DISTRIBUTIONS FOR WHICH $\operatorname{div} v = F$ HAS A CONTINUOUS SOLUTION

THIERRY DE PAUW AND WASHEK F. PFEFFER

Abstract. The equation $\operatorname{div} v = F$ has a continuous weak solution in an open set $U \subset \mathbb{R}^m$ if and only if the distribution F satisfies the following condition: $F(\varphi_i)$ converge to zero for each sequence $\{\varphi_i\}$ of test functions such that the supports of φ_i are contained in a fixed compact subset of U, and in the L^1 norm, $\{\varphi_i\}$ converges to zero and $\{\nabla\varphi_i\}$ is bounded.

If F is a distribution in $\mathbb{R}^m$, then a vector field $v \in L^1(\mathbb{R}^m;\mathbb{R}^m)$ is a solution of the equation $\operatorname{div} v = F$ whenever

$$F(\varphi) = -\int_{\mathbb{R}^m} v(x)\cdot\nabla\varphi(x)\,dx$$

for each test function $\varphi \in \mathcal{D}(\mathbb{R}^m)$. If such a v is continuous and $\varepsilon > 0$, we can find a $w \in C^1(\mathbb{R}^m;\mathbb{R}^m)$ so that $\bigl|v(x) - w(x)\bigr| < \varepsilon$ for each x in the ball $B(1/\varepsilon)$ of radius $1/\varepsilon$ about the origin. Selecting φ supported in $B(1/\varepsilon)$ and integrating by parts, we obtain

$$\begin{aligned}\bigl|F(\varphi)\bigr| &\le \left|\int_{B(1/\varepsilon)} \varphi \operatorname{div} w\right| + \left|\int_{B(1/\varepsilon)} (w - v)\cdot\nabla\varphi\right|\\ &\le |\varphi|_1 \sup_{x\in B(1/\varepsilon)} \bigl|\operatorname{div} w(x)\bigr| + \varepsilon|\nabla\varphi|_1,\end{aligned}$$

which implies a stronger continuity of F. In other words, the following continuity property of F is necessary for the equation $\operatorname{div} v = F$ to have a continuous solution.

Continuity. *Given $\varepsilon > 0$ there is a $\theta > 0$ such that*

$$\bigl|F(\varphi)\bigr| \le \theta|\varphi|_1 + \varepsilon|\nabla\varphi|_1 \tag{$*$}$$

for each $\varphi \in \mathcal{D}(\mathbb{R}^m)$ with $\operatorname{supp}\varphi \subset B(1/\varepsilon)$.

Our main result is Theorem 3.7 below, which asserts that this necessary continuity property is also sufficient. For historical reasons (see below), a distribution F satisfying the above continuity property is called a *strong charge*.

An example of a strong charge is the distribution associated with a function $f \in L^m_{\mathrm{loc}}(\mathbb{R}^m)$ (Proposition 2.9 below). H. Brezis and J. Bourgain [1, Proposition 1] proved that a continuous solution of $\operatorname{div} v = f$ exists for a $\mathbb{Z}^m$ periodic function $f \in L^m_{\mathrm{loc}}(\mathbb{R}^m)$. The continuity of v is the main point — establishing the existence of a solution $v \in L^\infty(\mathbb{R}^m;\mathbb{R}^m)$ is appreciably easier (Proposition 2.11 below). In

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The first author is a *chercheur qualifié* of the *Fonds National de la Recherche Scientifique* in Belgium.

general, neither a continuous nor essentially bounded solution is obtainable by solving the Poisson equation $\triangle u = f$ and letting $v := \nabla u$; a pertinent example is due to L. Nirenberg [1, Remark 7]. The absence of such a solution is related to the role of $p = m$ as the critical exponent for representing elements of $W^{1,p}$ by continuous functions [7, Chapter 5, Theorem 5].

We outline the proof of Theorem 3.7, which is inspired by the above mentioned proof of Brezis and Bourgain. The linear spaces $\mathcal{S}$ of all strong charges, and $\mathcal{C}$ of all continuous vector fields $v : \mathbb{R}^m \to \mathbb{R}^m$, are equipped with the Fréchet topologies of locally uniform convergence. For a $v \in \mathcal{C}$, we define a strong charge F_v by

$$F_v(\varphi) := -\int_{\mathbb{R}^m} v(x) \cdot \nabla\varphi(x)\, dx$$

for each $\varphi \in \mathcal{D}(\mathbb{R}^m)$, and observe that the linear map $\Gamma : v \mapsto F_v$ from $\mathcal{C}$ to $\mathcal{S}$ is continuous. Showing that

(i) $\Gamma(\mathcal{C})$ is a dense subspace of $\mathcal{S}$ (Lemma 3.1 below),
(ii) if $\Gamma^* : \mathcal{S}^* \to \mathcal{C}^*$ is the adjoint map of Γ, then $\Gamma^*(\mathcal{S}^*)$ is closed in the strong topology of $\mathcal{C}^*$ (Proposition 3.6 below),

completes the argument: (ii) and the Closed Range Theorem imply that $\Gamma(\mathcal{C})$ is closed in $\mathcal{S}$, and hence $\Gamma(\mathcal{C}) = \mathcal{S}$ by (i).

Because the space $\mathcal{S}$ is topologized so that its dual $\mathcal{S}^*$ is isomorphic to the linear space BV_c of all compactly supported BV functions in $\mathbb{R}^m$ (Proposition 3.2 below), the adjoint map Γ^* of Γ has an intuitive geometric meaning. Indeed, interpreting the continuous vector fields as $(m-1)$-forms and strong charges as m-forms, we can think of Γ as the exterior derivative; note that by definition, Γ is a weak divergence operator. Thus Γ^* is a boundary operator which maps $g \in BV_c$ to a compactly supported Radon measure Dg in $\mathbb{R}^m$; see equality (3.2) below. Clearly, Dg belongs to the dual space $\mathcal{C}^*$ of $\mathcal{C}$.

In the obvious way, the balls $B(i)$, $i = 1, 2, \dots$, determine seminorms s_i and c_i which define the topologies of locally uniform convergence in $\mathcal{S}$ and $\mathcal{C}$, respectively. Theorem 3.8 below shows that given a strong charge F and an integer $i \geq 1$, we can find a solution $v \in \mathcal{C}$ of $\operatorname{div} v = F$ so that $c_i(v)$ is as close to $s_i(F)$ as we wish.

If F is a strong charge, then the set $\Gamma^{-1}(F)$ of all continuous solutions of the equation $\operatorname{div} v = F$ has many elements. In Section 4 we consider continuous vector fields and strong charges that are invariant with respect to the orthogonal group $O(m)$, and produce constructively an isometry $\Upsilon : \mathcal{S}_{\text{inv}} \to \mathcal{C}_{\text{inv}}$ that is a right inverse of Γ (Proposition 4.3 below). The construction depends on showing that a rotation invariant strong charge on the sphere S^{m-1} is a multiple of a strong charge induced on S^{m-1} by the Hausdorff measure $\mathcal{H}^{m-1}$ in $\mathbb{R}^m$ (Proposition 4.2 below).

A strong charge is a special case of a *charge*, i.e., of a distribution F with the above continuity property where inequality $(*)$ is replaced by the inequality

$$\bigl|F(\varphi)\bigr| \leq \theta|\varphi|_1 + \varepsilon\bigl(|\nabla\varphi|_1 + |\varphi|_\infty\bigr).$$

Every distribution associated with an $f \in L^1_{\text{loc}}(\mathbb{R}^m)$ is a charge, called an *absolutely continuous charge*. Elaborating on the proof of Theorem 3.7, we show that each charge is the sum of a strong charge and an absolutely continuous charge (Theorem 5.2 below).

The concepts of charges and strong charges originate from our previous work on generalized Riemann integrals and the Gauss-Green theorem [12, 3, 2, 13, 5, 4]. In Section 6, we indicate how a substantial generalization of the classical Gauss-Green theorem (Theorem 6.5 below) can be obtained by means of charges and their derivatives. This version of the Gauss-Green theorem admits further generalizations that can be applied to removable sets of PDEs in divergence form [5, 4, Sections 4].

1. Preliminaries

The set of all real numbers is denoted by $\mathbb{R}$. In the Cartesian product $\mathbb{R}^n$ where $n \geq 1$ is an integer, we denote by $x \cdot y$ the usual inner product, which induces the norm $|x|$. The zero vector in $\mathbb{R}^n$ is denoted by $\mathbf{0}$. All functions we consider are real valued. For a map $f : A \to B$ and an $x \in A$, we use the symbols $f(x)$ and $\langle f, x \rangle$ interchangeably; both denote the value of f at x.

The ambient space of this paper is $\mathbb{R}^m$ where $m \geq 2$ is a fixed integer. Restricting to dimensions larger than one merely eliminates trivialities. The closure, interior, and diameter of a set $E \subset \mathbb{R}^m$ are denoted by $\operatorname{cl} E$, $\operatorname{int} E$, and $d(E)$, respectively. The open and closed balls of radius $r > 0$ centered at $x \in \mathbb{R}^m$ are denoted by $B(x, r)$ and $B[x, r]$, respectively. We write $B(r)$ instead of $B(\mathbf{0}, r)$, and $B[r]$ instead of $B[\mathbf{0}, r]$.

In $\mathbb{R}^m$ we use Lebesgue measure $\mathcal{L} := \mathcal{L}^m$ and the Hausdorff measure $\mathcal{H} := \mathcal{H}^{m-1}$. For $E \subset \mathbb{R}^m$, we write $|E|$ instead of $\mathcal{L}(E)$, and define the restricted measures $\mathcal{L} \llcorner E$ and $\mathcal{H} \llcorner E$ in the usual way [8, Section 1.1.1]. Unless specified otherwise, the words "measure," "measurable," and "negligible," as well as the expressions "almost everywhere" and "almost all" refer to Lebesgue measure $\mathcal{L}$. Symbols $\int f$ and $\int f(x)\,dx$ denote the Lebesgue integral $\int f \, d\mathcal{L}$.

Throughout, $U \subset \mathbb{R}^m$ is a fixed nonempty open set. For $1 \leq p \leq \infty$ and an integer $n \geq 1$, we give $L^p_{\mathrm{loc}}(U; \mathbb{R}^n)$ a topology induced by the seminorms

$$|f|_{p,K} := \left| f \restriction K \right|_p$$

where $f \in L^p_{\mathrm{loc}}(U; \mathbb{R}^n)$ and $K \subset U$ is a compact set. As there is an increasing sequence of compact subsets of U whose interiors cover U, the space $L^p_{\mathrm{loc}}(U; \mathbb{R}^n)$ is a Fréchet space. Clearly, $C(U; \mathbb{R}^m)$ topologized as a subspace of $L^\infty_{\mathrm{loc}}(U; \mathbb{R}^m)$ is a Fréchet space as well. We write $L^p_{\mathrm{loc}}(U)$ instead of $L^p_{\mathrm{loc}}(U; \mathbb{R})$, and denote by $L^p_c(U)$ the linear space of all functions in $L^p_{\mathrm{loc}}(U)$ whose support is a compact subset of U.

We denote by $BV(U)$ the linear space of all BV functions in U, and let

$$BV_c(U) := BV(U) \cap L^1_c(U) \quad \text{and} \quad BV^\infty_c(U) := BV(U) \cap L^\infty_c(U).$$

If $g \in BV(U)$, then $\|g\|$ is the total variation of the distributional gradient Dg of g.

The *essential boundary, perimeter* and *exterior normal* of a BV set E in U are denoted by $\partial_* E$, $\|E\|$ and ν_E, respectively. Note that $\|E\| = \mathcal{H}(\partial_* E) = \|\chi_E\|$ where χ_E is the indicator of E in U.

As usual, $\mathcal{D}(U)$ and $\mathcal{D}'(U)$ are the linear spaces of all test functions and all distributions in U, respectively. In accordance with the notation of the previous paragraph, we let $\|\varphi\| := |\nabla \varphi|_1$ for each $\varphi \in \mathcal{D}(U)$.

2. Definitions and basic properties

Definition 2.1. A distribution $F \in \mathcal{D}'(U)$ is called *fluxing*, or simply a *flux*, if the equation $\operatorname{div} v = F$ has a continuous solution, i.e., if there is a vector field $v \in C(U; \mathbb{R}^m)$ such that for each $\varphi \in \mathcal{D}(U)$,

$$F(\varphi) = -\int_U v(x) \cdot \nabla\varphi(x)\, dx \tag{2.1}$$

The linear space of all fluxing distributions in U is denoted by $\mathcal{F}(U)$. A distribution F defined by equality (2.1) is called the *flux* of v, denoted by F_v.

We say a sequence $\{f_i\}$ of functions defined on U is *compactly supported* if there is a compact set $K \subset U$ such that $\{f_i \neq 0\} \subset K$ for $i = 1, 2 \dots$. If the compact set K is specified a priori, we say that $\{f_i\}$ is *supported* in K. A sequence $\{A_i\}$ of subsets of U is called *compactly supported*, or *supported* in a compact set $K \subset U$, whenever the sequence $\{\chi_{A_i}\}$ has the respective property.

Observation 2.2. *If $F \in \mathcal{D}'(U)$ is a flux, then $\lim F(\varphi_i) = 0$ for every compactly supported sequence $\{\varphi_i\}$ in $\mathcal{D}(U)$ for which*

$$\lim |\varphi_i|_1 = 0 \quad \textit{and} \quad \sup \|\varphi_i\| < \infty. \tag{2.2}$$

Proof. Let $F = F_v$ for a $v \in C(U; \mathbb{R}^m)$, and let $\{\varphi_i\}$ be a sequence in $\mathcal{D}(U)$ supported in a compact set $K \subset U$ and satisfying conditions (2.2). Find a sequence $\{w_j\}$ in $C_c^1(\mathbb{R}^m; \mathbb{R}^m)$ converging to v uniformly in K, and observe

$$\begin{aligned}\bigl|F(\varphi_i)\bigr| &\le \int_K \bigl|v(x) - w_j(x)\bigr| \cdot \bigl|\nabla\varphi(x)\bigr|\, dx + \left|\int_K \varphi_i(x) \operatorname{div} w_j(x)\, d(x)\right| \\ &\le \bigl(\sup_n \|\varphi_n\|\bigr) \sup_{x\in K}\bigl|v(x) - w_j(x)\bigr| + |\varphi_i|_1 \sup_{x\in K}\bigl|\operatorname{div} w_j(x)\bigr|\end{aligned}$$

for $i, j = 1, 2, \dots$. Choosing a sufficiently large j and then a sufficiently large i, we can make $F(\varphi_i)$ arbitrarily small. ☐

Observation 2.2 motivates in part the following definition.

Definition 2.3. A linear functional $F : \mathcal{D}(U) \to \mathbb{R}$ is called

(i) a *charge* if $\lim F(\varphi_i) = 0$ for every compactly supported sequence $\{\varphi_i\}$ in $\mathcal{D}(U)$ for which

$$\lim |\varphi_i|_1 = 0 \quad \text{and} \quad \sup\bigl(\|\varphi_i\| + |\varphi_i|_\infty\bigr) < \infty;$$

(ii) a *strong charge* (abbreviated as *s-charge*) if $\lim F(\varphi_i) = 0$ for every compactly supported sequence $\{\varphi_i\}$ in $\mathcal{D}(U)$ for which

$$\lim |\varphi_i|_1 = 0 \quad \text{and} \quad \sup \|\varphi_i\| < \infty.$$

For each compact set $K \subset U$ and $n = 1, 2 \dots$, the convex sets

$$BV(K, n) := \bigl\{g \in BV_c^\infty(U) : \{g \neq 0\} \subset K \text{ and } \|g\| + |g|_\infty \le n\bigr\},$$

$$BV_s(K, n) := \bigl\{g \in BV_c(U) : \{g \neq 0\} \subset K \text{ and } \|g\| \le n\bigr\}$$

are compact subsets of $L^1(U)$ [8, Section 5.2, Theorem 4]. Give $BV_c^\infty(U)$ and $BV_c(U)$, respectively, the largest topology $\mathfrak{T}$ and $\mathfrak{T}_s$ for which all inclusion maps

$$BV(K,n) \hookrightarrow BV_c^\infty(U) \quad \text{and} \quad BV_s(K,n) \hookrightarrow BV_c(U)$$

are continuous. Since U is the union of an increasing sequence of compact sets, it follows from [13, Proposition 1.2.2] that the topologies $\mathfrak{T}$ and $\mathfrak{T}_s$ are locally convex, sequential, and sequentially complete. Moreover $\mathfrak{T}_s \subset \mathfrak{T}$, and $\mathcal{D}(U)$ is a dense subset of both $\big(BV_c^\infty(U), \mathfrak{T}\big)$ and $\big(BV_c(U), \mathfrak{T}_s\big)$ [8, Section 5.2, Theorem 2].

Observation 2.4. *A linear functional $F : \mathcal{D}(U) \to \mathbb{R}$ is, respectively, a charge or an s-charge if and only if it is $\mathfrak{T}$ or $\mathfrak{T}_s$ continuous. In particular, each charge has a unique $\mathfrak{T}$ continuous extension to $BV_c^\infty(U)$, and each s-charge has a unique $\mathfrak{T}_s$ continuous extension to $BV_c(U)$. These extensions are linear.*

Remark 2.5. Observe that the flux F_v of a locally bounded Borel vector field $v : U \to \mathbb{R}^m$, which need not be a charge, still extends to

$$F_v : g \mapsto -\int_U v \cdot d(Dg) : BV_c(U) \to \mathbb{R}.$$

In view of Observation 2.4, we always think of charges as **defined on** $BV_c^\infty(U)$, and of s-charges as **defined on** $BV_c(U)$. If F is a charge and E is a bounded BV set whose closure is contained in U, we let $F(E) := F(\chi_E)$. Note that

$$F_v(E) = -\int_U v \cdot d(D\chi_E) = \int_{\partial_* E} v \cdot \nu_E \, d\mathcal{H}.$$

Proposition 2.6. *If $F : BV_c(U) \to \mathbb{R}$ is a linear functional, then the following properties are equivalent.*

(i) *The functional F is an s-charge.*

(ii) *Given $\varepsilon > 0$ and compact set $K \subset U$, there is a $\theta > 0$ such that*

$$\big|F(g)\big| \le \theta |g|_1 + \varepsilon \|g\| \tag{2.3}$$

for each $g \in BV_c(U)$ with $\{g \neq 0\} \subset K$.

(iii) *For each compactly supported sequence $\{B_i\}$ of BV sets in U,*

$$\lim \frac{F(B_i)}{\|B_i\|} = 0 \quad \textit{whenever} \quad \lim |B_i| = 0.$$

Proof. (i) $\Rightarrow$ (ii). Suppose F is an s-charge, and choose an $\varepsilon > 0$ and a compact set $K \subset U$. There is an $\eta > 0$ such that $\big|F(g)\big| < \varepsilon/2$ for each $g \in BV_c(U)$ with $|g|_1 < \eta$, $\|g\| < 1$, and $\{g \neq 0\} \subset K$. Let $\theta := \varepsilon/(2\eta)$ and select a $g \in \mathcal{D}(U)$ with $\{g \neq 0\} \subset K$. With no loss of generality, we may assume $g \geq 0$; see [13, Theorem 1.8.12].

Let p and q be the smallest positive integers with $|g|_1/p < \eta$ and $\|g\|/q < 1$. Note $p \le |g|_1/\eta + 1$ and $q \le \|g\| + 1$. Since

$$s \mapsto \int_0^s \big|\{g > t\}\big| \, dt \quad \text{and} \quad s \mapsto \int_0^s \big\|\{g > t\}\big\| \, dt$$

are continuous increasing functions which map $[0, \infty]$ onto $\big[0, |g|_1\big]$ and $\big[0, \|g\|\big]$, respectively, there are points $0 = a_0 < \cdots < a_p = \infty$ and $0 = b_0 < \cdots b_q = \infty$ such that

$$\int_{a_{i-1}}^{a_i} \bigl|\{g > t\}\bigr|\,dt = \frac{1}{p}|g|_1 < \eta \quad \text{and} \qquad \int_{b_{j-1}}^{b_j} \bigl\|\{g > t\}\bigr\|\,dt = \frac{1}{q}\|g\| < 1$$

for $i = 1, \ldots, p$ and $j = 1, \ldots, q$. Order the set $\{a_0, \ldots, a_p, b_0, \ldots, b_q\}$ into a sequence $0 = c_0 < \cdots < c_r = \infty$. Then $r \le p + q - 1$, and

$$g_k := \max\bigl\{\min\{g, c_k\}, c_{k-1}\bigr\} - c_{k-1}, \quad k = 1, \ldots, r,$$

are BV functions vanishing outside K. As each $[c_{k-1}, c_k]$ is contained in some $[a_{i-1}, a_i] \cap [b_{j-1}, b_j]$, the previous inequalities imply $|g_k|_1 < \eta$ and $\|g_k\| < 1$. Since $g = \sum_{k=1}^{r} g_k$, we obtain

$$\begin{aligned}\bigl|F(g)\bigr| &\le \sum_{k=1}^{r}\bigl|F(g_k)\bigr| < r\frac{\varepsilon}{2} \le \frac{\varepsilon}{2}(p + q - 1)\\ &\le \frac{\varepsilon}{2}\left(\frac{1}{\eta}|g|_1 + \|g\| + 1\right) = \theta|g|_1 + \frac{\varepsilon}{2}\|g\| + \frac{\varepsilon}{2},\end{aligned}$$

and inequality (2.3) follows whenever $\|g\| \ge 1$. If $0 < \|g\| < 1$, we apply the previous result to $h := g/\|g\|$:

$$\bigl|F(g)\bigr| = \|g\| \cdot \bigl|F(h)\bigr| \le \|g\|\bigl(\theta|h|_1 + \varepsilon\|h\|\bigr) = \theta|g|_1 + \varepsilon\|g\|.$$

As the case $\|g\| = 0$ is trivial, the desired inequality is established.

(iii) ⇒ (ii). By [13, Proposition 2.1.7], given $\varepsilon > 0$ and a compact set $K \subset U$, there is a $\theta > 0$ such that

$$\bigl|F(B)\bigr| \le \theta|B| + \varepsilon\|B\|$$

for each BV set $B \subset K$. Now it follows from [13, Proposition 2.2.6 and Section 4.1] that F satisfies (ii).

The implications (ii) ⇒ (i) and (ii) ⇒ (iii) are obvious. □

Remark 2.7. Charges in U are characterized by an inequality similar to (2.3). Indeed, it follows from [13, Proposition 2.2.6 and Section 4.1] that a linear functional $F : BV_c^\infty(U) \to \mathbb{R}$ is a charge if and only if given $\varepsilon > 0$ and a compact set $K \subset U$, there is a $\theta > 0$ such that

$$\bigl|F(g)\bigr| \le \theta|g|_1 + \varepsilon\bigl(\|g\| + |g|_\infty\bigr)$$

for each $g \in BV_c^\infty(U)$ with $\{g \ne 0\} \subset K$. A direct proof of this fact is analogous to that of Proposition 2.6; see also [5, Proposition 2.4].

Remark 2.8. It follows from [13, Section 4.1] that charges are uniquely determined by their values on the indicators of bounded BV sets. As bounded BV sets can be approximated by finite unions of nondegenerate compact intervals [13, Proposition 1.10.3], charges, and a fortiori s-charges, are uniquely determined by their values on the indicators of nondegenerate compact intervals.

The linear spaces of all charges in U and all s-charges in U are denoted by $CH(U)$ and $CH_s(U)$, respectively. By Observation 2.2,

$$\mathcal{F}(U) \subset CH_s(U) \subset CH(U) \subset \mathcal{D}'(U).$$

If $f \in L^1_{\rm loc}(U)$, then the distribution $\Lambda(f)$ in $\mathcal{D}'(U)$ defined by

$$\big\langle \Lambda(f), \varphi \big\rangle := \int_U f(x)\varphi(x)\,dx$$

for each $\varphi \in \mathcal{D}(U)$ is a charge, called an *absolutely continuous charge* (abbreviated as *ac-charge*). Denoting by $CH_{ac}(U)$ the linear space of all ac-charges in U, we have a linear isomorphism

$$\Lambda : f \mapsto \Lambda(f) : L^1_{\rm loc}(U) \to CH_{ac}(U)$$

In particular, each $F \in CH_{ac}$ has an obvious linear extension to $L^\infty_c(U)$.

While easy examples show that neither of the spaces $CH_s(U)$ and $CH_{ac}(U)$ contains the other, they have a sizable intersection.

Proposition 2.9. $\Lambda\big[L^m_{\rm loc}(U)\big] \subset CH_s(U)$.

Proof. Choose an $f \in L^m_{\rm loc}(U)$, and let $F := \Lambda(f)$. For $g \in BV_c(U)$ and a measurable set $B \subset U$, the Hölder and Poincaré inequalities imply

$$\int_B |fg| \leq \left(\int_B |f|^m\right)^{\frac{1}{m}} \left(\int_U |g|^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}} \leq \kappa\|g\|\left(\int_B |f|^m\right)^{\frac{1}{m}} \tag{2.4}$$

where κ is a positive constant depending only on the dimension m [8, Section 5.6, Theorem 1,(i)]. In particular

$$\big|F(g)\big| \leq \kappa\|g\|\left(\int_{\{g\neq 0\}} |f|^m\right)^{\frac{1}{m}} < \infty, \tag{2.5}$$

and it follows that F is a linear functional on $BV_c(U)$. To show that F is an s-charge, select a sequence $\{g_i\}$ in $BV_c(U)$ supported in a compact set $K \subset U$, and assume that $\lim |g_i| = 0$ and $\sup \|g_i\| < \infty$. Applying inequality (2.4) to the set $B_\theta := \big\{x \in K : \big|f(x)\big| > \theta\big\}$ with $\theta \geq 0$, we obtain

$$\begin{aligned}\big|F(g_i)\big| &\leq \int_{K-B_\theta} |fg_i| + \int_{B_\theta} |fg_i| \leq \theta|g_i|_1 + \kappa\|g_i\|\left(\int_{B_\theta} |f|^m\right)^{\frac{1}{m}}\\ &\leq \theta|g_i|_1 + \kappa\big(\sup_n \|g_n\|\big)\left(\int_{B_\theta} |f|^m\right)^{\frac{1}{m}}.\end{aligned}$$

As $\lim_{\theta\to\infty}\big(\int_{B_\theta} |f|^m\big)^{1/m} = 0$, choosing a sufficiently large θ and then a sufficiently large i, we can make $F(g_i)$ arbitrarily small. □

Note. We proved Proposition 2.9 directly from the definition of s-charges. Using Proposition 2.6, the second part of the proof can be simplified by choosing a compactly supported sequence $\{B_i\}$ of BV sets in U, and applying inequality (2.5) to $g := \chi_{B_i}$. Indeed, we obtain

$$\big|F(B_i)\big| \leq \kappa\|B_i\|\left(\int_{B_i} |f|^m\right)^{1/m}$$

for $i = 1, 2, \ldots$, and hence $\lim\big[F(B_i)/\|B_i\|\big] = 0$ whenever $\lim |B_i| = 0$.

The next example shows that the inclusion $\Lambda\big[L^m_{\rm loc}(U)\big] \subset CH_s(U) \cap CH_{ac}(U)$ established in Proposition 2.9 is generally proper.

Example 2.10. Assume $m = 2$, and let $f(\xi, \eta) := \xi^{-\eta} + \eta^{-\xi}$ for each (ξ, η) in $U := (0,1)^2$. If $p \geq 1$ then

$$\xi^{-p\eta} + \eta^{-p\xi} \leq \big[f(\xi,\eta)\big]^p \leq 2^p(\xi^{-p\eta} + \eta^{-p\xi})$$

for each $(\xi, \eta) \in U$. Since for every $0 < a \leq 1/p$

$$\int_{[0,a]^2} (\xi^{-p\eta} + \eta^{-p\xi}) d\xi\, d\eta = \frac{2}{p} \int_{1-ap}^{1} t^{-1} a^t\, dt,$$

we see that $f \in L^p_{\mathrm{loc}}(U)$ if and only if $p = 1$. On the other hand, the formula

$$v(\xi,\eta) := \left(\frac{\xi^{1-\eta}}{1-\eta}, \frac{\eta^{1-\xi}}{1-\xi} \right)$$

for $(\xi, \eta) \in U$ defines a $v \in C^\infty(U; \mathbb{R}^2)$ with $\operatorname{div} v = f$. Integration by parts shows that $\Lambda(f)$ is the flux of v, and hence an s-charge according to Observation 2.2.

Proposition 2.11. *Given $f \in L^m(U)$, there is a $v \in L^\infty(U; \mathbb{R}^m)$ such that $\Lambda(f)$ is the flux F_v of v, and $|v|_\infty \leq \kappa |f|_m$ where κ is a constant depending only on the dimension m.*

Proof. Since it suffices to prove the proposition in each connected component of U, we may assume U is connected. Let $X := \big\{ \nabla\varphi : \varphi \in \mathcal{D}(U) \big\}$, and for $w \in X$, let

$$G(w) := \int_U f(x)\varphi(x)\, dx$$

where φ is the unique element of $\mathcal{D}(U)$ with $\nabla\varphi = w$. By the Hölder and Poincaré inequalities, there is a constant κ depending only on the dimension m such that

$$\big|G(w)\big| \leq |f|_m |\varphi|_{\frac{m}{m-1}} \leq \kappa |f|_m |w|_1,$$

for each $w \in X$. Applying Hahn-Banach theorem, extend G to a linear functional $H : L^1(U; \mathbb{R}^m) \to \mathbb{R}$ so that $\big|H(w)\big| \leq \kappa |f|_m |w|_1$ for each $w \in L^1(U; \mathbb{R}^m)$. Using the duality of L^p spaces, find a $v \in L^\infty(U; \mathbb{R}^m)$ so that $|v|_\infty \leq \kappa |f|_m$, and

$$H(w) = \int_U v(x) \cdot w(x)\, dx$$

for each $w \in L^1(U; \mathbb{R}^m)$. In particular, for each $\varphi \in \mathcal{D}(U)$,

$$\begin{aligned} \big\langle \Lambda(f), \varphi \big\rangle &= \int_U f(x)\varphi(x)\, dx = G(\nabla\varphi) = H(\nabla\varphi) \\ &= \int_U v(x) \cdot \nabla\varphi(x)\, dx = \langle F_v, \varphi \rangle. \end{aligned}$$

□

Remark 2.12. It follows from Proposition 2.11 that for each $f \in L^m(U)$, the equation $\operatorname{div} v = f$ has a solution in $L^\infty(U; \mathbb{R}^m)$. We included this result because it has a simple proof. Using a more elaborate argument, Brezis and Bourgain established the existence of a bounded continuous solution [1, Proposition 1]. The same, and more, follows from Section 3 below.

3. S-charges

A *Lipschitz domain* is an open set $\Omega \subset \mathbb{R}^m$ with Lipschitz boundary [8, Section 4.2.1]. Note that each Lipschitz domain is a locally BV set. If $\Omega \subset U$ is a Lipschitz domain and $g \in BV_c(U)$ with support in $B(r)$, then it follows from [16, Remark 5.10.2 and Lemma 5.10.4] that $g\chi_\Omega \in BV_c(U)$, and that

$$\|g\chi_\Omega\| \leq \kappa\bigl(|g|_1 + \|g\|\bigr)$$

where $\kappa > 0$ depends only on $\Omega \cap B(r)$.

Let F be an s-charge in U, and let $\Omega \subset U$ be a Lipschitz domain. In view of the previous paragraph and Proposition 2.6, the linear functional

$$F \llcorner \Omega : g \mapsto F(g\chi_\Omega) : BV_c(U) \to \mathbb{R}$$

is an s-charge in U. If $\operatorname{cl}\Omega \subset U$, we view $F \llcorner \Omega$ as an s-charge in $\mathbb{R}^m$; since $F(g\chi_\Omega)$ is defined for each $g \in BV_c(\mathbb{R}^m)$.

If $f : U \to \mathbb{R}$ is locally Lipschitz and $g \in BV_c(U)$, then $fg \in BV_c(U)$ and

$$\|fg\| \leq L|g|_1 + c\|g\|$$

where $L := \operatorname{Lip}(f \restriction \operatorname{supp} g)$ and $c := |f \restriction \operatorname{supp} g|_\infty$. Thus by Proposition 2.6,

$$F \llcorner f : g \mapsto F(fg) : BV_c(U) \to \mathbb{R}$$

is an s-charge in U whenever F is an s-charge in U.

We give $CH_s(U)$ a Fréchet topology induced by the seminorms

$$\|F\|_{s,K} := \sup\bigl\{F(g) : g \in BV(U),\ \{g \neq 0\} \subset K,\ \text{and } \|g\| \leq 1\bigr\}$$

where $F \in CH_s(U)$ and $K \subset U$ is a compact set. In view of Observation 2.2, there is a linear map

$$\Gamma : v \mapsto F_v : C(U; \mathbb{R}^m) \to CH_s(U),$$

which is continuous. Indeed, given a compact set $K \subset U$, we have

$$\bigl|F_v(g)\bigr| \leq \|g\| \cdot |v|_{\infty,K}$$

for every $g \in BV(U)$ with $\{g \neq 0\} \subset K$; thus $\|F_v\|_{s,K} \leq |v|_{\infty,K}$ for each compact set $K \subset U$. Note that $\mathcal{F}(U)$ is the image of Γ.

By Proposition 2.9, the restriction $\Lambda_s := \Lambda \restriction L^m_{\mathrm{loc}}(U)$ maps $L^m_{\mathrm{loc}}(U)$ to $CH_s(U)$. It follows from inequality (2.5) that there is a constant κ, depending only on the dimension m, such that for each $f \in L^m_{\mathrm{loc}}(U)$ and each compact set $K \subset U$,

$$\bigl\|\Lambda(f)\bigr\|_{s,K} \leq \kappa|f|_{m,K}. \tag{3.1}$$

In particular, the map $\Lambda_s : L^m_{\mathrm{loc}}(U) \to CH_s(U)$ is continuous.

Lemma 3.1. *Given $F \in CH_s(U)$, there is a sequence $\{v_i\}$ in $C^\infty(U; \mathbb{R}^m)$ such that the support of each $\operatorname{div} v_i$ is a compact subset of U, and*

$$\lim\bigl\|F - \Gamma(v_i)\bigr\|_{s,K} = 0$$

for every compact set $K \subset U$. In particular, the spaces $\mathcal{F}(U)$ and $\Lambda\bigl[\mathcal{D}(U)\bigr]$ are dense in $CH_s(U)$.

Proof. There are bounded Lipschitz domains Ω_i such that $\operatorname{cl}\Omega_i \subset \Omega_{i+1}$ and $U = \bigcup_{i=1}^{\infty} \Omega_i$. Since every compact set $K \subset U$ is contained in some Ω_i, we have

$$\lim \|F - F \llcorner \Omega_i\|_{s,K} = 0$$

for each compact set $K \subset U$. Thus it suffices to prove the lemma for an s-charge F such that $F = F \llcorner \Omega$ for a bounded Lipschitz domain Ω with $\operatorname{cl}\Omega \subset U$. Select a bounded Lipschitz domain Ω_0 with $\operatorname{cl}\Omega \subset \Omega_0$ and $\operatorname{cl}\Omega_0 \subset U$. There is a convergent (in the distributional sense) sequence $\{\eta_i\}$ of standard mollifiers such that the convolutions $\varphi_i := F * \eta_i$ have support in Ω_0. By [15, Theorems 6.30 and 6.32], each φ_i belongs to $C^\infty(\mathbb{R}^m)$, the s-charges $F_i := \Lambda(\varphi_i)$ converge to F in the distributional sense, and $F_i(g) = F(\eta_i * g)$ for every $g \in BV_c(\mathbb{R}^m)$.

For $i = 1, 2, \ldots$ and $x = (\xi_1, \ldots, \xi_m)$ in U, let

$$f_i(\xi_1, \ldots, \xi_m) := \int_{-\infty}^{\xi_1} \varphi_i(t, \xi_2, \ldots, \xi_m)\, dt.$$

Since $v_i := (f_i, 0, \ldots, 0)$ belongs to $C^\infty(U; \mathbb{R}^m)$ and $\operatorname{div} v_i = \varphi_i$, integration by parts shows that $F_i = \Gamma(v_i)$. As $F = F \llcorner \Omega_0$ and $F_i = F_i \llcorner \Omega_0$, it remains to prove $\lim \|F - F_i\|_{s,K} = 0$ for $K := \operatorname{cl}\Omega_0$. To this end choose an $\varepsilon > 0$, and use Proposition 2.6 to find a $\theta > 0$ so that

$$\bigl|F(g)\bigr| \le \theta |g|_1 + \varepsilon \|g\|$$

for each $g \in BV(\mathbb{R}^m)$ with $\{g \neq 0\} \subset K$. Select such a g, and let $g_x(z) := g(x - z)$ for all $x, z \in \mathbb{R}^m$. By an argument identical to the proof of [13, Lemma 4.2.1],

$$|g_x - g_y|_1 \le |x - y| \cdot \|g\|$$

for all $x, y \in \mathbb{R}^m$. This and Fubini's theorem yield

$$\begin{aligned}
|g - g * \eta_i|_1 &= \int_{\mathbb{R}^m} \left| g(x) \int_{\mathbb{R}^m} \eta_i(y)\, dy - \int_{\mathbb{R}^m} g_x(y) \eta_i(y)\, dy \right| dx \\
&\le \int_{\mathbb{R}^m} \eta_i(y) \left(\int_{\mathbb{R}^m} \bigl| g_{\mathbf{0}}(-x) - g_{-y}(-x) \bigr|\, dx \right) dy \\
&\le \int_{\mathbb{R}^m} \eta_i(y) |g_{\mathbf{0}} - g_{-y}|_1\, dy \le \int_{B(1/i)} \eta_i(y) |y| \cdot \|g\|\, dy \le \frac{1}{i} \|g\|.
\end{aligned}$$

Combining the above inequalities, we obtain

$$\begin{aligned}
\bigl|F(g) - F_i(g)\bigr| &= \bigl|F(g - g * \eta_i)\bigr| \le \theta |g - g * \eta_i|_1 + \varepsilon \|g - g * \eta_i\| \\
&\le \frac{\theta}{i} \|g\| + \varepsilon \bigl(\|g\| + \|g * \eta_i\| \bigr) = \|g\| \left(\frac{\theta}{i} + 2\varepsilon \right)
\end{aligned}$$

for $i = 1, 2, \ldots$, and the lemma follows from the arbitrariness of ε. ☐

The dual space of a topological vector space X is denoted by X^*.[1] Aside from the w*-topology on X^*, we will also use the *strong topology* defined by the uniform convergence on the family of all bounded subsets of X [6, Section 1.8.7 and 8.4].

Proposition 3.2. *There is a linear bijection* $\Phi : BV_c(U) \to CH_s(U)^*$ *defined by*

$$\bigl\langle \Phi(g), F \bigr\rangle := \langle F, g \rangle$$

for each $g \in BV_c(U)$ *and each* $F \in CH_s(U)$.

---

[1] For traditional reasons we still write $\mathcal{D}'(U)$ rather than $\mathcal{D}(U)^*$.

Proof. Clearly Φ is a linear map. Since

$$\Big|\big\langle \Phi(g), F\big\rangle\Big| = \big|\langle F, g\rangle\big| \le \|g\| \cdot \|F\|_{s,K}$$

for each $F \in CH_s(U)$, each compact set $K \subset U$, and each $g \in BV_c(U)$ with $\{g \neq 0\} \subset K$, we see that Φ maps $BV_c(U)$ to $CH_s(U)^*$. If $g \in BV_c(U)$ and $\Phi(g) = 0$, then

$$\int_B g(x)\,dx = \int_U \chi_B(x)g(x)\,dx = \big\langle \Lambda(\chi_B), g\big\rangle = \big\langle \Phi(g), \Lambda(\chi_B)\big\rangle = 0$$

for each bounded measurable set $B \subset U$. Consequently Φ is injective.

Let $T \in CH_s(U)^*$. As $\Lambda_s : L^m_{\mathrm{loc}}(U) \to CH_s(U)$ is continuous, $T \circ \Lambda_s \in L^m_{\mathrm{loc}}(U)^*$. Using the duality of L^p spaces [14, Theorem 6.16], find a $g \in L_c^{m/(m-1)}(U)$ so that

$$\big\langle T, \Lambda(f)\big\rangle = \langle T \circ \Lambda_s, f\rangle = \int_U f(x)g(x)\,dx$$

for every $f \in L^m_{\mathrm{loc}}(U)$. Now choose a $v \in C^1(U;\mathbb{R}^m)$ with $|v|_\infty \le 1$. Since

$$\big\langle \Lambda(\operatorname{div} v), h\big\rangle = \int_U h(x)\operatorname{div} v(x)\,dx \le \|h\|$$

for each $h \in BV_c(U)$, we infer $\big\|\Lambda(\operatorname{div} v)\big\|_{s,C} \le 1$ for every compact set $C \subset U$. By the continuity of T, there are a $c > 0$ and a compact set $K \subset U$ such that

$$\int_U g(x)\operatorname{div} v(x)\,dx = \big\langle T, \Lambda(\operatorname{div} v)\big\rangle \le c\big\|\Lambda(\operatorname{div} v)\big\|_{s,K} \le c.$$

Thus $g \in BV_c(U)$ by the arbitrariness of v, and for each $f \in L^m_{\mathrm{loc}}(U)$,

$$\big\langle \Phi(g), \Lambda(f)\big\rangle = \big\langle \Lambda(f), g\big\rangle = \int_U f(x)g(x)\,dx = \big\langle T, \Lambda(f)\big\rangle.$$

As $\Lambda\big[L^m_{\mathrm{loc}}(U)\big]$ is dense in $CH_s(U)$ by Lemma 3.1, we conclude $T = \Phi(g)$. ☐

A set $C \subset U$ is called *amiable* if it is compact, and for each connected component V of $U - C$, either $d(V) = \infty$ or $\partial V \cap \partial U \neq \emptyset$.

Lemma 3.3. *Each compact set $K \subset U$ is contained in an amiable set $C \subset U$.*

Proof. Denote by $\mathcal{W}$ the collection of all bounded connected components W of $U - K$ with $\partial W \subset K$, and by $\mathcal{V}$ the collection of all other connected components of $U - K$. Given $W \in \mathcal{W}$, observe that

$$\operatorname{dist}(W, \partial U) = \operatorname{dist}(\partial W, \partial U) \ge \operatorname{dist}(K, \partial U),$$

and as W is bounded, also $d(W) = d(\partial W) \le d(K)$. Consequently

$$d\Big(\bigcup \mathcal{W}\Big) \le 3d(K) \quad \text{and} \quad \operatorname{dist}\Big(\bigcup \mathcal{W}, \partial U\Big) \ge \operatorname{dist}(K, \partial U).$$

Since $\bigcup \mathcal{W}$ is a relatively closed subset of $U - K$, the previous inequalities imply that $C := K \cup \bigcup \mathcal{W}$ is a compact subset of U. If $V \in \mathcal{V}$ is bounded, then ∂V is a subset of $\partial(U - K) = \partial U \cup \partial K$, but not a subset of K. Thus V is either unbounded, or its boundary meets the boundary of U. But $\mathcal{V}$ is the collection of all connected components of $U - C$, and hence C is amiable. ☐

Observation 3.4. *Let $g \in BV_c(U)$, and let the support of Dg be contained in an amiable set $C \subset U$. Then the support of g is contained in C.*

Proof. Observe that g is constant in each connected component of $U - C$. If the support of g meets a connected component V of $U - C$, then $V \cap (\operatorname{supp} g)$ is a proper subset of V; since V is either unbounded or $\partial V \cap \partial U \neq \emptyset$. As V is open, $V \cap \{g \neq 0\}$ is also a proper subset of V, a contradiction. $\square$

Lemma 3.5. *Let* $\{g_i\}$ *be a sequence in* $BV_c(U)$ *such that*

$$\sup\left\{\int_U v \cdot d(Dg_i) : v \in B \text{ and } i = 1, 2, \ldots\right\} < \infty$$

for each bounded set $B \subset C(U; \mathbb{R}^m)$. *Then* $\{g_i\}$ *is compactly supported.*

Proof. There are open sets U_i such that $K_i := \operatorname{cl} U_i$ is contained in U_{i+1}, and $U = \bigcup_{i=1}^m U_i$. If the sequence $\{\operatorname{supp} Dg_i\}$ is not compactly supported, then we can construct inductively subsequences of $\{U_i\}$ and $\{g_i\}$, still denoted by $\{U_i\}$ and $\{g_i\}$, so that $\operatorname{supp} Dg_i$ meets the open set $U_{i+1} - K_i$. Consequently, there are $v_i \in C(U; \mathbb{R}^m)$ supported in $U_{i+1} - K_i$ such that $|v|_\infty \leq 1$ and $a_i := \int_U v_i \cdot d(Dg_i)$ is different from zero. Let $b_i = \max\{|a_1|^{-1}, \ldots, |a_i|^{-1}\}$. The bounded set

$$B := \left\{v \in C(U; \mathbb{R}^m) : |v|_{\infty, K_{i+1}} \leq ib_i \text{ for } i = 1, 2, \ldots\right\}$$

contains $w_i := (ib_i)v_i,\ i = 1, 2, \ldots$. As $\left|\int_U w_i \cdot Dg_i\right| \geq i$, we have a contradiction. Thus there is a compact set $K \subset U$ containing the support of each Dg_i. An application of Lemma 3.3 and Observation 3.4 completes the argument. $\square$

Proposition 3.6. *If* Γ^* *is the adjoint map of*

$$\Gamma : C(U; \mathbb{R}^m) \to CH_s(U),$$

then $\Gamma^*\left[CH_s(U)^*\right]$ *is sequentially closed in the strong topology of* $C(U; \mathbb{R}^m)^*$.

Proof. To simplify the notation, let $\mathcal{C} := C(U; \mathbb{R}^m)$ and $\mathcal{S} := CH_s(U)$. Observe

$$\left\langle \Gamma^*(S), v \right\rangle = \left\langle S, \Gamma(v) \right\rangle = \left\langle \Phi(g), \Gamma(v) \right\rangle = \left\langle \Gamma(v), g \right\rangle = -\int_U v \cdot d(Dg) \tag{3.2}$$

for $S \in \mathcal{S}^*,\ v \in \mathcal{C}$, and $g := \Phi^{-1}(S)$. Select a sequence $\{S_i\}$ in $\mathcal{S}^*$ so that $\left\{\Gamma^*(S_i)\right\}$ converges strongly to a $T \in \mathcal{C}^*$, and note that $\left\{\Gamma^*(S_i)\right\}$ is uniformly bounded on each bounded subset of $\mathcal{C}$. Applying (3.2) to $g_i := \Phi^{-1}(S_i)$, Lemma 3.5 implies that the sequence $\{g_i\}$ in $BV_c(U)$ is compactly supported. The set

$$B := \left\{v \in \mathcal{C} : |v|_\infty \leq 1\right\}$$

is a bounded subset of $\mathcal{C}$. Letting $\|R\| := \sup_{v \in B} \langle R, v \rangle$ for $R \in \mathcal{C}^*$, we have

$$\|R\| \leq \sup\left\{\langle R, v \rangle : v \in \mathcal{C} \text{ and } |v|_{\infty, K} \leq 1\right\} < \infty$$

for any compact set $K \subset U$. Since $\lim\left\|\Gamma^*(S_i) - T\right\| = 0$, there is a $c > 0$ such that

$$\begin{aligned}\|g_i\| &= \sup\left\{\int_U v \cdot d(Dg_i) : v \in C_c^1(U; \mathbb{R}^m) \text{ and } |v|_\infty \leq 1\right\} \\ &\leq \sup\left\{\left\langle \Gamma^*(S_i), v \right\rangle : v \in B\right\} = \left\|\Gamma^*(S_i)\right\| \leq c\end{aligned}$$

for $i = 1, 2, \ldots$. By Poincaré inequality, there is a constant $\kappa > 0$, depending only on the dimension m, such that $|g_i|_{\frac{m}{m-1}} \leq \kappa \|g_i\|$; in particular $g_i \in L^{\frac{m}{m-1}}(U)$. Since $L^{\frac{m}{m-1}}(U)$ is the dual space of $L^m(U)$, and since

$$\mathcal{V} := \left\{ h \in L^m(U) : |h|_m \leq \frac{1}{\kappa c} \right\}$$

is a neighborhood of zero in $L^m(U)$, the Banach-Alaoglu theorem [15, Section 3.15] shows that

$$\mathcal{K} := \left\{ f \in L^{\frac{m}{m-1}}(U) : \left| \int_U f(x)h(x)\,dx \right| \leq 1 \text{ for each } h \in \mathcal{V} \right\}$$

is w*-compact subset of $L^{\frac{m}{m-1}}(U)$. By the Hölder and Poincaré inequalities,

$$\left| \int_U g_i(x)h(x)\,dx \right| \leq |g_i|_{\frac{m}{m-1}} |h|_m \leq \kappa \|g_i\| \cdot |h|_m \leq \kappa c |h|_m \leq 1$$

for each g_i and each $h \in \mathcal{V}$. Thus the sequence $\{g_i\}$ has a w*-cluster point $g \in \mathcal{K}$. As $\{g_i\}$ is compactly supported, $\operatorname{supp} g$ is a compact subset of U; in particular $g \in L^1(U)$. Equality (3.2) implies

$$\lim \langle \Gamma^*(S_i), v \rangle = \lim \int_U g_i(x) \operatorname{div} v(x)\,dx = \int_U g(x) \operatorname{div} v(x)\,dx \tag{3.3}$$

for each $v \in C_c^1(U; \mathbb{R}^m)$; the last equality holds, since $\int_U g \operatorname{div} v$ is the cluster point of a convergent sequence $\left\{ \int_U g_i \operatorname{div} v \right\}$. Thus for $v \in C_c^1(U; \mathbb{R}^m)$ with $|v|_\infty \leq 1$,

$$\int_U g(x) \operatorname{div} v(x)\,dx \leq \sup \|g_i\| \leq c.$$

We infer $g \in BV_c(U)$, and let $S := \Phi(g)$. By equalities (3.3) and (3.2),

$$\begin{aligned} \langle T, v \rangle &= \lim \langle \Gamma^*(S_i), v \rangle = -\lim \int_U v \cdot d(Dg_i) \\ &= -\int_U v \cdot d(Dg) = \langle \Gamma^*(S), v \rangle \end{aligned}$$

for each $v \in C_c^1(U; \mathbb{R}^m)$. As $C_c^1(U; \mathbb{R}^m)$ is a dense subspace of $\mathcal{C}$, we see that $T = \Gamma^*(S)$ belongs to $\Gamma^*(\mathcal{S}^*)$. □

Theorem 3.7. $\mathcal{F}(U) = CH_s(U)$.

Proof. According to the Closed Range Theorem [6, Theorem 8.6.13], the following claims are equivalent:

(a) $\Gamma^* \big[CH_s(U)^* \big]$ is strongly closed in $C(U; \mathbb{R}^m)^*$;
(b) $\Gamma^* \big[CH_s(U)^* \big]$ is w*-closed in $C(U; \mathbb{R}^m)^*$;
(c) $\Gamma \big[C(U; \mathbb{R}^m) \big]$ is closed in $CH_s(U)$.

However, a careful look at the proof of implication $(a) \Rightarrow (b)$ presented in [6] reveals that the assumption "strongly closed" can be relaxed to "strongly sequentially closed". In view of this and Proposition 3.6, the space $\mathcal{F}(U) = \Gamma \big[C(U; \mathbb{R}^m) \big]$ is closed in $CH_s(U)$. As $\mathcal{F}(U)$ is dense in $CH_s(U)$ by Lemma 3.1, the theorem follows. □

Theorem 3.8. *Let $F \in CH_s(U)$. For each $\varepsilon > 0$ and each amiable set $K \subset U$, there is a $v \in C(U;\mathbb{R}^m)$ such that $\Gamma(v) = F$ and*

$$\|F\|_{s,K} \leq |v|_{\infty,K} \leq (1+\varepsilon)\|F\|_{s,K}.$$

Proof. The first inequality, which holds for any compact set $K \subset U$, is obvious. Choose an $\varepsilon > 0$ and an amiable set $K \subset U$. We simplify the notation by letting $|v| := |v|_{\infty,K}$ for each $v \in C(U;\mathbb{R}^m)$, and $\|F\| := \|F\|_{s,K}$. To avoid a triviality, assume that $\|F\| > 0$. It suffices to show that the nonempty convex sets

$$A := \bigl\{v \in C(U;\mathbb{R}^m) : |v| < (1+\varepsilon)\|F\|\bigr\},$$
$$B := \bigl\{v \in C(U;\mathbb{R}^m) : \Gamma(v) = F\bigr\}$$

have a nonempty intersection. Proceeding toward a contradiction suppose that $A \cap B = \emptyset$. As A is open, it follows from the Hahn-Banach theorem that there are $T \in \bigl[C(U;\mathbb{R}^m)\bigr]^*$ and $\gamma \in \mathbb{R}$ such that

$$\langle T, v\rangle < \gamma \leq \langle T, w\rangle \tag{3.4}$$

for each $v \in A$ and each $w \in B$ [15, Theorem 3.4, (a)]. Note $\gamma > 0$, because $v = 0$ belongs to A. For the reminder of the proof, select a $w \in B$. If $u \in \Gamma^{-1}(0)$, then $w + tu$ belongs to B for each $t \in \mathbb{R}$. Hence $t\langle T, u\rangle \geq \gamma - \langle T, w\rangle$ for each $t \in \mathbb{R}$, and consequently $\langle T, u\rangle = 0$. Therefore $\Gamma^{-1}(0) \subset T^{-1}(0)$. Since Γ is surjective, and hence open by the Open Mapping Theorem [15, Corollary 2.12, (a)], there is an $S \in \bigl[CH_s(U)\bigr]^*$ with $T = S \circ \Gamma$. The function $g := \Phi^{-1}(S)$ belongs to $BV_c(U)$, and

$$\int_U v \cdot d(Dg) = \bigl\langle \Gamma(v), g\bigr\rangle = \bigl\langle S, \Gamma(v)\bigr\rangle = \langle T, v\rangle \tag{3.5}$$

for each $v \in C(U;\mathbb{R}^m)$. If $v \in C(U;\mathbb{R}^m)$ and $\{v \neq 0\} \cap K = \emptyset$, then $|v| = 0$. Thus both tv and $-tv$ belong to A for each $t \in \mathbb{R}$, and inequality (3.4) implies $Tv = 0$. By equality (3.5), the support of Dg is contained in K, and by Observation 3.4, so is the support of g. Choose a positive $\eta < \varepsilon$ and a u in $C_c^1(\mathbb{R}^m;\mathbb{R}^m)$ with $|u|_\infty \leq 1$. Clearly $v := -(1+\eta)\|F\|u$ belongs to A, and by (3.5) and (3.4),

$$\begin{aligned}\int_U g(x)\operatorname{div} u(x)\,dx &= -\int_U u \cdot d(Dg) = \frac{1}{(1+\eta)\|F\|}\int_U v \cdot d(Dg)\\ &= \frac{1}{(1+\eta)\|F\|}\langle T, v\rangle < \frac{\gamma}{(1+\eta)\|F\|}.\end{aligned}$$

We infer $\|g\| \leq \gamma/\bigl[(1+\eta)\|F\|\bigr]$. As the support of g is contained in K, a contradiction follows from (3.4) and (3.5):

$$\gamma \leq \langle T, w\rangle = \bigl\langle \Gamma(w), g\bigr\rangle = \langle F, g\rangle \leq \|g\| \cdot \|F\| \leq \frac{\gamma}{1+\eta} < \gamma.$$

□

Let $K \subset \mathbb{R}^m$ be a compact set, and let $BV(K)$ be the linear space of all functions $g \in BV(\mathbb{R}^m)$ with $\{g \neq 0\} \subset K$. A linear functional $F : BV(K) \to \mathbb{R}$ is called an *s-charge* in K if given $\varepsilon > 0$, there is a $\theta > 0$ such that

$$\bigl|F(g)\bigr| \leq \theta|g_1| + \varepsilon\|g\|$$

for each $g \in BV(K)$. The linear space of all s-charges in K, denoted by $CH_s(K)$, is equipped with the Banach norm

$$\|F\|_s := \sup\bigl\{F(g) : g \in BV(K) \text{ and } \|g\| \leq 1\bigr\}$$

for $F \in CH_s(K)$. Given $K \subset U$, the restriction map

$$\rho_s : F \mapsto F \restriction BV(K) : CH_s(U) \to CH_s(K)$$

is linear and continuous. If Ω is a bounded Lipschitz domain, then $CH_s(\operatorname{cl}\Omega)$ is linearly homeomorphic to $\bigl\{F \in CH_s(\mathbb{R}^m) : F = F \llcorner \Omega\bigr\}$ topologized as a subspace of $CH_s(\mathbb{R}^m)$.

As the definitions of s-charges in an open set U and a in compact set K are similar, most of the properties established for s-charges in U hold also for s-charges in K, and the corresponding proofs are analogous. Since $CH_s(K)$ is a Banach space, proving properties of s-charges in K is often less technical.

Let $K \subset \mathbb{R}^m$ be a compact set. If $v \in C(K;\mathbb{R}^m)$, then the functional

$$F_v : g \mapsto \int_K v \cdot d(Dg) : BV(K) \to \mathbb{R}$$

is an s-charge in K, still called the *flux* of v. Topologizing $C(K;\mathbb{R}^m)$ by the Banach norm $|v|_\infty$, we have a continuous linear surjection

$$\Gamma_K : v \mapsto F_v : C(K;\mathbb{R}^m) \to CH_s(K)$$

(cf. Theorem 3.7), and the following diagram commutes

$$\begin{tikzcd}
C(\mathbb{R}^m;\mathbb{R}^m) \arrow[r, "\rho"] \arrow[d, "\Gamma"'] & C(K;\mathbb{R}^m) \arrow[d, "\Gamma_K"] \\
CH_s(\mathbb{R}^m) \arrow[r, "\rho_s"] & CH_s(K)
\end{tikzcd}$$

As the restriction map $\rho : v \mapsto v \restriction K$ is surjective, so is ρ_s; in particular

$$CH_s(K) = \bigl\{F \restriction BV(K) : F \in CH_s(\mathbb{R}^m)\bigr\}.$$

However, note that for an $F \in CH_s(\mathbb{R}^m)$, the inclusion

$$\bigl\{v \restriction K : v \in \Gamma^{-1}(F)\bigr\} \subset \Gamma_K^{-1}\bigl[F \restriction BV(K)\bigr]$$

may be proper. The next proposition, whose proof is analogous to that of Theorem 3.8, holds for any compact set $K \subset \mathbb{R}^m$.

Proposition 3.9. *Let F be an s-charge in a compact set K, and let $\varepsilon > 0$. There is a $v \in C(K;\mathbb{R}^m)$ such that $F = \Gamma_K(v)$ and*

$$\|F\|_s \leq |v|_\infty \leq (1+\varepsilon)\|F\|_s.$$

4. Rotation invariant charges

In this section we consider Γ restricted to a map from the space of all rotation invariant vector fields to the space of all rotation invariant s-charges, and construct a continuous right inverse of Γ.

Working with the standard orthonormal base in $\mathbb{R}^m$, we view the orthogonal group $O(m)$ as the multiplicative group of orthonormal matrices, and employ the usual matrix multiplication. Vectors and one-forms are viewed as one-column and one-row matrices, respectively. In particular, $x \in \mathbb{R}^m$ is a one-column matrix, and the gradient $\nabla\varphi$ of a $\varphi \in C^1(\mathbb{R}^m)$ is a one-row matrix; in this interpretation, $x \cdot \nabla\varphi(x) = \big[\nabla\varphi(x)\big]x$. The Haar probability on $O(m)$ is denoted by θ.

Throughout this section, select a positive $R \leq \infty$, and let

$$U := \big\{x \in \mathbb{R}^m : |x| < R\big\} \quad \text{and} \quad U_0 := \big\{x \in \mathbb{R}^m : 0 < |x| < R\big\}.$$

The group $O(m)$ acts linearly and continuously on the spaces $BV_c(U)$, $CH_s(U)$, and $C(U;\mathbb{R}^m)$ by the following rules:

$$\langle A \bullet g, x\rangle := \langle g, Ax\rangle, \quad \langle A \bullet F, g\rangle := \langle F, A \bullet g\rangle, \quad \langle A \bullet v, x\rangle := A^{-1}\langle v, Ax\rangle$$

for every $A \in O(m)$, $g \in BV_c(U)$, $F \in CH_s(U)$, $v \in C(U;\mathbb{R}^m)$, and $x \in U$. Let

$$\begin{aligned} CH_s^{\text{inv}}(U) &: = \big\{F \in CH_s(U) : A \bullet F = F\big\}, \\ C^{\text{inv}}(U;\mathbb{R}^m) &: = \big\{v \in C(U;\mathbb{R}^m) : A \bullet v = v\big\}, \end{aligned}$$

and give these spaces the subspace topology. If $v \in C^{\text{inv}}(U;\mathbb{R}^m)$ then $v(\mathbf{0}) = \mathbf{0}$, since $A^{-1}v(\mathbf{0}) = v(\mathbf{0})$ for each $A \in O(m)$. Observe

$$\begin{aligned} \big\langle \Gamma(A \bullet v), \varphi\big\rangle &= -\int_U \nabla\varphi(x)\big[A\langle v, A^{-1}x\rangle\big]\,dx = -\int_U \big[\nabla\varphi(Ay)A\big]v(y)\,dy \\ &= -\int_U \nabla(A \bullet \varphi)(y)v(y)\,dy = \big\langle \Gamma(v), A \bullet \varphi\big\rangle = \big\langle A \bullet \Gamma(v), \varphi\big\rangle \end{aligned} \tag{4.1}$$

for every $A \in O(m)$, $v \in C(U;\mathbb{R}^m)$, and $\varphi \in \mathcal{D}(U)$. Thus $\Gamma(A \bullet v) = A \bullet \Gamma(v)$, and it follows that Γ maps $C^{\text{inv}}(U;\mathbb{R}^m)$ into $CH_s^{\text{inv}}(U)$.

Observation 4.1. *The map $\Gamma : C^{\text{inv}}(U;\mathbb{R}^2) \to CH_s^{\text{inv}}(U)$ is surjective.*

Proof. If $F \in CH_s^{\text{inv}}(U)$ then by Theorem 3.7, there is a $v \in C(U;\mathbb{R}^2)$ such that $\Gamma(v) = F$. Defining a $w \in C^{\text{inv}}(U;\mathbb{R}^2)$ by the formula

$$w := \int_{O(m)} A \bullet v\,d\theta(A),$$

we have $\Gamma(w) = F$. Indeed for each $\varphi \in \mathcal{D}(U)$, Fubini's theorem and (4.1) yield

$$\begin{aligned} \big\langle \Gamma(w), \varphi\big\rangle &= \int_U \left[\int_{O(m)} (A \bullet v)(x)\,d\theta(A)\right] \cdot \nabla\varphi(x)\,dx \\ &= \int_{O(m)} \left[\int_U (A \bullet v)(x) \cdot \nabla\varphi(x)\,dx\right] d\theta(A) \\ &= \int_{O(m)} \big\langle \Gamma(A \bullet v), \varphi\big\rangle\,d\theta(A) = \int_{O(m)} \big\langle A \bullet \Gamma(v), \varphi\big\rangle\,d\theta(A) \\ &= \int_{O(m)} \langle A \bullet F, \varphi\rangle\,d\theta(A) = \int_{O(m)} \langle F, \varphi\rangle\,d\theta(A) = \langle F, \varphi\rangle. \end{aligned}$$

□

View the sphere $S_r := \partial B(r)$ as a Riemannian submanifold of $\mathbb{R}^m$, and denote by $T_x(S_r)$ its tangent space at $x \in S_r$. The measure $\mathcal{H}/\big\|B(r)\big\|$ defines an $O(m)$

invariant probability in S_r, denoted by σ_r. For $x \in S_r$ and $\varphi \in \mathcal{D}(S_r)$, let

$$|\nabla\varphi|(x) := \sup\bigl\{|X\varphi| : X \in T_x S_r \text{ and } |X| = 1\bigr\},$$

$$\|\varphi\| := \int_{S_r} |\nabla\varphi|(x)\, d\sigma_r(x).$$

With this notation at hand, we can introduce charges and s-charges in S_r by the obvious modification of Definition 2.3. Observation 2.4 readily translates to charges and s-charges in S_r, and a charge in S_r is determined by its values on BV sets in S_r (cf. Remark 2.8). A charge G in S_r is called *invariant* if

$$\langle G, A \bullet \varphi\rangle = \langle G, \varphi\rangle$$

for each $A \in O(m)$ and each $\varphi \in \mathcal{D}(S_r)$.

Proposition 4.2. *If G is an invariant charge in S_r, then*

$$\langle G, g\rangle = G(S_r) \int_{S_r} g(x)\, d\sigma_r(x)$$

for each $g \in BV(S_r)$.

Proof. In view of Observation 2.4, it suffices to prove the proposition when g is a test function. Choose a $\varphi \in \mathcal{D}(S_r)$, and for each $x \in S_r$, let

$$f(x) := \int_{O(m)} \varphi(Ax)\, d\theta(A).$$

Since $f(Bx) = f(x)$ for each $B \in O(m)$, and since $O(m)$ acts transitively on S_r, the function f equals a constant c. By Fubini's theorem

$$\begin{aligned} c = \int_{S_r} f(x)\, d\sigma_r(x) &= \int_{O(m)} \left[\int_{S_r} \varphi(Ax)\, d\sigma_r(x)\right] d\theta(A) \\ &= \int_{O(m)} \left[\int_{S_r} \varphi(x)\, d\sigma_r(x)\right] d\theta(A) = \int_{S_r} \varphi(x)\, d\sigma_r(x), \end{aligned}$$

and hence

$$G(f) = G(c\chi_{S_r}) = cG(S_r) = G(S_r) \int_{S_r} \varphi(x)\, d\sigma_r(x).$$

We complete the proof by showing that $G(\varphi) = G(f)$. To this end, consider collections $P := \bigl\{(E_1, A_1), \ldots, (E_p, A_p)\bigr\}$ such that $E_1, \ldots, E_p$ are disjoint Borel subsets of $O(m)$ whose union is $O(m)$, and $A_i \in E_i$ for $i = 1, \ldots, p$. Given such a collection P, define a test function $f_P := \sum_{i=1}^{p} (A_i \bullet \varphi)\theta(E_i)$, and observe

$$\begin{aligned} |f_P|_\infty &\le \sum_{i=1}^{p} |A_i \bullet \varphi|_\infty\, \theta(E_i) \le |\varphi|_\infty \sum_{i=1}^{p} \theta(E_i) = |\varphi|_\infty, \\ \|f_P\| &\le \sum_{i=1}^{p} \|A_i \bullet \varphi\|\, \theta(E_i) \le \|\varphi\| \sum_{i=1}^{p} \theta(E_i) = \|\varphi\|. \end{aligned} \tag{4.2}$$

The first inequality is obvious, and since $\bigl|\nabla(A_i \bullet \varphi)\bigr|(x) = |\nabla\varphi|(A_i x)$ for each $x \in S_r$ and $i = 1, \ldots, p$, the second one follows. Now the function $(A, x) \mapsto \varphi(Ax)$ is uniformly continuous on $O(m) \times S_r$. Thus making the diameter of each E_i sufficiently small, f_P approximates f uniformly with an arbitrary precision; in particular, f_P can be arbitrarily close to f in the L^1 norm of $L^1(S_r, \sigma_r)$. In view of Remark 2.7, this and inequalities (4.2) imply that $G(f_P)$ can be arbitrarily close to $G(f)$. Since

$$G(f_P) = \sum_{i=1}^{p} \theta(E_i) G(A_i \bullet \varphi) = G(\varphi) \sum_{i=1}^{p} \theta(E_i) = G(\varphi).$$

for each P, we obtain $G(f) = G(\varphi)$. $\square$

Proposition 4.3. *The map* $\Gamma : C^{\mathrm{inv}}(U;\mathbb{R}^m) \to CH_s^{\mathrm{inv}}(U)$ *has a linear right inverse* $\Upsilon : CH_s^{\mathrm{inv}}(U) \to C^{\mathrm{inv}}(U;R^m)$ *defined by the formula*

$$\big\langle \Upsilon(F), x \big\rangle = \begin{cases} \frac{\langle F, B(|x|)\rangle}{\|B(|x|)\|} \cdot \frac{x}{|x|} & \text{if } x \in U_0, \\ \mathbf{0} & \text{if } x = \mathbf{0}, \end{cases} \tag{4.3}$$

for each $F \in CH_s^{\mathrm{inv}}(U)$. *The equality* $\big|\Upsilon(F)\big|_{\infty,B[r]} = \|F\|_{s,B[r]}$ *holds for each positive* $r < R$; *in particular* Υ *is continuous.*

Proof. Clearly Υ is a linear map. Select an $F \in CH_s^{\mathrm{inv}}(U)$, and note the vector field $v := \Upsilon(F)$ belongs to $C^{\mathrm{inv}}(U;\mathbb{R}^m)$ by Proposition 2.6. We show that $F = F_v$. For $0 < r < R$ and a BV set E in S_r, the cone

$$C_E := \{sx : x \in E \text{ and } 0 \le s \le r\}$$

is a bounded BV subset of U. In view of Remark 2.8, we can define an invariant charge G_r in S_r by letting $G_r(E) := F(C_E)$ for each BV set E in S_r. By Proposition 4.2,

$$G_r(E) = G_r(S_r) \frac{\mathcal{H}(E)}{\|B(r)\|}$$

for every BV set E in S_r. Given $0 < t < r < R$ and a BV set E_r in S_r, the set $E_t := C_{E_r} \cap S_t$ is a BV set in S_t. If $C := C_{E_r} - C_{E_t}$, then

$$\begin{aligned} F(C) &= G_r(S_r)\frac{\mathcal{H}(E_r)}{\|B(r)\|} - G_t(S_t)\frac{\mathcal{H}(E_t)}{\|B(t)\|} \\ &= \int_{\partial_* C_{E_r}} v \cdot \nu_{C_{E_r}} \, d\mathcal{H} - \int_{\partial_* C_{E_t}} v \cdot \nu_{C_{E_t}} \, d\mathcal{H} = F_v(C). \end{aligned}$$

Cover U_0 by charts $(J_1, \phi_1), \dots, (J_n, \phi_n)$ where J_i are open subintervals of $\mathbb{R}^m$ and $\phi_i : J_i \to U_0$ are defined by means of the spherical coordinates. If K is a compact subinterval of J_i, we call $\phi_i(K)$ a "rectangle" in U_0. As the above calculation shows that $F(I) = F_v(I)$ for each "rectangle" I, we infer from Remark 2.8 that $F(E) = F_v(E)$ for each bounded BV set E with $\operatorname{cl} E \subset U_0$. If E is a bounded BV subset of U, then

$$F(E) = \lim_{r\to 0} F\big[E - B(r)\big] = \lim_{r \to 0} F_v\big[E - B(r)\big] = F_v(E)$$

and Remark 2.8 implies $F = F_v$. Since $\big|v(x)\big| \le \|F\|_{s,B[r]}$ for each $x \in B[r]$, we have $\big|\Upsilon(F)\big|_{\infty,B[r]} = |v|_{\infty,B[r]} \le \|F\|_{s,B[r]}$. The reverse inequality has been established prior to Lemma 3.1:

$$\|F\|_{s,B[r]} = \big|\langle \Gamma, \Upsilon(F)\rangle\big|_{\infty,B[r]} \le \big|\Upsilon(F)\big|_{\infty,B[r]}.$$

Noting that each compact subset of U is contained in $B[r]$ for some $r < R$ completes the argument. $\square$

Remark 4.4. We present a different proof of Proposition 4.3, which is available in dimension $m = 2$, but may not generalize to higher dimensions.

For $x = (\xi_1, \xi_2)$ in $\mathbb{R}^2$, let $\bar{x} = (-\xi_2, \xi_1)$. Given $v \in C^{\mathrm{inv}}(U; \mathbb{R}^m)$, there are continuous functions a_1, a_2 defined on $[0, R)$ such that $a_1(0) = a_2(0) = 0$, and

$$v(x) = a_1\bigl(|x|\bigr)x + a_2\bigl(|x|\bigr)\bar{x}$$

for each $x \in U$. Define vector fields $\pi_i v \in C^{\mathrm{inv}}(U; \mathbb{R}^2),\ i = 1, 2$, by

$$\pi_1 v(x) := a_1\bigl(|x|\bigr)x \quad \text{and} \quad \pi_2 v(x) := a_2\bigl(|x|\bigr)\bar{x}$$

for every $x \in U$. Interpreting derivatives in the distributional sense, observe that $\operatorname{div} \pi_2 v = 0$, and that $\operatorname{div} \pi_1 v = 0$ implies $ta_1'(t) + 2a_1(t) = 0$ for $0 < t < R$. In $(0, R)$, the continuous distributional solutions of the last equation are the same as the classical solutions $a_1(t) = ct^{-2}$ where $c \in \mathbb{R}$. As a_1 is bounded in the neighborhood of zero, $\operatorname{div} \pi_1 v = 0$ implies $\pi_1 v = 0$. Now

$$\begin{aligned}
\bigl\langle \Gamma(v), \varphi \bigr\rangle &= -\int_U \pi_1 v(x) \cdot \nabla\varphi(x)\, dx - \int_U \pi_2 v(x) \cdot \nabla\varphi(x)\, dx \\
&= \int_U \varphi(x) \operatorname{div} \pi_1 v(x)\, dx + \int_U \varphi(x) \operatorname{div} \pi_2 v(x)\, dx \\
&= \int_U \varphi(x) \operatorname{div} \pi_1 v(x)\, dx
\end{aligned}$$

for each $\varphi \in \mathcal{D}(U)$, and we conclude that $\Gamma(v) = 0$ implies $\pi_1 v = 0$.

Choose an $F \in CH_s^{\mathrm{inv}}(U)$, and use Observation 4.1 to find a $v \in C^{\mathrm{inv}}(U; \mathbb{R}^2)$ with $\Gamma(v) = F$. By the previous paragraph, $\pi_1 v \in C^{\mathrm{inv}}(U; \mathbb{R}^2)$ does not depend on the choice of v. Thus letting $\Upsilon(F) = \pi_1 v$ for any $v \in C^{\mathrm{inv}}(U; \mathbb{R}^2)$ with $\Gamma(v) = F$, we have defined a right inverse Υ of $\Gamma : C^{\mathrm{inv}}(U; \mathbb{R}^2) \to CH_s^{\mathrm{inv}}(U)$. Since

$$\bigl\langle F, B(r) \bigr\rangle = \int_{\partial B(r)} \pi_1 v \cdot \nu_{B(r)}\, d\mathcal{H} = r a_1(r) \bigl\| B(r) \bigr\|$$

for $0 < r < R$, the vector field $\Upsilon(F)$ is defined by formula (4.3).

5. Charges

Under the name "continuous additive functions", charges were introduced in [12] as a common generalization of ac-charges and fluxing distributions. They facilitate a definition of a multidimensional Riemann type integral that provides a Gauss-Green theorem for any differentiable vector field (cf. Section 6 below). In this section, we show that the common generalization given by charges is minimal: the space $CH(U)$ of all charges in U is the smallest linear space containing both $CH_{ac}(U)$ and $CH_s(U)$. The idea of the proof is similar to that of Theorem 3.7.

We give $CH(U)$ a Fréchet topology defined by the seminorms

$$\|F\|_K := \sup\bigl\{F(g) : g \in BV(U),\ \{g \neq 0\} \subset K,\ \text{and } |g|_\infty + \|g\| \leq 1\bigr\}$$

where $F \in CH(U)$, and $K \subset U$ is a compact set. Since $\|F\|_K \leq \|F\|_{s,K}$ for each s-charge F, the inclusion map $CH_s(U) \hookrightarrow CH(U)$ is continuous. However, $CH_s(U)$ is not topologized as a subspace of $CH(U)$.

The product topology in $L^1_{\mathrm{loc}}(U) \times C(U; \mathbb{R}^m)$ is defined by the seminorms

$$\bigl|(f, v)\bigr|_K := \max\bigl\{|f|_{1,K}\,, |v|_{\infty,K}\bigr\}.$$

A linear map $\Theta : L^1_{\mathrm{loc}}(U) \times C(U;\mathbb{R}^m) \to CH(U)$, defined by the formula

$$\big\langle \Theta(f,v), g \big\rangle := \Lambda(f) + \Gamma(v) = \int_U fg\, d\mathcal{L}^m - \int_U v \cdot d(Dg)$$

for $(f,v) \in L^1_{\mathrm{loc}}(U) \times C(U;\mathbb{R}^m)$ and $g \in BV_c^\infty(U)$, is continuous. Indeed,

$$\Big|\big\langle \Theta(f,v), g \big\rangle\Big| \le |g|_\infty \cdot |f|_{1,K} + \|g\| \cdot |v|_{\infty,K} \le \big(|g|_\infty + \|g\|\big)\big|(f,v)\big|_K$$

whenever $K \subset U$ is compact and $\{g \neq 0\} \subset K$, and hence $\big\|\Theta(f,v)\big\|_K \le \big|(f,v)\big|_K$.

Proposition 5.1. *If Θ^* is the adjoint map of Θ, then $\Theta^*\big[CH(U)^*\big]$ is sequentially closed in the strong topology of $\big[L^1_{\mathrm{loc}}(U) \times C(U;\mathbb{R}^m)\big]^*$.*

Proof. To simplify the notation, let $\mathcal{C} := C(U;\mathbb{R}^m)$, and write BV_c^∞, L^1_{loc}, and CH instead of $BV_c^\infty(U)$, $L^1_{\mathrm{loc}}(U)$, and $CH(U)$, respectively. By [13, Theorem 4.3.5], there is a linear bijection $\Psi : BV_c^\infty \to CH^*$ such that

$$\big\langle \Psi(g), F \big\rangle = \langle F, g \rangle.$$

for each $g \in BV_c^\infty$ and each $F \in CH$. Observe that

$$\begin{aligned} \big\langle \Theta^*(S), (f,v) \big\rangle &= \big\langle S, \Theta(f,v) \big\rangle = \big\langle \Psi(g), \Theta(f,v) \big\rangle \\ &= \big\langle \Theta(f,v), g \big\rangle = \int_{\mathbb{R}^m} f(x)g(x)\, dx - \int_{\mathbb{R}^m} v \cdot d(Dg) \end{aligned} \tag{5.1}$$

for $S \in CH^*$, $(f,v) \in L^1_{\mathrm{loc}} \times \mathcal{C}$, and $g := \Psi^{-1}(S)$. Select a sequence $\{S_i\}$ in CH^* so that $\big\{\Theta^*(S_i)\big\}$ converges strongly to a T in $(L^1_{\mathrm{loc}} \times \mathcal{C})^*$, and note that $\big\{\Theta^*(S_i)\big\}$ is uniformly bounded on each bounded subset of $L^1_{\mathrm{loc}} \times \mathcal{C}$. Applying (5.1) to $g_i = \Psi^{-1}(S_i)$ and $(0,v)$, Lemma 3.5 implies that the sequence $\{g_i\}$ in BV_c^∞ is supported in a compact set $C \subset U$. The set

$$B := \big\{(f,v) \in L^1_{\mathrm{loc}} \times \mathcal{C} : |f|_1 \le 1 \text{ and } |v|_\infty \le 1\big\}$$

is a bounded subset of $L^1_{\mathrm{loc}} \times \mathcal{C}$. Letting $\|R\| := \sup\big\{\big\langle R, (f,v) \big\rangle : (f,v) \in B\big\}$ for $R \in (L^1_{\mathrm{loc}} \times \mathcal{C})^*$, we have

$$\|R\| \le \sup\Big\{\big\langle R, (f,v) \big\rangle : (f,v) \in L^1_{\mathrm{loc}} \times \mathcal{C} \text{ and } \big|(f,v)\big|_K \le 1\Big\} < \infty$$

for any compact set $K \subset U$. Since $\lim\big\|\Theta^*(S_i) - T\big\| = 0$, there is a $c > 0$ such that $\big\|\Theta^*(S_i)\big\| \le c$ for $i = 1, 2, \ldots$. From

$$\begin{aligned} |g_i|_\infty &= \sup\Big\{\int_U f(x)g_i(x)\, dx : f \in L^1(U) \text{ and } |f|_1 \le 1\Big\}, \\ \|g_i\| &= \sup\Big\{\int_U v \cdot d(Dg_i) : v \in C_c^1(U;\mathbb{R}^m) \text{ and } |v|_\infty \le 1\Big\}, \end{aligned}$$

and equality (5.1), we obtain

$$|g_i|_\infty + \|g_i\| \le \sup\Big\{\big\langle \Theta^*(S_i), (f,v) \big\rangle : (f,v) \in B\Big\} = \big\|\Theta^*(S_i)\big\| \le c.$$

Now $L^\infty(U)$ is the dual of $L^1(U)$, and $\mathcal{V} := \big\{h \in L^1(U) : |h|_1 \le 1/c\big\}$ is a neighborhood of zero in $L^1(U)$. According to the Banach-Alaoglu theorem,

$$\mathcal{K} := \Big\{f \in L^\infty(U) : \Big|\int_U f(x)h(x)\, dx\Big| \le 1 \text{ for each } h \in \mathcal{V}\Big\}$$

is w*-compact subset of $L^\infty(U)$. Every g_i belongs to $BV_c^\infty \subset L^\infty(U)$, and

$$\left|\int_{\mathbb{R}^m} g_i(x)h(x)\,dx\right| \leq |g_i|_\infty \cdot |h|_1 \leq 1$$

for each $h \in \mathcal{V}$. Thus the sequence $\{g_i\}$ has a w*-cluster point $g \in \mathcal{K}$. As $\{g_i\}$ is supported in C, the support of g is a subset of C. The sequence $\{\Theta^*(S_i)\}$ converges strongly to T, and a fortiori, it w*-converges to T. Equality (5.1) implies

$$\begin{aligned}\lim\bigl\langle\Theta^*(S_i),(f,0)\bigr\rangle &= \lim\int_U f(x)g_i(x)\,dx = \int_U f(x)g(x)\,dx,\\ \lim\bigl\langle\Theta^*(S_i),(0,v)\bigr\rangle &= \lim\int_U g_i(x)\operatorname{div}v(x)\,dx = \int_U g(x)\operatorname{div}v(x)\,dx\end{aligned}\tag{5.2}$$

for each $(f,v) \in L^1(U) \times C_c^1(U;\mathbb{R}^m)$; the last equalities hold, since the right hand sides are cluster points of convergent sequences $\bigl\{\int_U fg_i\bigr\}$ and $\bigl\{\int_U g_i \operatorname{div} v\bigr\}$. For each $v \in C_c^1(U;\mathbb{R}^m)$ with $|v|_\infty \leq 1$, the second equality in (5.2) implies

$$\int_U g(x)\operatorname{div}v(x)\,dx = \lim\int_U g_i(x)\operatorname{div}v(x)\,dx \leq \sup\|g_i\| \leq c.$$

We infer $g \in BV_c^\infty$, and let $S := \Psi(g)$. By equalities (5.2) and (5.1),

$$\begin{aligned}\bigl\langle T,(f,v)\bigr\rangle &= \lim\bigl\langle\Theta^*(S_i),(f,v)\bigr\rangle\\ &= \lim\bigl\langle\Theta^*(S_i),(f,0)\bigr\rangle + \lim\bigl\langle\Theta^*(S_i),(0,v)\bigr\rangle\\ &= \lim\int_U f(x)g_i(x)\,dx - \lim\int_U v\cdot d(Dg_i)\\ &= \int_U f(x)g(x)\,dx - \int_U v\cdot d(Dg) = \bigl\langle\Theta^*(S),(f,v)\bigr\rangle\end{aligned}$$

for each (f,v) in $L^1(U) \times C_c^1(U;\mathbb{R}^m)$. As $L^1(U) \times C_c^1(U;\mathbb{R}^m)$ is a dense subspace of $L^1_{\text{loc}} \times \mathcal{C}$, we see that $T = \Theta^*(S)$ belongs to $\Theta^*(CH^*)$. ☐

Theorem 5.2. *Each charge is the sum of an ac-charge and an s-charge.*

Proof. As in the proof of Theorem 3.6, we deduce from Proposition 5.1 and the Closed Range Theorem that $\Theta\bigl[L^1_{\text{loc}}(U)\times C(U;\mathbb{R}^m)\bigr]$ is a closed subspace of $CH(U)$. By [13, Proposition 4.2.2], the space $CH_{ac}(U) = \Theta\bigl[L^1_{\text{loc}}(U) \times \{0\}\bigr]$ is dense in $CH(U)$. Consequently

$$CH_{ac}(U) + \mathcal{F}(U) = \Theta\bigl[L^1_{\text{loc}}(U) \times C(U;\mathbb{R}^m)\bigr] = CH(U),$$

and the theorem follows from Theorem 3.7. ☐

Remark 5.3. From [13, Proposition 4.2.2] and Lemma 3.1, we see that both spaces $CH_{ac}(U)$ and $CH_s(U)$ are dense in $CH(U)$.

6. The Gauss-Green theorem

According to Definition 2.1, the distributional divergence of $v \in C(U;\mathbb{R}^m)$ is defined as the flux F_v of v. In this framework the Gauss-Green theorem is a mere tautology, which gains its usual meaning when the distribution F_v is given by a function $f \in L^1_{\text{loc}}(U)$ [11, Proposition 4.1]. This is a well-known case: the flux F_v is an ac-charge whose density f is obtained by derivating F_v with respect to a suitable derivation basis. However, one may wish to look at a more general situation when

F_v is not an ac-charge, but still has a density f obtained by derivation. Then f is not in $L^1_{\mathrm{loc}}(U)$, and two questions arise.

(i) When is F_v determined uniquely by its density f?

(ii) If F_v is determined uniquely by its density f, then how can F_v be recovered from f?

Answers to these questions lead to extensions of the classical Gauss-Green theorem — a topic to which we devote the remainder of our paper.

For a bounded BV set A contained in U, let

$$r(A) := \begin{cases} \frac{|A|}{d(A)\|A\|} & \text{if } |A| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We say that a sequence $\{A_i\}$ of bounded BV sets contained in U *tends* to $x \in U$ if x belongs to each A_i, $\lim d(A_i) = 0$, and $\inf r(A_i) > 0$. A charge F in U is *derivable* at $x \in U$ whenever a finite limit

$$DF(x) := \lim \frac{F(A_i)}{|A_i|}$$

exists for each sequence $\{A_i\}$ of bounded BV sets contained in U that tends to x. The number $DF(x)$, called the *derivative* of F at x, does not depend on a particular sequence $\{A_i\}$. If $v \in C(U;\mathbb{R}^m)$ is differentiable at $x \in U$, the it is easy to verify that the flux F_v of v is derivable at x and $DF_v(x) = \operatorname{div} v(x)$.

Denote by $CH_D(U)$ the linear space of all charges in U that are derivable at almost all $x \in U$, and by $L^0(U)$ the space of all measurable functions defined on U. According to Luzin's theorem for charges [10], the map

$$D_U : F \mapsto DF : CH_D(U) \to L^0(U),$$

is surjective, and we call it the *derivation* in U. By our choice of derivation basis, the derivation D_U is a natural transformation of the functors $\boldsymbol{CH}_D : U \mapsto CH_D(U)$ and $\boldsymbol{L}^0 : U \mapsto L^0(U)$ defined on the category $\mathbf{Lip}_{\mathrm{loc}}$ of open subsets of $\mathbb{R}^m$ and proper local lipeomorphisms [13, Section 4.6]. The map D_U has a nontrivial kernel $CH_{sing}(U) := D_U^{-1}(0)$, whose elements are called *singular charges* in U. Consequently, D_U has no **natural** right inverse. Notwithstanding, we may find a **functorial** subspace $X(U)$ of $CH_D(U)$ so that $X(U) \cap CH_{sing}(U) = \{0\}$, in which case the restriction $D_U \restriction X(U)$ is a bijection from $X(U)$ onto $\mathcal{I}_X(U) := D_U(X)$. We denote the inverse map

$$(D_U \restriction X)^{-1} : \mathcal{I}_X(U) \to X(U)$$

by $I_{X,U}$, and call it the *integration* in U induced by X. Clearly, the integration $I_{X,U}$ is a natural transformation of the functors $\boldsymbol{X} : U \to X(U)$ and $\boldsymbol{\mathcal{I}}_X : U \to \mathcal{I}_X(U)$ defined on $\mathbf{Lip}_{\mathrm{loc}}$.

The following are classical examples of the procedure we described.

(1) Letting $X(U) := CH_{ac}(U)$, we obtain $\mathcal{I}_X(U) = L^1_{\mathrm{loc}}(U)$ and $I_{X,U}$ is the Lebesgue integration in U.

(2) Let $X(U) := CH_{DD}(U)$ be the linear space of all charges in U that are derivable at **each** $x \in U$. Then $X(U) \cap CH_{sing}(U) = \{0\}$ by [13, Section 2.6], and

the resulting integration $I_{X,U}$ generalizes the *Newton integral* of elementary calculus.

Since neither of the spaces $CH_{ac}(U)$ and $CH_{DD}(U)$ contains the other, it is inviting to look for a functorial space $X(U) \subset CH_D(U)$ such that

$$CH_{ac}(U) + CH_{DD}(U) \subset X(U) \quad \text{and} \quad X(U) \cap CH_{sing}(U) = \{0\}.$$

While such a space $X(U)$ is by no means unique, practical considerations limit the choices. We seek an $X(U)$ that is large and well behaved — a delicate balancing act still open for investigation. Below we describe a particular definition of $X(U)$ that proved useful in applications.

A *gage* on a set $E \subset \mathbb{R}^m$ is a nonnegative function δ defined on E such that the measure $\mathcal{H} \llcorner \{\delta = 0\}$ is σ-finite (see Remark 6.6 below for the motivation). Given $F \in CH(U)$ and $E \subset U$, let

$$V_* F(E) := \sup_{\eta > 0} \inf_{\delta} \sup \sum_{i=1}^{p} \bigl| F(A_i) \bigr|$$

where δ is a gage on E and the supremum is taken over all collections

$$\bigl\{ (A_1, x_1), \ldots, (A_p, x_p) \bigr\}$$

such that $A_1, \ldots, A_p$ are disjoint BV sets in U, and $x_i \in A_i$, $d(A_i) < \delta(x_i)$, and $r(A_i) > \eta$ for $i = 1, \ldots, p$.

It is not difficult to prove that $V_* F : E \mapsto V_* F(E)$ is a Borel regular measure in U [13, Proposition 3.5.1]. It follows from [13, Proposition 3.5.3] that $V_* F$ restricted to BV subsets of a compact interval $J \subset U$ is the least additive function larger than or equal to $|F \llcorner J|$. In particular $\bigl| F(J) \bigr| \le V_* F(J)$ for each compact interval $J \subset U$. An easy argument reveals that F is an ac-charge if and only if $V_* F$ is absolutely continuous and locally finite [13, Proposition 3.6.1]. This fact suggests the following definition.

Definition 6.1. An $F \in CH(U)$ is called an *ac$_*$-charge* if the measure $V_* F$ is absolutely continuous.

Denoting by $CH_*(U)$ the linear space of all ac$_*$-charges, it is immediate that $CH_{ac}(U) \subset CH_*(U)$; in fact, it follows from Theorems 5.2 and 3.7 that

$$CH_*(U) = CH_{ac}(U) + \mathcal{F}(U) \cap CH_*(U).$$

A direct verification of the inclusion $CH_{DD}(U) \subset CH_*(U)$ is straightforward [13, Theorem 3.6.7]. Establishing the functoriality of $\boldsymbol{CH}_* : U \mapsto CH_*(U)$ on the category $\mathbf{Lip}_{\mathrm{loc}}$ is not difficult, but requires some work [13, Section 4.6]. On the other hand, proving the next fundamental theorem is hard. We refer the interested reader to [13, Sections 3.5 and 3.6].

Theorem 6.2. *$CH_*(U) \subset CH_D(U)$ and*

$$V_* F(E) = \int_E \bigl| DF(x) \bigr| \, dx$$

for each $F \in CH_*(U)$ and each measurable set $E \subset U$.

If $F \in CH_*(U) \cap CH_{sing}(U)$, then Theorem 6.2 yields $\big|F(J)\big| \leq V_*F(J) = 0$ for each compact interval $J \subset U$. From this and Remark 2.8, we obtain the following essential corollary.

Corollary 6.3. *$CH_*(U) \cap CH_{sing}(U) = \{0\}$.*

The next theorem, proved in [13, Section 4.5], is important for applications [13, Sections 5.2 and 5.3]. It indicates a good behavior of the space $CH_*(U)$.

Theorem 6.4. *Let $F \in CH_*(U)$ and $g \in BV^\infty_{\mathrm{loc}}(U)$. Then $F \llcorner g \in CH_*(U)$ and*

$$D(F \llcorner g)(x) = DF(x)g(x)$$

for almost all $x \in U$.

A vector field $v : U \to \mathbb{R}^m$ is called *pointwise Lipschitz* in a set $E \subset U$ if

$$\limsup_{y\to x} \frac{\big|v(y) - v(x)\big|}{|y - x|} < \infty$$

for each $x \in E$. By Stepanoff's theorem [9, Theorem 3.1.9], a vector field v that is pointwise Lipschitz in $E \subset U$ is differentiable at almost all $x \in E$; in particular, the classical $\operatorname{div} v$ is defined almost everywhere in E. Now we can generalize the classical Gauss-Green theorem.

Theorem 6.5. *Let $E \subset U$ be such that the measure $\mathcal{H} \llcorner E$ is σ-finite, and let $v \in C(U; \mathbb{R}^m)$ be pointwise Lipschitz in $U - E$. Then within $CH_*(U)$, the flux F_v of v is uniquely determined by the classical* $\operatorname{div} v$. *If* $\operatorname{div} v$ *belongs to $L^1_{\mathrm{loc}}(U)$, then*

$$F_v(A) = \int_A \operatorname{div} v(x)\, dx$$

for each bounded BV set A with $\operatorname{cl} A \subset U$.

Proof. Since v is pointwise Lipschitz almost everywhere in U, Stepanoff's theorem implies $DF_v(x) = \operatorname{div} v(x)$ for almost all $x \in U$. However, more is true. Utilizing that v is pointwise Lipschitz in $U - E$ and that the measure $\mathcal{H} \llcorner E$ is σ-finite, it is easy to find gages on negligible sets which demonstrate the absolute continuity of the measure V_*F_v. Consequently $F_v \in CH_*(U)$, and the first claim follows from Corollary 6.3. If $\operatorname{div} v$ belongs to $L^1_{\mathrm{loc}}(U)$, then the charge $G : A \mapsto \int_A \operatorname{div} v(x)\, dx$ belongs to $CH_{ac}(U)$, and hence to $CH_*(U)$. By the classical derivability result,

$$DG(x) = \operatorname{div} v(x) = DF_v(x)$$

for almost all $x \in U$, and another application of Corollary 6.3 completes the argument. □

Remark 6.6. The simplicity of the previous proof is due to an application of Corollary 6.3. Of course, if $\operatorname{div} v$ belongs to $L^1_{\mathrm{loc}}(U)$, then the conclusion of Theorem 6.5 tells us that F_v is an ac-charge. However, proving this directly from the assumptions of Theorem 6.5 is appreciably harder than proving that F_v is an ac$_*$-charge. The latter proof is facilitated by our definition of gages.

If under the assumptions of Theorem 6.5, $\operatorname{div} v$ does not belong to $L^1_{\mathrm{loc}}(U)$, we must address question (ii) concerning the recovery of F_v from $\operatorname{div} v$. The answer is affirmative: each $F \in CH_*(U)$ can be recovered from DF by means of an averaging process akin to the generalized Riemann integral of Henstock and Kurzweil [13, Section 5.5].

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Université Catholique de Louvain, Département de mathématiques, 2 chemin du cyclotron, B-1348 Louvain-la-Neuve, Belgium

E-mail address: `depauw@math.ucl.ac.be`

Department of Mathematics, University of California, Davis, CA 95616

E-mail address: `wfpfeffer@ucdavis.edu`