Mixed Integer Programming Models with Multiple Machines, Sites and Shared Set-Ups

Laurence A. Wolsey

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The Challenge

Find ways to use what we have learnt on modeling single item problems to tackle more complicated problems with

- multiple items
- multiple production levels
- multiple machines
- more complicated variants

Find ways to share costs.

Extend Tight Formulations for Single Discrete Time Item Models

Multi-item models.

- Multiple Machines - Sequence-dependent changeovers
  Sharing Machines (with Céline Gicquel)
- Two-Level Supply Chains
  One or more production sites, multiple client areas
  Sharing Fixed Production and Transportation Costs
  Sharing Production Facilities (with Rafael Melo)

Outline

- Multi-item multi-machine model
- Modeling start-ups and changeovers
- Identical machine model
- Computational Results
- Sharing machines: finding core allocations
- 2-Level supply chains

Production with Sequence Dependent Changeovers

- In this sequencing problem the time periods are short, so at most one item is produced per period.
- The order of production is important as there are significant changeover costs every time that one switches production from one item to another.
- There are also storage costs per item and period.
- The problem is to satisfy the demands while minimizing the sum of the storage and switch-over costs.
- Discrete/all or nothing production.

The multi-item/**multi-machine**/changeover cost problem

I is the number of items, K is the number of machines. At most one item is produced per period.

$$\min \sum_{i,k} p_t^{ik} y_t^{ik} + \sum_i h_t^i s_t^i + \sum_{i,j,k,t} q^{ijk} \chi_t^{ijk}$$
$$s_t^i + \sum_k C^{ik} y_t^{ik} = d_t^i + s_t^i \ \forall\ i, t$$
$$\sum_i y_t^{ik} \leq 1 \ \forall\ k, t$$
$$\chi_t^{ijk} \geq y_{t-1}^{ik} + y_t^{jk} - 1 \ \forall\ i, j, k, t$$
$$s \in \mathbb{R}_+^1, y \in \{0,1\}, \chi \in \{0,1\}$$

$y_t^{ik} = 1$ if production of item i on machine k in period t.
$\chi_t^{ijk} = 1$ if switch from i in $t-1$ to j in t on machine k
s_t^i is stock of item i at the end of period t

Normalization

Assume identical machines (production rates and costs)
Normalize so that $C^i = 1$ for all items.
$d_t^i \in \{0, 1, \ldots, K\}$
Dummy item when no production
$\sum_i y_t^{ik} = 1$ for all k, t

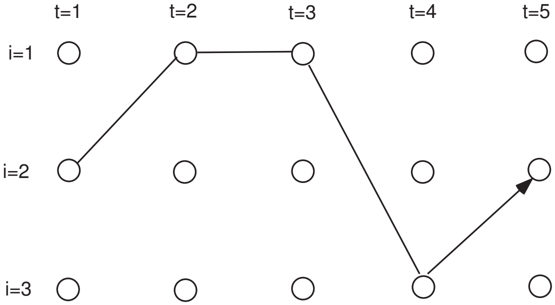
Changeover variables: Formulation

$\chi_t^{ij} = 1$ if i is set up in period $t-1$ and item j is set up in t, and $\chi_t^{ij} = 0$ otherwise.

Basic formulation:

$$X^{CH} = \{(\chi, y) \in \{0,1\} : \sum_i y_t^i = 1 \; \forall \; t$$
$$\chi_t^{ij} \geq y_{t-1}^i + y_t^j - 1 \; \forall \; i,j,t$$
$$\chi_t^{ij} \leq y_{t-1}^i, \quad \chi_t^{ij} \leq y_t^j \; \forall \; i,j,t\}$$

Improved Path Formulation for each Machine



$$\sum_i y_1^{ik} = 1$$

$$\sum_i \chi_t^{ijk} = y_t^{jk} \ \forall \, j, t$$

$$\sum_j \chi_t^{ijk} = y_{t-1}^{ik} \ \forall \, j, t > 1$$

Link to Start-Up Variables

Start-up variables: $z_t^{ik} = 1$ if start producing i in t
Switch-off variables: $w_t^{ik} = 1$ if stop producing i in t

$$\begin{aligned}
z_t^{jk} &= \sum_{i:i\neq j} \chi_t^{ijk} \\
w_{t-1}^{ik} &= \sum_{j:j\neq i} \chi_t^{ijk} \\
z_t^{jk} + \chi_t^{jjk} &= y_t^{jk} \\
w_{t-1}^{ik} + \chi_t^{iik} &= y_{t-1}^{ik}
\end{aligned}$$

Start-Ups and Single-item Lot-sizing

$$
\begin{aligned}
s_{t-1} + \sum_k y_t^{ik} &= d_t^i + s_t^i \\
z_t^{ik} &\geq y_t^{ik} - y_{t-1}^{ik} \\
z_t^{ik} &\leq y_t^{ik}
\end{aligned}
$$

Valid inequalities of Van Eijl (when $k = 1$ and $d_t^i \in \{0, 1\}$).
Generalized by Gicquel.

Results for pb24-K3-I10-T40: 3 machines, 10 items, 40 periods

i) Initial Disaggregated with Path
ii) Path +valid Inequalities
iii) Path + Extended Formulation

m	n	int	LP	XLP	BLB	BIP	Gap	secs
5803	18923	2673	2393	2915	3089	9966	69%	600
7200	18923	2673	5784	5870	5921	6010	1.5%	600
17600	46308	2673	5991	5991	6002	6002	0%	83

K Multiple Identical Machines

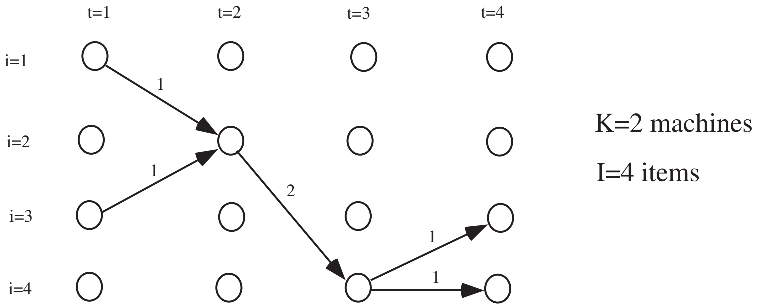
Use integer variables between 0 and K. This leads to a much more compact model.

$$Y_t^i = \sum_k y_t^{ik},\ Z_t^i = \sum_k z_t^{ik},\ \Omega_t^{ij} = \sum_k \chi_t^{ijk}$$

$$\min \textstyle\sum_i p_t^{ik} Y_t^i + \sum_i h_t^i s_t^i + \sum_{i,j,t} q^{ij} \Omega_t^{ij}$$
$$s_t^i + Y_t^i = d_t^i + s_t^i \ \forall\ i, t$$
$$\textstyle\sum_i Y_t^i = K \ \forall\ t$$
$$\textstyle\sum_{i \neq j} \Omega_t^{ij} = W_{t-1}^i$$
$$\textstyle\sum_{j \neq i} \Omega_t^{ij} = Z_t^i$$
$$Z_t^j + \Omega_t^{jj} = Y_t^j$$
$$W_t^i + \Omega_t^{ii} = Y_{t-1}^i$$
$$s \in \mathbb{R}_+^1, Y, Z, W, \Omega \in \{0, 1, \ldots, K\}$$

Flow of K Units

- Any flow (real or integer) of K units decomposes into K unit flows (real or integer) respectively
- Linear Programming values are identical
- Integer Programming values are identical



Single Item Start-up Model?

What about start-ups?

$$\begin{aligned} S_{t-1} + Y_t &= D_t + S_t \\ Z_t &\geq Y_t - Y_{t-1} \\ Z_t &\leq Y_t \\ S \in \mathbb{R}^T_+, Y_t, Z_t &\in \{0, 1, \ldots, K\} \end{aligned}$$

Lasdon-Terjung model (1972)
Extended formulation due to Eppen and Martin - very large
Vanderbeck and Wolsey - cutting planes -special purpose separation heuristics

Using Single Item Inequalities

Suppose that demand over the interval $[t, \ldots, l]$ is p units.
Suppose that the u^{th} unit falls in period t_u

Proposition (Gicquel and W.)
The $DLS - CC - SC$ inequalities

$$S_{t-1} + \sum_{u=1}^{p} (Y_{t+u-1} + Z_{t+u} + \cdots + Z_{t_u}) \geq p$$

are valid for the Lasdon-Terjung model.

Results for pb24-K3-I10-T40: 3 machines, 10 items, 40 periods

i) Initial with Path

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iii) Path + Extended Formulation

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5803	18923	2673	2393	2915	3089	9966	69	600
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m	n	int	LP	XLP	BLB	BIP	Gap	secs
2201	6581	891	2393	4673	5049	6315	20.1	600
4598	6581	891	5784	5926	6002	6002	0	94
13956	33966	891	5991	6000	6002	6002	0	17

Sharing Identical Machines

Each player $k = 1, \cdots, K$ owns a machine and has a demand vector d_t^{ik}

$Q \subset \{1, \cdots, K\}$ is a subset of the players.

$$v(Q) = \sum_{k \in Q} [\sum_{i,t} p_t^i y_t^{ik} + \sum_{i,t} h_t^i s_t^i + \sum_{i,j,t} q^{ij} \chi_t^{ijk}]$$

$$s_{t-1}^i + \sum_{k \in Q} y_t^{ik} = \sum_{k \in Q} d_t^{ik} + s_t^i$$

$$(y^k, z^k, \chi^k) \in X^k \; k \in Q$$

Finding a point in the core (if any)

Calculate $v(Q)$ for all $\phi \subset Q \subset \{1, \cdots, K\}$

$$
\begin{aligned}
\delta^* &= \max \delta \\
\sum_{i \in Q} x_i + \delta &\leq v(Q) \quad \phi \subset Q \subset \{1, \cdots, K\} \\
x &\in \mathbb{R}^K_+
\end{aligned}
$$

Point in the core if $\delta^* \geq 0$.

Results for $P2 - K3 - I10 - T40$

$$
\begin{aligned}
v(1) &= 3994 \\
v(2) &= 3854 \\
v(3) &= 3565 \\
v(1,2) &= 5314 \\
v(1,3) &= 5230 \\
v(2,3) &= 5141 \\
v(1,2,3) &= 6002
\end{aligned}
$$

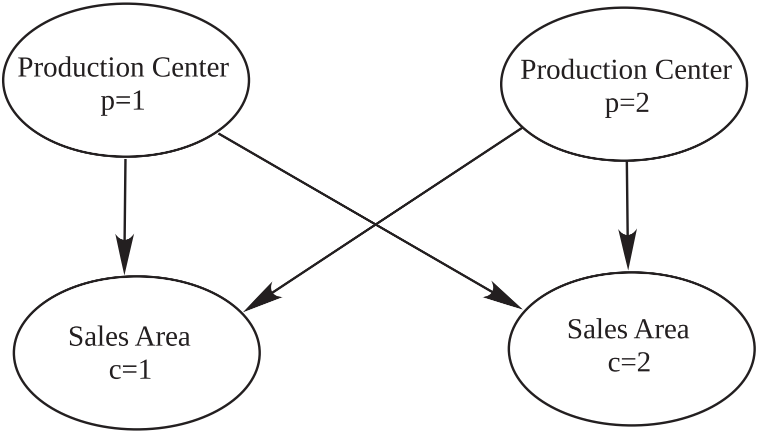
Solving the LP give $\delta^* = 1227$, so the core is indeed nonempty. An optimal allocation is

$$x = (2088, 1999, 1915).$$

The gain of each player is well in excess of $\delta^* = 1227$.
Shapley value is $x = (2139.17, 2024.67, 1838.17)$.

The multi-item/multi-warehouse/multi-client problem

Two-level problem with multiple items, warehouses and clients. $I \geq 1$ is the number of items, $P \geq 1$ is the number of production sites (warehouses) and $C \geq 1$ is the number of clients.



A basic MIP formulation is

$$\min \sum_{i,p,t} p_t^{0ip} x_t^{0ip} + \sum_{i,p,t} q_t^{0ip} y_t^{0ip} + \sum_{i,p,c,t} p_t^{1ipc} x_t^{1ipc} + \sum_{p,c,t} f_t^{1pc} Y_t^{1pc}$$

$$s_{t-1}^{0ip} + x_t^{0ip} = \sum_{c=1}^{C} x_t^{1ipc} + s_t^{0ip} \quad \text{for } 1 \le i \le I,\ 1 \le p \le P,\ 1 \le t \le T,$$

$$s_{t-1}^{1ic} + \sum_{p=1}^{P} x_t^{1ipc} = d_t^{i,c} + s_t^{1ic} \quad \text{for } 1 \le i \le I,\ 1 \le c \le C, 1 \le t \le T,$$

$$x_t^{0ip} \le M y_t^{0ip} \quad \text{for } 1 \le i \le I,\ 1 \le p \le P, 1 \le t \le T,$$

$$\sum_{i=1}^{I} x_t^{1ipc} \le Q Y_t^{1pc} \quad \text{for } 1 \le p \le P,\ 1 \le c \le C, 1 \le t \le T,$$

$$s^0, x^0 \in \mathbb{R}_+^{I \times P \times T},\ s^1 \in \mathbb{R}^{I \times C \times T},\ x^1 \in \mathbb{R}^{I \times P \times C \times T},$$

$$y^0 \in \{0,1\}^{I \times P \times T},\ Y^1 \in \{0,1\}^{P \times C \times T}.$$

Some Computational Results: Single Production Site, Uncapacitated

Group	Dimensions
G4	$I = 5, \ C = 10, \ T = 15$
G5	$I = 5, \ C = 20, \ T = 18$
G6	$I = 20, \ C = 10, \ T = 15$

Table: Instances

	Standard			Echelon stock				MC		New
	LP	XLP	Gap	LP	XLP	Gap	Sec	LP	Sec	Sec
G4	68	83	21	99.3	99.8	0	41	100	2	13
G5	72	86	20	99.3	99.6	3.5	–	100	22	129
G6	58	80	22	99.4	99.7	3.7	–	100^a	75	247

Table: Results for instances with multiple items and multiple clients

Next Steps

- Conclusion: Must use Multi-commodity Formulation - add index c, t to each variable!
- Use a multi-item family set-up model to deal with vehicle capacities
- Use an approximate multi-commodity formulation to handle larger problems

Computation: Multiple Production Sites

2 sites, 10 items, 20 clients, 6 periods

	LP	XLP	BLB	BIP	gap	secs	nodes
Xpress	4799	8763	8866	12084	26.6	300	-
mc	12081	12081	12081	12081	0	24	-

Table: NI=10,NP=2,NC=20,NT=6

	LP	XLP	BLB	BIP	gap	secs	nodes
Xpress	2335	2927	3606	6875	47.6	300	-
mc	6432	6437	6437	6773	5.0	300	-

Table: NI=3,NP=3,NC=20,NT=8

Typical Results: Cooperative Game Theory

- One Warehouse, Multiple retailers
  Continuous Time, Constant Demand, Infinite Horizon
  Retailers share fixed costs of warehouse.
  Power of 2 Policies.
  Resulting games are convex - nonempty core
  S. Anily and M.Haviv (Operations Research)
- Discrete Time
  Single item uncapacitated lot-sizing. Sharing of ficed costs. Game is balanced - nonempty core. Economic lot-sizing games, W. van den Heuvel, P. Borm, H. Hamers, EJOR 176, 1117-1130 (2007).
- Game-theoretic analysis of cooperation among supply chain agents: review and extensions, M. Nagarajan and G. Sosic, EJOR 187, 719-745 (2008).

A Couple of Further Remarks

- For test instances of the discrete-time one warehouse, multiple retailer problem, one can easily calculate the cost function function (using the multi-commodity reformulation or other)for a set of retailers and numerically look test for a non-empty core.
- One can also look at cost sharing between retailers ordering multiple items.
- There are theoretical questions about the game arising from the simple Wagner-Whitin lot-sizing model with non-speculative costs.

THANK YOU FOR YOUR ATTENTION