proof Nash equilibrium, the abstract stable set.

in three single-candidate districts, each one coinciding with a specific group of the population. Two sources of uncertainty are assumed, one affecting the three groups differently and the other the whole population uniformly. The main result is that in majoritarian countries, as opposed to proportional ones, the electoral competition is stiffer in a key district, leading to a redistribution targeted to a specific group at the expense of programs benefiting the majority of voters.

As regards the game-theoretical literature, our paper shares its methodological premise with the literature that, pursuing ideas first broached by Farquharson (1969), studies the refinements of the Nash equilibrium concept in voting games, from Moulin (1979) who bases the definition of dominance solvable voting schemes on the sophisticated voting principle, to De Sinopoli (2000) who, analyzing various refinements, such as perfection and properness, in plurality voting games, shows that only Mertens' stability satisfies the sophisticated voting principle.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 addresses the multidistrict majoritarian election, and Section 4 the proportional election. We devote Section 5 to the comparison of the policy outcomes under the two systems. Section 6 concludes the paper.

2 The model

Consider a society electing a parliament of k members.

The policy space. The unidimensional policy space $\mathbb{X}$ is a closed interval of the real line. Assume, without loss of generality, $\mathbb{X} = [0, 1]$.

Parties. There are two parties, indexed by $p \in P = \{L, R\}$. Each party p is characterized by a policy position $\theta_p \in \mathbb{X}$, with $\theta_L < \theta_R$.

Voters. There is a finite set of voters $N = \{1, 2, ..., n\}$. Each voter $i \in N$ has a most preferred policy - her bliss point- $\theta_i \in \mathbb{X}$. Voters' preferences are single peaked and symmetric. Denote as $u_i(X)$ player i's utility function over the policy space. Given the set of parties P, each voter i casts her vote for one of them. Hence, the pure strategy set of voter i is given by $S_i = \{L, R\}$, and denote $S = S_1 \times S_2 \times ... \times S_n$. A mixed strategy of player i is a vector $\sigma_i = (\sigma_i^L, \sigma_i^R)$ where each σ_i^p represents the probability that player i votes for party $p \in P$. As usual, the mixed strategy which assigns probability one to a pure strategy will be denoted by such a pure strategy.

The electoral rule. Voters vote to elect a parliament composed by k representatives. Given a pure strategy combination $s \in S$, the electoral rule determines the composition of the parliament, i.e. the seats allocated to each party. We consider two different electoral rules: majority rule and proportional rule. Let $\varphi : S \to \{0, 1, ..., k\}$ be the function that maps votes into the number of seats allocated to party L. The number of seats allocated to party R is then $k - \varphi(s)$.

The policy outcome. The final policy outcome is the result of a bargaining process between parties. We do not explicitly model this bargaining process. We assume that it depends only on the number of seats each party has in the parliament. In other words, we assume the existence of a function $X(.)$, mapping the number of seats obtained by party L into the policy space, i.e., $X : \{0, 1, ..., k\} \to \mathbb{X}$. We assume that $X(.)$ is a decreasing function, i.e. the higher the number of seats party L obtains in parliament, the more leftist the policy.

Given the electoral rule φ and the policy outcome function X, the utility that voter $i \in N$ gets under the pure strategy combination s is:

$$U_i(s) = u_i(X(\varphi(s))).$$

Given a mixed strategy combination $\sigma = (\sigma_1, ..., \sigma_n)$, the probability that $s = (s_1, s_2, ..., s_n)$ occurs is

$$\sigma(s) = \prod_{i \in N} \sigma_i^{s_i},$$

since players make their choices independently of each other.

The expected utility player i obtains under the mixed strategy combination σ is:

$$U_i(\sigma) = \sum \sigma(s) U_i(s).$$

We shall write $\sigma = (\sigma_{-i}, \sigma_i)$, where $\sigma_{-i} = (\sigma_1, ...\sigma_{i-1}, \sigma_{i+1}, ...\sigma_n)$ denotes the $(n-1)$ $-$tuple of strategies of the players other than i. Furthermore, s_i denotes the mixed strategy σ_i assigning probability one to the pure strategy s_i.

For $j \in \{1, 2, ..., k\}$, define $\alpha_j = \frac{X(j)+X(j-1)}{2}$. A voter with bliss point equal to α_j is indifferent between a parliament with j or $j-1$ members of party L. For simplicity, we assume no such voter exists[5].

An outcome is a probability distribution over policies. We call "pure" an outcome that assigns probability one to a given policy, and we denote it by that policy.

3 The Multidistrict Majoritarian Election

Consider an electorate composed of voters populating a number of districts. Each district elects one representative from one of the two existing political parties to the national parliament by majority rule. Formally, the electorate has been divided into k districts, indexed by $d \in D = \{1, 2, ..., k\}$. Let N_d be the set of voters in district d, i.e. $N_1, N_2, ..., N_k$ is the partition of N in the k districts[6]. We assume that in each district

[5] We will discuss later what happens if such an assumption is not satisfied.

[6] Notice that we denote sets with uppercase letters, and their cardinality with lowercase letters. Hence, N_1 is the set of voters and n_1 the number of voters in district 1.

d there is an odd number of voters n_d. Let $m_d \in M = \{m_1, ..., m_k\}$ be the bliss point of the median voter in district d, and, without loss of generality, assume that $m_1 \leq m_2 \leq ... \leq m_k$. Let us define the distribution[7] $F^m(\theta) = \{\# m_d \in M \text{ s.t. } m_d \leq \theta\}$, which gives the number of median voters with a bliss point not higher than θ.

In each district voters elect a representative belonging either to party L or to party R by majority rule. Given a pure strategy combination $s = (s_1, s_2, ..., s_n)$, let $s_d = (s_i)_{i \in N_d}$ be the pure strategy combination of the voters in district d. District d is won by the party which gets more votes and let $D^L(s)$ be the districts where L wins, hence the electoral rule φ^M is simply:

$$\varphi^M(s) = \# D^L(s).$$

We begin with an example showing that the solution concept has to be chosen carefully in our framework.

Example 1. Consider an electorate composed of six voters, equally divided in two districts. Each district elects one representative from one of the two existing political parties to the parliament by majority rule. The two parties, L and R, have policy positions represented by $\theta_L = 0.1$ and $\theta_R = 0.9$. The preferred policies of the three voters in the first district are 0.31, 0.4 and 0.69, respectively. The preferred policies of the three voters in the second district are 0.31, 0.6 and 0.69, respectively. Once the election is over, each party will try to implement its preferred policy, but the

[7] The assumption about the oddness of the number of voters in each district makes sure that the electoral result does not end in a tie. This implies two things. First, a pure strategy combination leads to - what we have defined as- a "pure" outcome. Second, the median is uniquely defined. We could have skipped this assumption using a deterministic tie-breaking rule and defining the median accordingly.

presence in parliament of a representative of the other party will lead to a compromise. Specifically, the policy outcome will be given by the policy position of party R if both representatives are from party R, half way between the policy positions of the two parties if each party obtains one representative and equal to the policy position of party L if both representatives are from party L, i.e. the policy outcomes are $X(0) = 0.9 > X(1) = 0.5 > X(2) = 0.1$. Every voter i's utility is given by minus the distance between her preferred policy and the policy outcome X. Everybody voting L is a pure strategy Nash equilibrium, with outcome $X(2) = 0.1$, since no voter can change the outcome by voting R, when everybody else is voting L. Since the most leftist voter in each district strictly prefers a parliament with one representative from the left to a parliament with two representatives from the left - 0.31 is closer to 0.5 than to 0.1-, and the most rightist voter in each district prefers a parliament with one representative from the right to a parliament with two representatives from the right - 0.69 is closer to 0.5 than to 0.9- , there are no dominated strategies. Hence, there is a pure strategy undominated equilibrium where everybody votes L, with outcome $X(2) = 0.1$. Analogously, there is a pure strategy undominated equilibrium where everybody votes R, with outcome $X(0) = 0.9$.

This shows that the very choice of the solution concept is a delicate issue in the present framework. Not even the undominated equilibrium concept, which is often used in voting models with two parties to narrow down the multiplicity associated with the Nash solution concept, is of much help. Next, we define an intuitive solution concept which turns out to lead to a unique equilibrium outcome.

3.1 District Sincerity

A strategy combination is district sincere if, given the strategies of the players in the other districts, every voter who strictly prefers the left (respectively, the right) to win in her district votes for the left (the right). Formally, given σ, we shall write $\sigma = \left(\sigma^{-d}, \sigma^{d}\right)$, where $\sigma^{-d} = (\sigma_i)_{i \in N \diagup N_d}$ denotes the $(n - n_d)-$tuple of strategies of the players outside the district d while $\sigma^{d} = (\sigma_i)_{i \in N_d}$ denotes the n_d-tuple of strategies of the players in the district d. Moreover, let L^d $\left(\text{resp. } R^d\right)$ denote the n_d-tuple of pure strategies of the players in the district d where everybody votes for party[8] L (R).

Definition 1 ***District Sincerity.*** *A strategy combination σ is district sincere if for every district d and for every player i in district d the following holds:*

$$\begin{aligned} U_i\left(\sigma^{-d}, L^d\right) - U_i\left(\sigma^{-d}, R^d\right) &> 0 \quad \textit{then } \sigma_i = L \\ U_i\left(\sigma^{-d}, L^d\right) - U_i\left(\sigma^{-d}, R^d\right) &< 0 \quad \textit{then } \sigma_i = R \end{aligned}$$

Notice[9] that every district sincere strategy combination is an equilibrium, because a player affects the outcome only if she is pivotal in her district and district sincerity implies that the outcome is affected in the "right" direction.

[8] Hence, L wins district d.

[9] For simplicity, we have written the definition of district sincerity with the n_d-tuple of strategies of the players in district d given by everybody voting for L (resp. R). At the cost of a heavier notation, we could have written any n_d-tuple of strategies leading to a victory of L (R) in district d.

The following formula provides an easy way to compute the number of districts won by the left in a district sincere equilibrium[10],

$$\bar{d}^{M}=\begin{cases}0 & if\ m_1>\alpha_1\\ \max d \text{ s.t } m_d\leq\alpha_d & if\ m_1\leq\alpha_1.\end{cases}\tag{1}$$

In words, given all districts d such that the median voter location m_d is on the left of α_d (i.e. the average of the outcomes when L wins d and $(d-1)$ districts), take the right-most of them. Since the districts are ordered according to the bliss points of their median voters, from the left-most to the right-most and the policy outcome is decreasing in the number of districts carried by the left, which implies that the indifference condition between a parliament with d and $d-1$ leftist representatives, i.e. α_d, is decreasing in d, the number of districts won by the left and, hence, the electoral result, is uniquely pinned down by the number of median voters with a bliss point m_d to the left of their "own" indifference condition, α_d.

Example 1 (Continued). Consider the undominated equilibrium in which everybody votes for the left, with outcome $X(2)=0.1$. This cannot be a district sincere equilibrium since all voters prefer a parliament with one leftist representative to a parliament with two leftist representatives. Analogously, for the undominated equilibrium in which everybody votes for the right, with outcome $X(0)=0.9$, all voters prefer a parliament with one rightist representative to a parliament with two rightist representatives. The only composition of the parliament compatible with district sincerity has one seat for the left and one for the right. To see this, observe that the median voter in the first district, i.e. the voter with bliss point at 0.4, prefers a

[10] We have assumed that no bliss point is equal to α_1, hence $m_1\neq\alpha_1$, and analogously $m_d\neq\alpha_d$. However, since we are going to discuss in some cases also what happens if these conditions do not hold, we prefer to define $\bar{d}$ independently from these conditions.

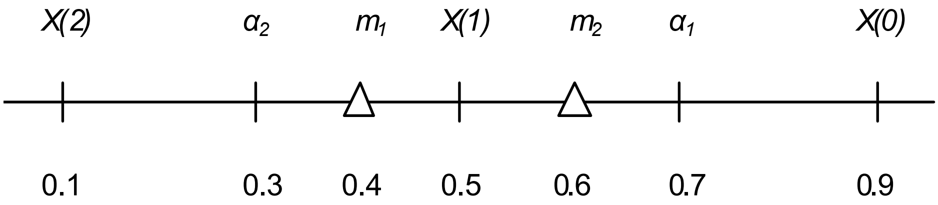

Figure 1: Unique outcome $X(1)$.

parliament with one representative per party to a parliament with two representatives of either party and, likewise, the median voter in the second district, i.e. the voter with bliss point at 0.6, prefers a parliament with one representative per party to a parliament with two representatives of either party. The indifference point between a parliament with zero or one representative from the left is $\alpha_1 = \frac{0.9+0.5}{2} = 0.7$ and the indifference point between a parliament with one or two representatives from the left is $\alpha_2 = \frac{0.5+0.1}{2} = 0.3$. Using formula (1), we see that $\bar{d}^M = 1$, since $m_1 = 0.4 < \alpha_1 = 0.7$ and $m_2 = 0.6 > \alpha_2 = 0.3$. The associated policy outcome is $X(1) = 0.5$ (see Figure 1). The pure strategy combination whereby the three voters in district one vote for the left and the three voters in district two vote for the right supports the outcome $X(1) = 0.5$ and is a district sincere equilibrium. Indeed, consider voters in district one. The strategy combination of the voters in district two implies that the right carries district two. In such a case, all voters in district one prefer the left to carry district one. Similarly, for voters in district two. Suppose there is an equilibrium with outcome $X(0) = 0.9$. In a district sincere equilibrium, if the right carries district two, all voters in district one prefer the left to carry district one, since all voters in district one prefer $X(1)$ to $X(0)$, which contradicts $X(0) = 0.9$ being a district sincere outcome. Similarly, for the other outcome $X(2) = 0.1$.

Next, we prove that the uniqueness of the outcome holds generally when district sincerity is adopted as a solution concept and that district sincerity, far from being an arbitrary solution concept, is related to perfection.

3.2 Equilibrium

3.2.1 Uniqueness of the District Sincere Outcome

First, we prove that the unique pure strategy district sincere outcome is the outcome where party L wins exactly $\bar{d}^M$ districts, as defined in the formula (1) above.

Proposition 1 $X\left(\bar{d}^M\right)$ *is the unique pure strategy district sincere equilibrium outcome*[11]*.*

Proof. We first prove that there exists a pure strategy district sincere equilibrium (PDSE) with outcome $X\left(\bar{d}^M\right)$.

Consider the following strategy combination $\bar{s} = (\bar{s}_1, ..., \bar{s}_n)$ with:

$$\bar{s}_i = L \;\; if \; i \in d \leq d^M \; and \; \theta_i < \alpha_{\bar{d}^M} \; or \; i \in d > d^M \; and \; \theta_i < \alpha_{\bar{d}^M+1}$$

$$\bar{s}_i = R \;\; if \; i \in d \leq d^M \; and \; \theta_i > \alpha_{\bar{d}^M} \; or \; i \in d > d^M \; and \; \theta_i > \alpha_{\bar{d}^M+1}$$

(i.e., in every district $d \leq \bar{d}^M$, every voter i with $\theta_i < \alpha_{\bar{d}^M}$ votes for party L, and every voter i with $\theta_i > \alpha_{\bar{d}^M}$ votes for party R; in every district $d > \bar{d}^M$: every voter i with $\theta_i < \alpha_{\bar{d}^M+1}$ votes for L, and every voter i with $\theta_i > \alpha_{\bar{d}^M+1}$ votes for party R). Notice that under $\bar{s}$ party L wins every district $d \leq \bar{d}^M$, because in such a case $m_d < \alpha_d$, while R wins all the district $d > \bar{d}^M$, since in such a case $m_d > \alpha_d \geq \alpha_{\bar{d}^M+1}$,

[11] In case $m_{\bar{d}} = \alpha_{\bar{d}}$, we would have two possible outcomes $X(\bar{d})$ and $X(\bar{d}-1)$.

hence the outcome of $\bar{s}$ is $X(\bar{d}^M)$. Furthermore $\bar{s}$ is district sincere, since in every district where L wins voters who prefer $X(\bar{d}^M)$ to $X(\bar{d}^M - 1)$ vote for L and the others for R, while in the district where R wins voters vote according to their preferences over $X(\bar{d}^M)$ and $X(\bar{d}^M + 1)$.

We now prove that no other PDSE outcome exists. Suppose we have a PDSE with $\hat{d} \neq \bar{d}^M$ districts won by L. District-sincerity implies that in districts won by L, every voter i with $\theta_i < \alpha_{\hat{d}}$ votes for L, and every voter i with $\theta_i > \alpha_{\hat{d}}$ votes in favor of party R. Moreover, in districts in which R is getting the majority, voter i with $\theta_i < \alpha_{\hat{d}+1}$ votes for L, and voter i with $\theta_i > \alpha_{\hat{d}+1}$ votes for R. Suppose first that $\hat{d} < \bar{d}^M$, then it must be $\alpha_{\bar{d}^M} \leq \alpha_{\hat{d}+1} < \alpha_{\hat{d}}$ and hence district sincerity implies that party L gets at least $\bar{d}^M$ districts, which contradicts $X\left(\hat{d}\right)$ being a district sincere equilibrium outcome. *Mutatis mutandis*, $\hat{d} > \bar{d}^M$ implies $\alpha_{\hat{d}+1} < \alpha_{\hat{d}} \leq \alpha_{\bar{d}^M+1}$ and this with district sincerity and the fact that $\alpha_{\bar{d}^M+1} < m_{\bar{d}^M+1}$implies party R wins at least $\left(k - \bar{d}^M\right)$ districts, and, hence, party L wins at most $\bar{d}^M$ districts contradicting $X\left(\hat{d}\right)$ being a district sincere equilibrium outcome.■

Given the assumption that no voter is located in α_j $(j = 1, ..., k)$, if σ is district sincere and assigns probability one to a given policy, then σ is a pure strategy combination. Hence, we have:

Corollary 2 $X\left(\bar{d}^M\right)$ *is the unique "pure" outcome induced by district sincere equilibria.*

3.2.2 Perfect Equilibria

Next, we show that district sincerity is related to trembling-hand perfection, the solution concept introduced by Selten (1975).

Definition 2 *A completely mixed strategy σ^{ε} is an ε-perfect equilibrium if*

$$\begin{aligned} \forall i &\in N, \;\; \forall s_i, s_i^{'} \in S_i \\ if \;\; U_i\left(s_i, \sigma_{-i}^{\varepsilon}\right) &> U_i\left(s_i^{'}, \sigma_{-i}^{\varepsilon}\right) \;\; then \\ \sigma_i^{\varepsilon}\left(s_i^{'}\right) &\leq \varepsilon. \end{aligned}$$

A strategy combination σ is a perfect equilibrium if there exists a sequence $\{\sigma^{\varepsilon}\}$ of ε-perfect equilibria converging (for $\varepsilon \to 0$) to σ.

Since a dominated strategy is never a best reply to a completely mixed strategy of the opponent and, hence, in every ε-perfect equilibrium it is played with probability less than ε, the perfect equilibrium concept is a refinement of the undominated equilibrium concept. The next proposition shows that, in this model, it is a refinement also of district sincerity. Intuitively, if a voter has a positive probability of being pivotal in her district as a consequence of the errors of the other voters and her equilibrium strategy is not district sincere, such a voter cannot be playing a best reply, since district sincerity requires her to behave as if she were pivotal in her district.

Proposition 3 *Every perfect equilibrium σ is district sincere.*

Proof. Let $f_i(\sigma)$ denote the probability player i is pivotal under the strategy combination σ in her district d. Clearly, we can write:

$$U_i(L, \sigma_{-i}) - U_i(R, \sigma_{-i}) = f_i(\sigma)\left[U_i\left(\sigma^{-d}, L^d\right) - U_i\left(\sigma^{-d}, R^d\right)\right] \tag{2}$$

and, if $\sigma >> 0$, then $f_i(\sigma)$ is strictly positive. Suppose now σ is not district sincere. This implies there exists a district d and a player $i \in N_d$ such that either $U_i\left(\sigma_{-d}, L^d\right) - U_i\left(\sigma_{-d}, R^d\right) > 0$ and $\sigma_i(R) > 0$ or $U_i\left(\sigma_{-d}, L^d\right) - U_i\left(\sigma_{-d}, R^d\right) < 0$ and $\sigma_i(L) >$

0. Let us consider the first case. Take a sequence of completely mixed strategy combinations σ^{ε} converging to σ. Sufficiently close to σ, $f_i(\sigma^{\varepsilon})$ is strictly positive as well as $\left[U_i\left(\sigma^{\varepsilon^{-d}}, L^d\right) - U_i\left(\sigma^{\varepsilon^{-d}}, R^d\right)\right]$ and hence R is not a best reply for player i. It follows that if σ^{ε} is a sequence of $\varepsilon-$perfect equilibria, $\sigma_i^{\varepsilon}(R) \leq \varepsilon$, and hence $\sigma_i(R) = 0$. *Mutatis mutandis* the second case.■

Propositions 1 and 3 directly imply that the only possible pure strategy perfect equilibrium outcome of the model can be $X\left(\bar{d}^M\right)$. Since not every pure district sincere equilibrium is perfect, we still have to prove that there exists a pure strategies perfect equilibrium whose outcome is $X\left(\bar{d}^M\right)$. This is accomplished considering $\bar{s}$ as defined in the proof of Proposition 1. From (2), it is immediate that $\bar{s}$ is a best reply to every strategy combination sufficiently close to it, hence perfect[12]. Then, we have:

Proposition 4 *$X\left(\bar{d}^M\right)$ is the unique pure strategy perfect equilibrium outcome.*

Moreover, from Corollary 2, Propositions 3 and 4 immediately follows that:

Corollary 5 *$X\left(\bar{d}^M\right)$ is the unique "pure" outcome induced by perfect equilibria.*

We go back to our example to convince the reader that, while the *outcome* is unique, uniqueness in terms of *equilibrium strategies* cannot be hoped for. Furthermore, we show that a mixed strategy equilibrium exists, supporting the district sincere equilibrium outcome with some probability (positive, but different from one). Hence, the uniqueness of the outcome must rely either on the use of pure strategies, or,

[12]This shows also that $\bar{s}$ is a strictly perfect equilibrium (Okada, 1981) and a stable set as defined in Kholberg and Mertens (1986). Notice that $\bar{s}$ is an absorbing retract (Kalai and Samet, 1984) and, hence, also a stable set accordingly to the definition of Mertens (1989).

when mixed strategies are allowed, on limiting the analysis to outcomes assigning probability one to a given policy.

Example 1 (Concluded). We know that $X(1) = 0.5$ is the unique pure strategy district sincere equilibrium outcome and, according to Proposition 3, it is the only pure strategy perfect equilibrium outcome. Observe, though, that there are two different pure strategy district sincere and perfect equilibria[13]. In one, discussed previously, every voter in district one votes for the left and every voter in district two votes for the right; in the other, every voter in district one votes for the right and every voter in district two votes for the left. The latter is a district sincere equilibrium since voters in district one prefer the right to carry their district if the strategy combination of the voters in district two implies that the left carries district two. Similarly, for voters in district two. The game has also a mixed equilibrium ($\bar{\sigma}$) in which voters with bliss point at 0.31 vote for the left, voters with bliss point at 0.69 vote for the right, while the median voter in district one plays the mixed strategy $\frac{1}{3}L + \frac{2}{3}R$ and the median voter in district two plays $\frac{2}{3}L + \frac{1}{3}R$. Under $\bar{\sigma}$, $X(0)$ occurs with probability $\frac{2}{9}$, $X(2)$ with probability $\frac{2}{9}$ and $X(1)$ with probability $\frac{5}{9}$. It is easy to verify that $\bar{\sigma}$ is district sincere. Consider voters in district one. The strategy combination of the voters in district two implies that the left wins with probability equal to $\frac{2}{3}$ in district two. In such a case the median voter of district one is indifferent between a leftist or a rightist winning in her district, while the voter with bliss point at 0.31 strictly prefers[14] that district one is won by the left, while the voter with bliss point at 0.69 will prefer[15] that district one is won by the right. Similarly, for voters in district two.

[13] Both of them are also strictly perfect and stable.

[14] Since $\frac{1}{3}(-0.19) + \frac{2}{3}(-0.21) > \frac{1}{3}(-0.59) + \frac{2}{3}(-0.19)$.

[15] Since $\frac{1}{3}(-0.21) + \frac{2}{3}(-0.19) > \frac{1}{3}(-0.19) + \frac{2}{3}(-0.59)$.

Now, we can prove that $\bar{\sigma}$ is perfect and that even applying stronger solution concepts than perfection, such as strategic stability (Mertens, 1989), we cannot eliminate it. Notice that $\bar{\sigma}$ is also quasi-strict (this follows from $\bar{\sigma}$ being district sincere and from the fact that, given that in each district the median voter randomizes and the other two voters vote one for L and one for R, voters are pivotal with positive probability). It follows that it is isolated since the equilibria near $\bar{\sigma}$ can be studied simply analyzing the following 2×2 game (Γ) among the two median voters (the row player being the one in district one).

	L	R
L	$-0.3, -0.5$	$-0.1, -0.1$
R	$-0.1, -0.1$	$-0.5, -0.3$

This game has two pure strategy equilibria (L, R), (R, L) and a mixed one

$$\left(\frac{1}{3}L + \frac{2}{3}R, \frac{2}{3}L + \frac{1}{3}R\right),$$

which correspond to $\bar{\sigma}$. Since $\left(\frac{1}{3}L + \frac{2}{3}R, \frac{2}{3}L + \frac{1}{3}R\right)$ is isolated and quasi-strict then it is a strongly stable equilibrium of Γ (see van Damme, 1991:55, Th. 3.4.4). Moreover, because the other players are using their strict best reply in $\bar{\sigma}$, it follows that $\bar{\sigma}$ is a strongly stable equilibrium (Kojima et *al*., 1985) of the voting game, and, hence, a Mertens' stable set.

4 Proportional Representation

We turn now to a society composed of voters electing a number of representatives from two political parties to the national parliament by proportional rule. We analyze the case of one, national district electing k representatives. We assume, without loss of generality, that the voters' bliss policies are ordered so that $\theta_1 \leq \theta_2 \leq ... \leq \theta_n$, and are distributed in the national district according to the distribution $F(\theta) = \{\#i \in N \text{ s.t. } \theta_i \leq \theta\}$, which gives the number of voters with a bliss point not higher than θ.

There are various rules used in proportional systems to transform votes into seats. We use a general one, allowing for any majority premium. To obtain d representatives, $d = 0, 1, ..., k$, party L needs at least ν_d votes[16], with $\nu_{d+1} > \nu_d$, $\forall d$ - i.e. to elect exactly d representatives party L needs a number of votes in between $[\nu_d, \nu_{d+1})$.

Given a pure strategy combination $s = (s_1, s_2, ..., s_n)$ let $N_d^L(s)$ be the set of citizens voting for party L under s, and let us define by $n_d^L(s)$ its cardinality. Hence, there exists a unique d^* such that $n_d^L(s) \in [\nu_{d^*}, \nu_{d^*+1})$, and the electoral rule φ^P is simply:

$$\varphi^P(s) = d^*.$$

4.1 Equilibrium

When the election is held in a single, national district obviously district sincerity cannot help. We thus focus directly on trembling-hand perfection as a solution concept.

[16] As in the multidistrict majoritarian case (see footnote 9), we rely on the use of a deterministic rule to pin down the allocation of seats .

As for the multidistrict majoritarian election, we define

$$\bar{d}^P = \begin{cases} 0 & if\ F(\alpha_1) < \nu_1 \\ \max d \text{ s.t } F(\alpha_d) \geq \nu_d & if\ F(\alpha_1) \geq \nu_1 \end{cases} \tag{3}$$

In words, $\bar{d}^P$ is the maximum number of seats for the left party such that the number of voters whose bliss points are on the left of α_d (that is the outcome averaging a parliament with d and $d-1$ seats for L) is greater or equal to the minimum number of votes needed to elect d representatives for party L. The number of seats for the left and, hence, the electoral result, can be computed counting the number of voters who prefer a parliament with d to a parliament with $d-1$ leftist representatives and checking whether such a number is enough to elect exactly d representatives according to the electoral rule. Since the policy outcome is decreasing in the number of seats carried by the left and more votes are required to elect more representatives of the left to the parliament, $\bar{d}^P$ is pinned down uniquely. Before proving that $X\left(\bar{d}^P\right)$ is the unique pure strategy perfect equilibrium outcome, we discuss an example.

Example 2. Consider an electorate composed of six voters electing two representatives from two political parties to the national parliament by proportional rule. At least two votes are required to elect one representative and at least five votes to elect two representatives. The two parties, L and R, have policy positions represented by $\theta_L = 0.1$ and $\theta_R = 0.9$. The preferred policies - i.e. the bliss points- of the six voters are 0.31, 0.31, 0.4, 0.6, 0.69 and 0.69 respectively. The policy outcomes are: $X(0) = 0.9 > X(1) = 0.5 > X(2) = 0.1$. Every voter i's utility is given by minus the distance between her bliss point and the policy outcome X. In this case there are six voters with bliss point to the left of $\alpha_1 = 0.7$ and no voters with bliss point to the left of $\alpha_2 = 0.3$. Two votes are needed to elect one representative, i.e. $\nu_1 = 2$, and five votes to elect two representatives, i.e. $\nu_2 = 5$. According to formula (3), the number of seats won by the left is $\bar{d}^P = 1$, since $6 > 2$ and $0 < 5$, and the policy outcome is 0.5. The strategy combination whereby the two voters with bliss point at 0.31 vote L and all the others vote R supports the outcome $X(1) = 0.5$. To see that it is perfect, consider the completely mixed strategy combination σ^ε with $\sigma_i^\varepsilon = (1 - \varepsilon^6) L + \varepsilon^6 R$ for the first two voters and $\sigma_i^\varepsilon = (1 - \varepsilon) R + \varepsilon L$ for the remaining four. Voting L is a strict best reply for the first two voters to the strategy combination we are considering, since if one of them votes R, the outcome moves to $X(0) = 0.9$ which is farther away from 0.31 than $X(1) = 0.5$, and it remains so for near-by strategies. Each of the other voters is pivotal between the election of one or two candidates for the left with a probability which is infinitely greater than every other probability in which her vote matters. Since these voters are located to the right of $\alpha_2 = 0.3$, they prefer R to L and, hence, σ^ε is an $\varepsilon-$perfect equilibrium. Therefore the original strategy combination is perfect. To see that the perfect equilibrium outcome $X(1) = 0.5$ is unique, suppose $X(0) = 0.9$ is another outcome induced by a perfect equilibrium strategy combination, σ^0. Observe that, since for every sequence of completely mixed strategy combinations converging to σ^0, for every player, the probability of being pivotal between $X(1)$ and $X(0)$ is infinitely greater than the probability of being pivotal between $X(2)$ and $X(1)$, it must be that under σ^0 all voters vote L (since all voters prefer $X(1)$ to $X(0)$), which is incompatible with the left winning zero seats. A symmetric argument applies to $X(2) = 0.1$.

Intuitively, trembling-hand perfection helps, since the chance for a voter to be decisive, as a result of trembles by the other voters, between a parliament with $\bar{d}^P$ leftist representatives or one less is infinitely higher than in any other circumstance in which her vote is decisive. When only consecutive electoral outcomes - one more or one less seat for the left- turn out to be really crucial, since the policy outcome moves farther to the right for every extra seat lost by the left, a "critical" number of seats is bound to emerge in equilibrium.

4.1.1 Uniqueness of the Perfect Equilibrium Outcome

The next Proposition proves that the perfect equilibrium outcome is unique.

Proposition 6 $X\left(\bar{d}^P\right)$ *is the unique pure strategy perfect equilibrium outcome and the unique "pure" outcome induced by perfect equilibria.*

Proof. We first prove that there exists a perfect equilibrium with the unique "pure" outcome $X\left(\bar{d}^P\right)$.

We have to analyze three cases[17]:

i) $\bar{d}^P \neq k$ and $\theta_{\nu_{\bar{d}^P}} > \alpha_{\bar{d}^P+1}$

Consider the following strategy combination $\bar{s} = (\bar{s}_1, ..., \bar{s}_n)$ with:

$$\bar{s}_i = L \;\; if \; i \in [1, 2, ..., \nu_{\bar{d}^P}]$$

$$\bar{s}_i = R \;\; if \; i \in [\nu_{\bar{d}^P} + 1, ..., n]$$

Notice that under $\bar{s}$ exactly $\bar{d}^P$ seats are won by L. Now we show that $\bar{s}$ is perfect

Notice that L is a strict best reply for every $i \in [1, 2, ..., \nu_{\bar{d}^P}]$, since if one of them votes for R instead L the outcome moves from $X\left(\bar{d}^P\right)$ to $X\left(\bar{d}^P - 1\right)$ which is worst for them since they are located to the left of $\alpha_{\bar{d}^P}$. Consider the completely mixed strategy combination σ^ε :

$$\sigma_i^\varepsilon = (1 - \varepsilon^n)\, L + \varepsilon^n R \;\; if \; i \in [1, 2, ..., \nu_{\bar{d}^P}]$$

[17] In order to avoid duplication of the proof, if $\bar{d}^P = 0$, let $\theta_0 = 0$ and, hence, refer to (ii).

$$\sigma_i^{\varepsilon} = (1-\varepsilon) R + \varepsilon L \ \ if \ i \in [\nu_{\bar{d}^P} + 1, ..., n]$$

We claim that, for ε sufficiently close to zero, σ^{ε} is an $\varepsilon - perfect$ equilibrium. Since L is a strict best reply to $\bar{s}$ for $i \in [1, 2, ..., \nu_{\bar{d}^P}]$, it remains so for close-by strategies. Notice that the probability a player $i \in [\nu_{\bar{d}^P} + 1, ..., n]$ is "pivotal" between the election of $\bar{d}^P$ and $\bar{d}^P + 1$ of L candidates is infinitely greater than every other probability in which her vote matters. Since all these players are located to the right of $\alpha_{\bar{d}^P+1}$, R is preferred for them to L and, hence, σ^{ε} is an $\varepsilon - perfect$ equilibrium. Therefore $\bar{s}$ is perfect.

ii) $\bar{d}^P \neq k$ and $\theta_{\nu_{\bar{d}^P}} < \alpha_{\bar{d}^P+1}$

Let $\tilde{n}$ the largest i such that $\theta_i < \alpha_{\bar{d}^P+1}$. By the definition of $\bar{d}^P$ and because $\theta_{v_{\bar{d}^P}} < \alpha_{\bar{d}^P+1}$, we have $\tilde{n} \in [v_{\bar{d}^P}, v_{\bar{d}^P+1})$. Consider the following strategy combination $\tilde{s}$:

$$\tilde{s}_i = L \ \ if \ i \in [1, 2, ..., \tilde{n}]$$

$$\tilde{s}_i = R \ \ if \ i \in [\tilde{n} + 1, ..., n]$$

Notice that under $\tilde{s}$ exactly $\bar{d}^P$ seats are won by L. Now we show that $\tilde{s}$ is perfect. To this end consider the completely mixed strategy combination σ^{ε} :

$$\sigma_i^{\varepsilon} = (1-\varepsilon^n) L + \varepsilon^n R \ \ if \ i \in [1, 2, ..., \tilde{n}]$$

$$\sigma_i^{\varepsilon} = (1-\varepsilon) R + \varepsilon L \ \ if \ i \in [\tilde{n} + 1, ..., n]$$

We claim that, for ε sufficiently close to zero, σ^{ε} is an $\varepsilon - perfect$ equilibrium. Notice that the probability a player is "pivotal" between the election of $\bar{d}^P$ and $\bar{d}^P + 1$ of L candidates is infinitely greater than every other probability in which her vote matters.

Because for all the players located to the left (right) of $\alpha_{\bar{d}^P+1}$, $L(R)$ is preferred to $R(L)$, σ^{ε} is an $\varepsilon - perfect$ equilibrium. Therefore $\tilde{s}$ is perfect

iii) $\bar{d}^P = k$

Let $\breve{n}$ the largest i such that $\theta_i < \alpha_k$. By the definition of $\bar{d}^P$ we have that $\breve{n} \geq \nu_k$. Consider the following strategy combination $\breve{s}$:

$$\breve{s}_i = L \ \ if \ i \in [1, 2, ..., \breve{n}]$$

$$\breve{s}_i = R \ \ if \ i \in [\breve{n} + 1, ..., n]$$

Notice that under $\breve{s}$ all the k seats are won by L. Moreover for every completely mixed strategy combination close to $\breve{s}$, the probability a player is "pivotal" between the election of k and $k-1$ of L candidates is infinitely greater than every other probability in which her vote matters. Hence, $\breve{s}$ is perfect.

Now we prove that no other "pure" outcome is induced by a perfect equilibrium. Suppose we have a perfect equilibrium σ^{δ} which induces $X(\delta)$ as the policy outcome. Since, for every sequence of completely mixed strategy combinations converging to σ^{δ}, for every player, the probability of the event "being pivotal between $X(\delta+1)$ and $X(\delta)$" is infinitely greater than the probability of the event "being pivotal between $X(\delta+j)$ and $X(\delta+1+j)$" ($j = 1, .., k-\delta-1$) and the probability of the event "being pivotal between $X(\delta)$ and $X(\delta-1)$" is infinitely greater than the the probability of the event "being pivotal between $X(\delta-j)$ and $X(\delta-1-j)$" ($j = 1, .., k-\delta-1$) we must have:

(α) $\forall i \ s.t. \ \theta_i < \alpha_{\delta+1} \qquad \sigma_i^{\delta} = L$

(β) $\forall i \ s.t. \ \theta_i > \alpha_{\delta} \qquad \sigma_i^{\delta} = R$

Suppose $\delta < \bar{d}^P$. This would imply that $\alpha_{\delta+1} \geq \alpha_{\bar{d}^P}$, and, by ($\alpha$), it follows that in

σ^{δ} party L would receive at least $\nu_{\bar{d}^P}$ votes contradicting the fact that just δ of its candidates are elected.

Suppose $\delta > \bar{d}^P$. Notice that $\delta > \bar{d}^P$ implies that $\alpha_{\bar{d}^P+1} \geq \alpha_{\delta}$ and the above condition (β) implies that in σ^{δ} party R takes at least all the votes of the voters located to the right of $\alpha_{\bar{d}^P+1}$. By the definition of $\bar{d}^P$, it follows that, even if all the others voters vote for L, the leftist party cannot win $\bar{d}^P + 1$ seats, which contradicts $\delta > \bar{d}^P$.■

5 Comparison of Electoral Systems

It is interesting to compare the equilibrium outcome in the single district proportional and the multidistrict majority system. Such a comparison is made straightforward by our uniqueness results. For the sake of the comparison, in this section, we specify a particular electoral rule dictating the minimum number of votes required to elect a member of parliament when the single district proportional system is adopted. The minimum number of votes needed to elect d members of parliament with the single district proportional system is

$$\frac{n}{k}(d-1) + \frac{1}{2}\frac{n}{k}. \tag{4}$$

In case the multidistrict majority system is used, the electoral rule requires the leftist party to obtain at least half of a district votes in order to carry the district.

When a multidistrict majority system is adopted, the electoral outcome may depend on how voters are distributed across districts. Since the electoral outcome is independent of voters' distribution across districts when a single district proportional system is adopted, the comparison of electoral outcomes between the two systems is bound to be affected by the distribution of voters across districts. In order to come to grips with such an issue, we consider two extreme distributions of voters across districts. First, we look at a situation of homogeneity across districts. This case represents a society where districts of the multidistrict majority system are similar to each other and similar to the single district of the proportional system, in terms of the political preferences of their voters. More specifically, districts are homogeneous in the sense that their median voters have the same preferences which coincide with the preferences of the median voter of the single district in the proportional system. Second, we examine a case of heterogeneity across districts. In this alternative society, districts of the multidistrict majority system are characterized by diverse political orientations, with some districts being a stronghold of the leftist party some of the rightist party and some being inhabited by voters with more mixed political orientations. Specifically we consider a situation of extreme heterogeneity across districts where - equally sized- districts have been ordered according to the political preferences of their voters, with the first district being inhabited by the first $\frac{n}{k}$ most leftist voters, the second by the next $\frac{n}{k}$ most leftist voters and the following districts being inhabited each by $\frac{n}{k}$ increasingly more rightist voters.

We find that in the case of homogeneity across districts, the outcome may differ depending on which electoral system is adopted. A single district proportional system favours a more moderate outcome, since it protects minorities dispersed in different districts more than a multidistrict majority system. In the case of extreme heterogeneity across districts, the outcomes are instead the same independently of the electoral system. Differences in electoral outcomes are a joint product of the electoral system and the distribution of voters. In very polarized societies where leftist voters are concentrated in some districts and rightist voters in others the choice of the electoral system - proportional vs. multidistrict majority- will tend not to affect the political outcome, while in very homogeneous societies where electoral districts are similar to each other in terms of the political preferences of their voters, the outcome will tend to be more moderate when elections are held with a proportional system than when elections are held with a multidistrict majority system. This is fairly intuitive since with a lower concentration of like-minded voters, in a multidistrict majority system fewer votes are wasted on a candidate who would win anyway.

In between these two polar cases, when the electorate is divided between a homogeneous group inhabiting some districts and a very polarized group inhabiting the remaining districts, the outcome of an election held with a proportional system will be more moderate than a majoritarian system. It is not possible, however, to argue that the proportional system is more moderate in all circumstances. We conclude with an example showing that there are cases where the majoritarian system may lead to a more moderate outcome than the proportional one. This can happen when an overwhelmingly, say, leftist electorate is concentrated in some districts, leading to landslide victories for the left in those districts, while the minority is sufficiently well distributed elsewhere for the right to win several districts by small margins.

5.1 Homogeneity across Districts

We first consider a situation in which each district of the multidistrict majority system has the same median voter as the single district of the proportional system, i.e. $m_d = m$ for all d. The following example points out that the two systems may give rise to different outcomes in this case.

Example 3. Consider a society with six voters electing a parliament of two members, i.e. $n = 6$, $k = 2$, two parties with policies $\theta_L = 0$ and $\theta_R = 1$ respectively and the following symmetric outcome function $X(0) = 1 > X(1) = 0.5 > X(2) = 0$. The indifference point between a parliament with one or two representatives from the left is $\alpha_2 = 0.25$ and the indifference point between a parliament with one or no representative from the left is $\alpha_1 = 0.75$. Four of the six voters are leftist, having zero as their preferred policy, i.e. their bliss points are $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$, and the remaining two are rightist, having one as their preferred policy, i.e. $\theta_5 = \theta_6 = 1$. If the multidistrict majority system is adopted, two districts - inhabited by three voters each- elect a member of parliament each. A party carries a district if it obtains at least two votes in the district. District 1 is inhabited by two voters with bliss point at 0 and one voter with bliss point at 1, i.e. the three voters in district one are $\theta_1 = \theta_2 = 0$ and $\theta_5 = 1$, and district 2 is inhabited by two voters with bliss point at 0 and one voter with bliss point at 1, i.e. the three voters in district two are $\theta_3 = \theta_4 = 0$ and $\theta_6 = 1$. Observe that the median voter in each of the two district is a voter with 0 as her preferred policy, i.e. $m_1 = m_2 = 0$. If the single district proportional system is adopted, the six voters all belong to the single district and the electoral rule (4) prescribes that at least $3(d - 0.5)$ votes are needed to elect d representatives. Observe that the median voter in the single district is a voter with 0 as her preferred policy, i.e. $m = 0$. The unique district sincere equilibrium outcome of the multidistrict majority system is $X(2) = 0$, i.e. the leftist party obtains two members of parliament and implements its preferred policy. Indeed, observe that $\alpha_2 = 0.25 > m_2 = 0$. On the other hand the unique perfect equilibrium outcome of the proportional system is $X(1) = 0.5$, i.e. the leftist party obtains one member of parliament and implements a moderate policy. Indeed, observe that $F(\alpha_1) = 4 > 3(0.5) = 1.5$ and $F(\alpha_2) = 4 < 3(1.5) = 4.5$.

In the multidistrict majority system two votes are enough to carry a district and thus four votes are enough to elect two members of parliament. The electoral rule of the proportional system, however, requires more than four votes to elect two members of parliament. The result of the election is markedly different in the two cases, with a two-nil victory for the left in the multidistrict majority system and a one-one draw in the proportional system. The policies implemented, which depend on the parliamentary strength of a party, differ as well in the two cases, with a more moderate policy in the second case. The example suggests that in a multidistrict majority system - with fairly homogeneous districts- a party may obtain a landslide victory in terms of seats in parliament without a corresponding landslide victory in terms of the number of votes, while in a proportional system there would be a closer relationship between number of seats in parliament and number of votes. The proportional system tends to moderate the electoral outcome. This happens because a minority of voters dispersed in different districts will be able to elect fewer members of parliament in a multidistrict majority system than in a single district proportional system. Since the final policy that is implemented is closer to a party preferred policy the stronger its parliamentary force, the single district proportional system is conducive to a more moderate policy outcome. The following proposition proves that this intuition carries over to less special situations. In order to be able to compare leftist and rightist policies to moderate ones in a sensible way, we assume that the outcome function is symmetric around the mid point of the policy interval. We prove that the equilibrium policy outcome - if the single district proportional system is adopted as an electoral system- is not farther away from the mid point of the policy interval than the equilibrium policy outcome in case the multidistrict majority system is adopted[18].

[18] We remind the reader that $X\left(\overline{d}^{P}\right)$ is the unique perfect equilibrium outcome in the single district proportional and $X\left(\overline{d}^{M}\right)$ the unique district sincere equilibrium outcome in the multidistrict majority systems.

Proposition 7 *Assume that $m_d = m$,$\forall d$, and that $X(d)$ is symmetric*[19] *around* 0.5*, then:*

a. if $X\left(\overline{d}^M\right) \leq 0.5$, $X\left(\overline{d}^M\right) \leq X\left(\overline{d}^P\right) \leq 0.5$*;*

b. if $X\left(\overline{d}^M\right) > 0.5$, $0.5 \leq X\left(\overline{d}^P\right) \leq X\left(\overline{d}^M\right)$.

Proof. *Part a.* We first prove that $X\left(\overline{d}^M\right) \leq X\left(\overline{d}^P\right)$. Given that $X\left(\overline{d}^M\right) \leq 0.5$, suppose, contrary to the thesis, that $X\left(\overline{d}^P\right) < X\left(\overline{d}^M\right)$, i.e. $\overline{d}^P > \overline{d}^M$ and $\alpha_{\overline{d}^P} < \alpha_{\overline{d}^M}$. Since $X\left(\overline{d}^M\right)$ is the unique district sincere equilibrium outcome, it has to be that $\alpha_{\overline{d}^P} < m_{\overline{d}^P}$, or $X\left(\overline{d}^P\right)$ would be the district sincere equilibrium outcome. Since by assumption $m_d = m$, $\forall d$, then $\alpha_{\overline{d}^P} < m$. Since m is the median, $F\left(\alpha_{\overline{d}^P}\right) < \frac{n}{2}$. Since $X\left(\overline{d}^P\right)$ is the equilibrium outcome in the proportional election, $F\left(\alpha_{\overline{d}^P}\right) \geq \frac{n}{k}\left(\overline{d}^P - 0.5\right)$. These two together imply

$$\frac{n}{2} > F\left(\alpha_{\overline{d}^P}\right) \geq \frac{n}{k}\left(\overline{d}^P - 0.5\right).$$

For $\frac{n}{2} > \frac{n}{k}\left(\overline{d}^P - 0.5\right)$ to hold, it has to be that $\overline{d}^P < \frac{k+1}{2}$, which directly implies $\overline{d}^P \leq \frac{k}{2}$. Symmetry of $X(d)$ around $\frac{1}{2}$ implies $\overline{d}^M \geq \frac{k}{2}$. We obtain $\frac{k}{2} \geq \overline{d}^P > \overline{d}^M \geq \frac{k}{2}$, which is a contradiction. We conclude that $\overline{d}^P \leq \overline{d}^M$ and thus $X\left(\overline{d}^P\right) \geq X\left(\overline{d}^M\right)$. We are left to show that $X\left(\overline{d}^P\right) \leq 0.5$. Assume, contrary to the thesis, that $X\left(\overline{d}^P\right) > 0.5$, hence $\overline{d}^M > \overline{d}^P$ and by symmetry of $X(d)$ around 0.5 we have $\alpha_{\overline{d}^P+1} \geq 0.5$. Notice that if k is even $\overline{d}^P + 1 \leq \frac{k}{2}$ and if k is odd $\overline{d}^P + 1 \leq \frac{k+1}{2}$. In both cases $\overline{d}^P + 1 \leq \frac{k+1}{2}$. Notice that $F\left(\alpha_{\overline{d}^P+1}\right) > \frac{n}{2}$, since $\alpha_{\overline{d}^P+1} \geq \alpha_{\overline{d}^M} \geq m$. Then, we have

$$F\left(\alpha_{\overline{d}^P+1}\right) > \frac{n}{2} = \frac{n}{k}\left(\frac{k+1}{2} - 0.5\right) \geq \frac{n}{k}\left(\left(\overline{d}^P + 1\right) - 0.5\right),$$

[19] i.e. $X(k-j) = 1 - X(j)$, $j = 0, 1, 2, ..., k$.

which contradicts (3). We conclude that $X\left(\overline{d}^M\right) \leq X\left(\overline{d}^P\right) \leq 0.5$.

Part b. We first prove that $X\left(\overline{d}^P\right) \leq X\left(\overline{d}^M\right)$. Given that $X\left(\overline{d}^M\right) > 0.5$, suppose, contrary to the thesis, that $X\left(\overline{d}^P\right) > X\left(\overline{d}^M\right)$, i.e. $\overline{d}^P < \overline{d}^M$ and $\alpha_{\overline{d}^P} > \alpha_{\overline{d}^M}$. Since $X\left(\overline{d}^M\right)$ is the district sincere equilibrium outcome it has to be that $m_{\overline{d}^M} \leq \alpha_{\overline{d}^M}$. Since $m_d = m$,$\forall d$, $m \leq \alpha_{\overline{d}^M}$. Since m is the median, $\frac{n}{2} \leq F(m)$. These together imply:

$$\frac{n}{2} \leq F(m) \leq F\left(\alpha_{\overline{d}^M}\right).$$

Observe that $\frac{n}{k}\left(\overline{d}^M - 0.5\right) < \frac{n}{2}$, since by symmetry of $X(d)$, $\overline{d}^M < \frac{k}{2}$. Then $\overline{d}^M > \overline{d}^P$ and such that $\frac{n}{k}\left(\overline{d}^M - 0.5\right) \leq F\left(\alpha_{\overline{d}^M}\right)$, contradicting that $X\left(\overline{d}^P\right)$ is the equilibrium outcome in the proportional election. We conclude that $\overline{d}^P \geq \overline{d}^M$ and thus $X\left(\overline{d}^P\right) \leq X\left(\overline{d}^M\right)$.

We are left to show that $0.5 \leq X\left(\overline{d}^P\right)$. Suppose $0.5 > X\left(\overline{d}^P\right)$, i.e. $\overline{d}^P \geq \frac{k+1}{2}$. District sincerity implies $\alpha_{\overline{d}^P} < m$, since $\alpha_{\overline{d}^M} \geq \alpha_{\overline{d}^P}$ and $\alpha_{\overline{d}^M} < m$. This, in turn, implies $F\left(\alpha_{\overline{d}^P}\right) < \frac{n}{2}$, since m is the median voter. Observe that $\frac{n}{2} \leq \frac{n}{k}\left(\overline{d}^P - 0.5\right)$ when $\overline{d}^P \geq \frac{k+1}{2}$. Thus

$$F\left(\alpha_{\overline{d}^P}\right) < \frac{n}{k}\left(\overline{d}^P - 0.5\right),$$

which contradicts (3). We conclude that $X\left(\overline{d}^M\right) \geq X\left(\overline{d}^P\right) \geq 0.5$.■

5.2 Heterogeneity across Districts

We now consider a situation of extreme heterogeneity across districts. We have in mind a society where some districts are the stronghold of the leftist party and some others of the rightist party. Specifically, the k districts of the multidistrict majority system are inhabited by the same odd number of voters, $n_d = \frac{n}{k}$, for all d. Moreover, districts have been ordered according to the political preferences of their voters, with the first district being inhabited by the first $\frac{n}{k}$ most leftist voters, the second by the next $\frac{n}{k}$ most leftist voters and the following districts being inhabited each by $\frac{n}{k}$ increasingly more rightist voters. Thus median voters in each district are ordered, with $m_1 \leq m_2 \leq ... \leq m_d \leq ... \leq m_k$.

Example 4. Consider a society identical to the one presented in Example 3 except for the distribution of voters in the two districts of the multidistrict majority system. In this alternative society, district 1 is inhabited by three leftist voters, with bliss points $\theta_1 = \theta_2 = \theta_3 = 0$ and median voter $m_1 = 0$, while district 2 is inhabited by one leftist voter and two rightist voters, i.e. by voters with bliss points $\theta_4 = 0$, $\theta_5 = \theta_6 = 1$ and median voter $m_2 = 1$. The unique district sincere equilibrium outcome of the multidistrict majority system is $X(1) = 0.5$. i.e. the leftist party obtains one member of parliament and implements a moderate policy. Indeed, observe that $\alpha_1 = 0.75 > m_1 = 0$ and $\alpha_2 = 0.25 < m_2 = 1$. The unique perfect equilibrium outcome of the proportional system is $X(1) = 0.5$. Indeed, observe that $F(\alpha_1) = 4 > 3(0.5) = 1.5$ and $F(\alpha_2) = 4 < 3(1.5) = 4.5$.

The example presents a society were leftist voters are more concentrated in one district of the multidistrict majority system. One of their votes is - so to speak- wasted, in the sense that the leftist candidate in the first district would be elected even with only two votes in her favour, while an extra vote would be useful to elect the leftist candidate in the second district. The following proposition proves that such an intuition carries over to more general situations and the two electoral systems - i.e. the single district proportional and multidistrict majority system- give rise to the same equilibrium outcome when districts are equally sized and ordered from left to right.

Proposition 8 *If districts are equally sized and ordered from left to right, then*

$X\left(\overline{d}^P\right) = X\left(\overline{d}^M\right)$.

Proof. *a.* Contrary to the thesis, suppose first $X\left(\overline{d}^P\right) > X\left(\overline{d}^M\right)$, i.e. $\overline{d}^P < \overline{d}^M$ and $\alpha_{\overline{d}^P} > \alpha_{\overline{d}^M}$. Recall that $\overline{d}^M$ is the maximum d satisfying $\alpha_d \geq m_d$. Furthermore, $F\left(\alpha_{\overline{d}^M}\right) \geq F\left(m_{\overline{d}^M}\right)$. Since districts are ordered and of equal size, the total number of voters up to and including the median voter of a generic district d is at least equal to the number of voters in all previous districts - $\frac{n}{k}(d-1)$- plus half of the voters in that district - $\frac{n}{k}\frac{1}{2}$-, i.e. $F(m_d) > \frac{n}{k}(d-1) + \frac{n}{k}\frac{1}{2} = \frac{n}{k}(d-0.5)$ for all d, where the strict inequality sign follows from the fact that $\frac{n}{k}$ is odd. Hence, it follows

$$F\left(\alpha_{\overline{d}^M}\right) > \frac{n}{k}\left(\overline{d}^M - 0.5\right).$$

This contradicts $X\left(\overline{d}^P\right)$ being the equilibrium outcome in the proportional election, since we found a higher d satisfying $F(\alpha_d) \geq \frac{n}{k}(d-0.5)$. We conclude that $X\left(\overline{d}^P\right) \leq X\left(\overline{d}^M\right)$.

b. Contrary to the thesis, suppose now $X\left(\overline{d}^P\right) < X\left(\overline{d}^M\right)$, i.e. $\overline{d}^P > \overline{d}^M$ and $\alpha_{\overline{d}^P} < \alpha_{\overline{d}^M}$. Since $\overline{d}^M$ is by definition the maximum d satisfying $\alpha_d \geq m_d$ and $\overline{d}^P > \overline{d}^M$, it follows $\alpha_{\overline{d}^P} < m_{\overline{d}^P}$. Given that districts are equally sized and ordered, the total number of voters strictly to the left of the median voter of a generic district d is strictly smaller than $\frac{n}{k}(d-1) + \frac{n}{k}\frac{1}{2}$, thus $F(\alpha_d) < \frac{n}{k}(d-0.5)$ for d such that $\alpha_d < m_d$. Since $\alpha_{\overline{d}^P} < m_{\overline{d}^P}$, it follows

$$F\left(\alpha_{\overline{d}^P}\right) < \frac{n}{k}\left(\overline{d}^P - 0.5\right),$$

which contradicts (3). We conclude that $X\left(\overline{d}^P\right) = X\left(\overline{d}^M\right)$.■

5.3 Between the Extremes

The result that the single district proportional system leads to a more moderate outcome than the multidistrict majoritarian system can be extended to intermediate situations where some districts are homogeneous and others are extremely heterogeneous. We have in mind a country which is fairly homogeneous, except for some specific areas which are very polarized. Let J be the set of districts where the median voters have the same bliss point, m, which is also the bliss point of the median voter of the entire population, and its complement, $\overline{J}$, the set of districts ordered according to how leftist their median voters are, as in section 5.2. We can prove the following

Proposition 9 *Assume that i) districts are equally sized, ii) $m_j = m$, for $j \in J$, iii) districts in $\overline{J}$ are ordered from left to right and iv) $X(d)$ is symmetric around* 0.5, *then:*

a. if $X\left(\overline{d}^M\right) \leq 0.5$, $X\left(\overline{d}^M\right) \leq X\left(\overline{d}^P\right) \leq 0.5$;

b. if $X\left(\overline{d}^M\right) > 0.5$, $0.5 \leq X\left(\overline{d}^P\right) \leq X\left(\overline{d}^M\right)$.

Proof. *Part a.* Suppose, contrary to the thesis, that $X\left(\overline{d}^P\right) < X\left(\overline{d}^M\right)$, i.e. $\overline{d}^P > \overline{d}^M$ and $\alpha_{\overline{d}^P} < \alpha_{\overline{d}^M}$. Since $X\left(\overline{d}^M\right)$ is the unique district sincere equilibrium outcome, it has to be that $\alpha_{\overline{d}^P} < m_{\overline{d}^P}$ If $\alpha_{\overline{d}^P} < m$, the (first part of the) proof of Proposition 7 applies; if $m < \alpha_{\overline{d}^P}$, the (second part of the) proof of Proposition 8 applies. The proof that $X\left(\overline{d}^P\right) \leq 0.5$ follows from the (first part of the) proof of Proposition 7. Part b follows similarly from the proofs of Proposition 7 and 8.■

5.4 Reversal

We can show, by means of one last example, that it is not possible to prove that, in every circumstance, the single district proportional system will necessarily lead to more moderate outcomes than the multidistrict majoritarian system. Interestingly, for some distribution of voters, the results on the comparison of the two systems may be turned around. The following example shows that if a minority of voters is sufficiently well distributed across the districts, the electoral result under a multidistrict majoritarian system may be more moderate than under a single district proportional system.

Example 5. Consider a society with twelve voters electing a parliament of four members, i.e. $n = 12$, $k = 4$, two parties with policies $\theta_L = 0$ and $\theta_R = 1$ respectively and the following symmetric outcome function

$$X(0) = 1 > X(1) = 0.75 > X(2) = 0.5 > X(3) = 0.25 > X(4) = 0.$$

Eight of the twelve voters are leftist, having zero as their preferred policy, i.e. their bliss points are $\theta_i = 0$ for $i = 1, ..., 8$, and the remaining four are rightist, having one as their preferred policy, i.e. $\theta_h = 1$ for $h = 9, ..., 12$. If the multidistrict majority system is adopted, four districts - inhabited by three voters each- elect a member of parliament each. A party carries a district if it obtains at least two votes in the district. District 1 and 2 are inhabited by three voters with bliss point at 0 each, i.e. the three voters in district 1 are $\theta_1 = \theta_2 = \theta_3 = 0$ and in district 2 $\theta_4 = \theta_5 = \theta_6 = 0$. District 3 and 4 are inhabited by one voter with bliss point at 0 and two voters with bliss point at 1, i.e. the three voters in district 3 are $\theta_7 = 0$ and $\theta_9 = \theta_{10} = 1$ and in district 4 $\theta_8 = 0$ and $\theta_{11} = \theta_{12} = 1$. Observe that the median voter in the former two districts is a voter with 0 as her preferred policy, i.e. $m_1 = m_2 = 0$, while the median voter in the latter two is a voter with 1 as her preferred policy, i.e. $m_3 = m_4 = 1$. If the single district proportional system is adopted, the twelve voters all belong to the single district and the electoral rule (4) prescribes that at least $3(d - 0.5)$ votes are needed to elect d

representatives. Observe that the median voter in the single district is a voter with 0 as her preferred policy, i.e. $m = 0$. The unique district sincere equilibrium outcome of the multidistrict majority system is $X(2) = 0.5$, i.e. the leftist party obtains two members of parliament and a moderate policy is implemented. Indeed, observe that $\alpha_2 = 0.625 > m_2 = 0$ and $\alpha_3 = 0.375 < m_3 = 1$. On the other hand the unique perfect equilibrium outcome of the proportional system is $X(3) = 0.25$, i.e. the leftist party obtains three member of parliament and implements a more leftist policy. Indeed, observe that $F(\alpha_3) = 8 > 3(2.5) = 7.5$ and $F(\alpha_4) = 8 < 3(3.5) = 10.5$.

This example shows that under a multidistrict majoritarian system the outcome may be more moderate than under a single district proportional system. A parliament with a clear majority for the left, were the election held with a single district proportional system, would turn into a hung parliament with a multidistrict majoritarian system. This happens because, due to the distribution of voters in the districts, the left wins by a big margin in two districts, while the right wins by a smaller margin in the remaining two. Hence, despite a very leftist electorate, the election would end in a tie with a multidistrict majority system. With a single district proportional system, the result of the election would, instead, reflect more closely the overall composition of the electorate. Here a tension between moderation and representation emerges.

6 Conclusion

We have studied a model where rational citizens cast their votes for candidates belonging to two parties to be elected to a national parliament. The distinguishing feature of our approach has been the assumption that the ability of a party to implement a policy close to its preferred one is determined by the its strength in parliament, i.e. by the number of seats it has in parliament. We have focused on the two most popular electoral rules in modern democracies, multidistrict majority and purely proportional representation. In both systems we have proved the existence of a unique pure strategies perfect equilibrium outcome. Thanks to the uniqueness of the outcome, the model allows a comparison of the two systems. Given the distribution of voters, we can determine which system leads to a more moderate outcome.

We can think of at least three main directions in which our model can be taken. First, the institutional setting can be enriched, allowing citizens to vote not only for the Parliament, but also for an executive, the President, say by majority rule. The policy outcome would be the combination of the President's preferred policy and of the position of the Parliament, as defined in this paper. Second, parties could be allowed to design districts strategically to maximize their number of seats in the next Parliament. Finally, strategic party competition could be addressed, having a stage in which policy motivated parties could choose policy positions to maximize their number of seats in the next Parliament. We will tackle these issues in future research endeavors.

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