SIZE MINIMIZING SURFACES WITH INTEGRAL COEFFICIENTS

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ABSTRACT. We prove a new existence theorem pertaining to the Plateau problem in 3 dimensional Euclidean space. We compare the approach of E.R. Reifenberg with that of H. Federer and W.H. Fleming.

CONTENTS

1. INTRODUCTION

1.1. **Foreword.** The Plateau problem can be stated informally like this: Given a *boundary* $B \subseteq \mathbb{R}^3$ we seek a surface $S \subseteq \mathbb{R}^3$ *spanning* B and having least *area* among all such surfaces. Solving the problem partly consists in making sense of the italicized words. One expects that the minimizing surfaces model soap films, which are the objects J. Plateau was interested in, [17]. In his classical book [2] R. Courant reports on the work of J. Douglas where surfaces are understood as continuous mappings. This setting is shown to be an actual restriction for instance in W.H. Fleming's paper [11].

We start by recalling why an application of the direct method of the calculus of variations may be a troublesome task. Indeed some minimizing sequence may have "thin tentacles", or "filigree", that will not contribute for a lot of area but yet might persist for some substantial part of the limit (e.g. in the sense of Hausdorff distance). Think of B being a (planar) circle and let S denote the 2 dimensional flat disk bounded by B. Referring to the observation that the nearest point projection on the plane containing B and S has Lipschitz constant 1 (and therefore does not increase area) we infer that S is the unique area minimizer in any reasonable sense[1]. Choose for S_j, $j = 1, 2, \ldots$, the flat disk S from which j nonoverlapping

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[1] In the present paper we consider only area induced by the Euclidean metric of $\mathbb{R}^3$.

small disks have been removed and replaced with "curvy conical" surfaces (the tentacles) whose vertices are points $a_1,\dots,a_j$ chosen in advance. This can be done in order for the total contribution in area of the tentacles to be bounded by j^{-1}, so that $S_1,S_2,\dots$ is indeed a minimizing sequence. The reader may enjoy tickling their imagination by staring at Figures 1.3.j, $j=1,\dots,4$, in [16]. Letting $a_1,a_2,\dots$ be a dense sequence is space we see that we can arrange for the Hausdorff limit of that particular minimizing sequence to be the whole space $\mathbb{R}^3$. Therefore the required semicontinuity of area does not hold. One way to circumvent the problem is to modify the minimizing sequence (cutting off the tentacles and patching the holes with controlled disks); another way consists in considering a weaker concept of convergence of surfaces so that the filigree disappear in the limit. We explain below the two points of view.

Two nearly simultaneous theories were published in 1960. One by E.R. Reifenberg [18], and the other by H. Federer and W.H. Fleming [10]. Both dealt with general dimensions and codimensions — and as a matter of fact this was one of their main striking features —, yet in this paper we will purposedly restrict ourselves to 2 dimensional surfaces in $\mathbb{R}^3$. We now turn to giving a brief account of these contributions.

1.2. **The approach of E.R. Reifenberg.** Assume for the sake of illustration that $B\subseteq\mathbb{R}^3$ is a simple closed Jordan curve, and $S\supseteq B$ is a compact set. We say that B *bounds* S if the homomorphism $H_1(B;G)\to H_1(S,G)$ induced in homology by the inclusion map is trivial. Upon a moment of reflection it should be clear that this indeed says that S fills the hole in B (see the intriguing example [18, Appendix, Example 9] though). The definition also readily depends on G, a fixed "coefficients group". Furthermore, as we shall see soon enough the choice of a particular homology theory is not indifferent. In this setting *area* is understood as the 2 dimensional spherical measure $\mathscr{S}^2$ (see [9, 2.10.2(2)] for a definition[2]). Letting $S_1,S_2,\dots$ be any sequence of competitors (i.e. compact sets bounded by B in the above sense) converging to some S in Hausdorff distance we first want to make sure that the boundary condition is preserved in the limit. This will be the case if we consider Čech homology groups in the definition of "B bounds S" (see Proposition 3.1). After possibly projecting the sets S_j, $j=1,2,\dots$, on the convex hull of B we infer from the Blaschke selection principle that some subsequence $S_{j(1)},S_{j(2)},\dots$ converges in Hausdorff distance. Before referring to this principle E.R. Reifenberg performs a careful cutting and pasting surgery on the sets S_j, $j=1,2,\dots$, in order that semicontinuity of area holds along the modified minimizing sequence $\tilde{S}_j$, $j=1,2,\dots$. Checking that the sets $\tilde{S}_j$ verify the same boundary condition as S_j turns out to rely on the Exactness Axiom of Eilenberg-Steenrod (among many other things of course). This axiom is verified when G is a compact abelian group (see [8, Chap. IX, Theorem 7.6]) but not necessarily otherwise (in particular exactness does not hold when $G=\mathbb{Z}$, see [8, Chap. X, §4]). Thus existence theory in this setting is restricted to the case when G is compact and abelian, and in fact E.R. Reifenberg concentrates on $G=\mathbb{Z}_2$ and G the group of reals modulo 1.

We are now ready to state a corollary of E.R. Reifenberg's work. Letting $B\subseteq\mathbb{R}^3$ be a closed simple Jordan curve and G be a compact abelian group, the following

[2]In case S is $(\mathscr{H}^2,2)$ rectifiable then $\mathscr{S}^2(S)=\mathscr{H}^2(S)$ where the latter is the 2 dimensional Hausdorff measure of S, [9, 3.2.26].

variational problem admits a minimizer:

$$(\mathscr{P}_{R,G,B})\begin{cases}\text{minimize } \mathscr{S}^2(S)\\ \text{among compact sets } S\supseteq B\\ \qquad\qquad\qquad \text{such that } \check{H}_1(i_{B,S}):\check{H}_1(B;G)\to\check{H}_1(S;G) \text{ is trivial}\end{cases}$$

where $i_{B,S}$ denotes the inclusion $B\to S$. Moreover Reifenberg proves that if S^* is a (proper) minimizer of the problem then in a neighborhood of $\mathscr{S}^2$ almost every $x\in S^*\setminus B$ the set S^* is a topological disk. In a subsequent analysis [19] he was able to improve this regularity result to showing that at such point S^* is in fact a real analytic graph.

1.3. **The approach of H. Federer and W.H. Fleming.** Here boundaries and surfaces are meant as currents. An m dimensional current in $\mathbb{R}^3$ is a continuous linear form on the space $\mathscr{D}^m(\mathbb{R}^3)$ of smooth differential forms of degree m with compact support. A current $T\in\mathscr{D}_m(\mathbb{R}^3)$ is called rectifiable whenever the following holds. There exist

(1) A bounded $\mathscr{H}^m$ measurable $(\mathscr{H}^m,m)$ *rectifiable* set $M\subseteq\mathbb{R}^3$;
(2) An $\mathscr{H}^m$ measurable *orientation* $\xi:M\to\wedge_m\mathbb{R}^3$;
(3) An $\mathscr{H}^m$ measurable *multiplicity* $\theta:M\to\mathbb{Z}\setminus\{0\}$;

such that

$$\mathbf{M}(T):=\int_M|\theta|d\mathscr{H}^m<\infty \tag{1}$$

and

$$\langle T,\omega\rangle=\int_M\langle\omega,\xi\rangle\theta d\mathscr{H}^m$$

whenever $\omega\in\mathscr{D}^m(\mathbb{R}^3)$. By M being $(\mathscr{H}^m,m)$ *rectifiable* we mean that $\mathscr{H}^m(M)<\infty$ and there are finitely many or countably many m dimensional submanifolds of class C^1, $M_1,M_2,\ldots$, such that $\mathscr{H}^m(M\setminus\cup_jM_j)=0$. This implies M has an m dimensional approximate tangent space $\operatorname{Tan}^m(M,x)$ at $\mathscr{H}^m$ almost every $x\in M$ (see e.g. [9, 3.2.16,3.2.19] or [20, 11.4,11.6]). At such points $x\in M$ an *orientation* $\xi(x)$ consist in a unit m vector generating $\operatorname{Tan}^m(M,x)$. The integer *multiplicity* θ can be thought of as the number of sheets passing through a point. The combinatorics of "sheets" and "multiplicities" accounts for the way the boundary of T is computed: The boundary operator ∂ of currents is defined by duality of exterior differentiation, thereby generalizing Stokes' theorem for smooth orientable surfaces M. In this context the area of a 2 dimensional rectifiable current T is understood as the *mass* $\mathbf{M}(T)$ defined in (1) — the $\mathscr{H}^2$ measure of the underlying set M counting multiplicities.

The group of m dimensional rectifiable currents in $\mathbb{R}^3$ is denoted $\mathscr{R}_m(\mathbb{R}^3)$. We say $T\in\mathscr{R}_m(\mathbb{R}^3)$ is an *integral current* if also $\partial T\in\mathscr{R}_{m-1}(\mathbb{R}^3)$[3]. The group of m dimensional integral currents in $\mathbb{R}^3$ is denoted $\mathbf{I}_m(\mathbb{R}^3)$. The compactness theorem relevant to the Plateau problem is the following.

1.1. **Theorem** (Federer-Fleming)**.** *Let $T_1,T_2,\ldots$ be a sequence of 2 dimensional integral currents in $\mathbb{R}^3$ whose supports are all contained in some fixed compact set, and such that $\sup_j\mathbf{M}(T_j)+\mathbf{M}(\partial T_j)<\infty$. There then exists a subsequence $T_{j(1)},T_{j(2)},\ldots$ converging weakly* to a 2 dimensional integral current T in $\mathbb{R}^3$.*

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[3] The condition is void when $m=0$.

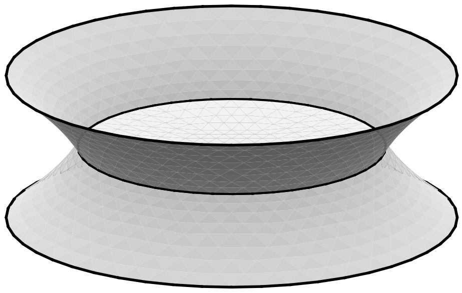

FIGURE 1. Size minimizing but not mass minimizing

The weak* convergence to a *current* of some subsequence of $T_1, T_2, \dots$ follows from the uniform mass bound together with the Banach-Alaoglu theorem and the separability of $\mathscr{D}^2(\mathbb{R}^3)$. Thus the deep content of the theorem is that the limit T is rectifiable as well. We notice that the boundary operator ∂ is continuous with respect to weak* convergence and that mass is lower semicontinuous. The latter follows from the following formula:

$$\mathbf{M}(T) = \sup\{\langle T, \omega\rangle : \omega \in \mathscr{D}^m(\mathbb{R}^3) \text{ and } \|\omega(x)\| \leqslant 1 \text{ for all } x \in \mathbb{R}^3\}$$

where $\|\cdot\|$ is a suitable norm on $\wedge_m \mathbb{R}^3$. Thus the following variational problem admits a minimizer:

$$(\mathscr{P}_{FF,\partial T_0}) \begin{cases} \text{minimize } \mathbf{M}(T) \\ \text{among } T \in \mathscr{R}_2(\mathbb{R}^3) \text{ with } \partial T = \partial T_0 \end{cases}$$

Here $T_0 \in \mathscr{R}_2(\mathbb{R}^3)$ is fixed. The filigree disappear automatically in the weak* limit due to cancelations of orientations of nearby points in "horizontal sections of the tentacles".

1.4. **Mass versus size.** Mass minimizing currents model some but not all soap films. For instance if ∂T_0 consists of two similarly oriented circles lying in close parallel planes then the mass minimizer is the sum of the two oriented flat disks bounded by these circles rather than the singular surface shown on Figure 1. This is because in order to meet the boundary requirement, a current supported in that surface must have multiplicity 2 on the middle disk and hence has larger mass than the two multiplicity 1 disks as is implied by the triangular inequality. As a matter of fact if $T^* \in \mathscr{R}_2(\mathbb{R}^3)$ is a mass minimizer (a solution of $(\mathscr{P}_{FF,\partial T_0})$) then $\operatorname{spt}(T^*) \setminus \operatorname{spt}(\partial T^*)$ is a real analytic submanifold of $\mathbb{R}^3$ according to a theorem of W.H. Fleming [12].

The surface shown in Figure 1 can be realized as a soap film. As a current it minimizes *size* rather than mass. The size of a rectifiable current $T \in \mathscr{R}_2(\mathbb{R}^3)$ is the area of the underlying rectifiable set without counting multiplicities, i.e.

$$\mathbf{S}(T) = \mathscr{H}^2(M)\,.$$

The Plateau problem can now be put as follows.

$$(\mathscr{P}_{FF,\mathbf{S},\partial T_0}) \begin{cases} \text{minimize } \mathbf{S}(T) \\ \text{among } T \in \mathscr{R}_2(\mathbb{R}^3) \text{ such that } \partial T = \partial T_0 \end{cases}$$

Unfortunately a size minimizing sequence need not be bounded in mass (see the example given in [7, Introduction, p. 407-408]) so that the compactness theorem 1.1 does not apply. The paper [7] proves the existence of currents which minimize an energy interpolating between mass and size. In case ∂T_0 is a smooth submanifold contained in the boundary of a convex body then F. Morgan has proved existence of a size minimizer [15] (this result holds in general dimension and codimension 1, the proof shows that some size minimizing sequence is uniformly bounded in mass). However existence of a size minimizing current is unknown for instance when the given boundary curve ∂T_0 is a trefoil knot (see next section though).

Note that, akin to size, the energy being minimized by E.R. Reifenberg does not take the multiplicity into account. Since in the rectifiable currents setting one allows for integer multiplicities, it should now be clear that minimizing size among rectifiable currents corresponds to minimizing $\mathscr{S}^2$ in the Reifenberg context with coefficient group $G = \mathbb{Z}$. In the first case existence is unknown because of a lack of compactness. In the second case it is unknown because the suitable cut and paste procedure is not available to produce a bald minimizing sequence.

2. Result

In the remainder of this paper B denotes a smooth compact 1 dimensional submanifold of $\mathbb{R}^3$. We also let $B_0 \in \mathscr{R}_1(\mathbb{R}^3)$ be a 1 dimensional rectifiable current such that $\operatorname{spt} B_0 = B$, i.e. B_0 consists in a choice of an orientation and multiplicity of each component of B. Furthermore we let L denote the subgroup of $\check{H}_1(B;\mathbb{Z})$ generated by B_0 (see Remark 3.4). Given a set $S \subseteq \mathbb{R}^3$ containing B we say that the kernel of the homomorphism $\check{H}_1(i_{B,S}) : \check{H}_1(B;\mathbb{Z}) \to \check{H}_1(S;\mathbb{Z})$ is the *algebraic boundary of* S. We will be interested in sets S whose algebraic boundary contains L.

We can then consider the following two formulations of the Plateau problem. First, in E.R. Reifenberg's context:

$$(\mathscr{P}_{R,\mathbb{Z},L}) \begin{cases} \text{minimize } \mathscr{H}^2(S) \\ \text{among compact } (\mathscr{H}^2, 2) \text{ rectifiable sets } S \text{ containing } B \\ \qquad\qquad\qquad\qquad \text{and whose algebraic boundary contains } L \end{cases}$$

Second, in the setting introduced by H. Federer and W.H. Fleming:

$$(\mathscr{P}_{FF,\mathbf{S},B_0}) \begin{cases} \text{minimize } \mathbf{S}(T) \\ \text{among } T \in \mathscr{R}_2(\mathbb{R}^3) \text{ with } \partial T = B_0 \end{cases}$$

Our result reads as follows.

2.1. **Theorem.** *With the same notations as above one has:*

(A) $\inf(\mathscr{P}_{R,\mathbb{Z},L}) = \inf(\mathscr{P}_{FF,\mathbf{S},B_0})$*;*
(B) *The variational problem* $(\mathscr{P}_{R,\mathbb{Z},L})$ *admits a minimizer.*

Proving conclusion (A) amounts to constructing a minimizing sequence for one problem which is also a minimizing sequence for the other problem. Note this is not trivial since it is not clear whether a competitor for one problem is also a competitor for the other. In other words it is not immediately obvious that any set S whose algebraic boundary contains L supports a rectifiable current whose boundary is B_0, and vice versa. The question boils down to the comparison of the

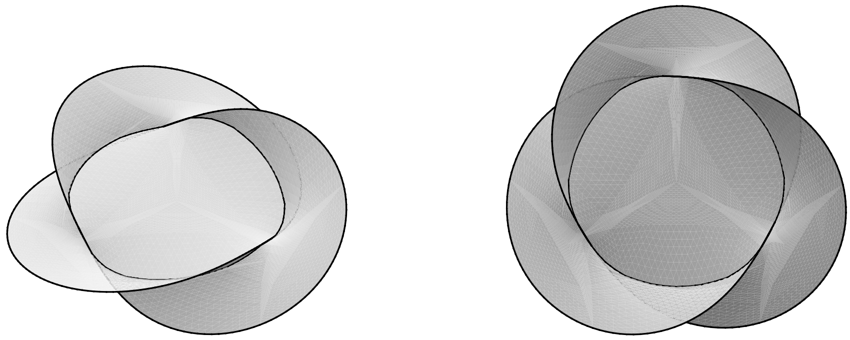

FIGURE 2. A conjectured size minimizer

Čech homology group $\check{H}_1(S;\mathbb{Z})$ with the homology group $\overline{\mathbf{H}}_1(S)$ corresponding to the chain complex of integrals currents supported in S,

$$\mathbf{I}_3(S) \xrightarrow{\ \partial\ } \mathbf{I}_2(S) \xrightarrow{\ \partial\ } \mathbf{I}_1(S) \xrightarrow{\ \partial\ } \mathbf{I}_0(S) \longrightarrow \mathbb{Z}$$

In the next section we will state a theorem to that effect which has been proved in [5]. In order that this theorem applies, the minimizing sequence $S_1, S_2, \ldots$ needs to be slightly modified first, so that each S_j is replaced with some locally acyclic[4] $\tilde{S}_j$ of nearly the same area. The relevant approximation procedure is developed in [3].

As a matter of fact it is so far unknown whether the Čech homology group $\check{H}_1(S^*;\mathbb{Z})$ and the integral currents homology group $\overline{\mathbf{H}}_1(S^*)$ coincide in case S^* is a minimizer whose existence is stated in (B). According to Proposition 3.3 it is sufficient to prove that S^* is locally acyclic at each of its points $x \in S^*$. That this holds true at interior points $x \in S^* \setminus B$ is a consequence of J. Taylor's regularity theorem [21]. At boundary points though the situation is more complex and full regularity has not yet been proved (there are *ten* conjectured possible tangent cones arising at boundary points according to G. Lawlor and F. Morgan [14], see also [16, Figure 13.9.3])[5].

The lack of known boundary regularity makes it impossible to apply F. Morgan's method in [15, Theorem 2.11] (see also [7, Remark 2.3.5]) to prove the existence part (B) of our result. Recall that the point is to find a minimizing sequence which doesn't grow tentacles, in order that Hausdorff measure be lower semicontinuous along that sequence. To achieve this goal we will apply the Federer-Fleming compactness theorem for integral currents and find a sequence $T_1, T_2, \ldots$ of currents which minimize size plus a small amount of mass (this penalization was studied in [7]). The key property that yields lower semicontinuity is the monotonicity of each such current. We face again the problem that $S_j = \operatorname{spt} T_j$ does not need to be a proper competitor in the setting of E.R. Reifenberg, yet the ad hoc approximation procedure evoked above fills the gap and we are able to produce the sought for minimizing sequence $\tilde{S}_1, \tilde{S}_2, \ldots$ without armful filigree.

[4]With respect to integral currents homology.

[5]Potential fun includes the case when the size minimizing set is a subset of that shown in Figure 1, and the new smooth boundary curve wanders on different sheets near the singular circle, interesecting that circle (when it changes from one sheet to another) along a Cantor set of positive length.

Before we start writing on a more technical basis it is worth mentioning at least one example of a (conjectured) minimizer. Here we let B be a smooth nicely symmetric trefoil knot in $\mathbb{R}^3$. We conjecture that the minimizing set S^* of problem $(\mathscr{P}_{R,\mathbb{Z},L})$ is the one shown (from two different angles) in Figure 2. It is made of one "disk" in the middle to which three quarter moons[6] are attached along portions of the given boundary B and portions of a singular curve $C \subseteq S^* \setminus B$. Along the singular curve C three sheets meet at equal angles of 120°. Some portions of the boundary knot B touch one sheet of S^*, and some touch two sheets of S^* (corresponding to the tangent cone number 5 in [16, Figure 13.9.3], and to the tangent cone number 4 in that list at "transition points" between B and C). In order that the boundary condition be met one needs to give the middle disk a multiplicity 2, whereas the quarter moons each have mutliplicity 1. I do not know of any technique to establish that the surface shown in Figure 2 is indeed a minimizer — calibrations apply mainly to proving mass minimization.

3. Toolkit

We let $\check{H}(\cdot;\mathbb{Z})$ denote the Čech homology functor with coefficients in $\mathbb{Z}$ defined on the category of compact pairs and their continuous maps (see [8, Chap. IX] and [5, 2.2]). The following is (essentially) taken from [18].

3.1. **Proposition.** *Assume that:*

(1) $B \subseteq \mathbb{R}^3$ *is compact;*
(2) $S_j \subseteq \mathbb{R}^3$, $j = 1, 2, \ldots$, *are compact and* $B \subseteq S_j$ *for every* j*;*
(3) $L \subseteq \check{H}_1(B;\mathbb{Z})$ *is a subgroup and* $L \subseteq \ker \check{H}_1(i_{B,S_j})$ *for every* $j = 1, 2, \ldots$ *where* i_{B,S_j} *denotes the inclusion map;*
(4) $S_1, S_2, \ldots$ *converge in Hausdorff distance to some compact set* $S \subseteq \mathbb{R}^3$.

Then $L \subseteq \ker \check{H}_1(i_{B,S})$.

Proof. We define a decreasing sequence of compact sets $S'_j = S \cup (\cup_{k=1}^{\infty} S_k)$, $j = 1, 2, \ldots$, and we notice that its inverse limit S_∞ (see [8, Chap. VIII, Definition 3.1]) is homeomorphic to S. In fact letting B_∞ denote the inverse limit of the inverse system "B" (indexed by $j = 1, 2, \ldots$ and having each term constant equal to B) there are canonical homeomorphisms f_B and f_S such that the following diagram commutes (horizontal arrows denote inclusion)

$$\begin{array}{ccc} B & \longrightarrow & S \\ {\scriptstyle f_B}\big\downarrow & & {\scriptstyle f_S}\big\downarrow \\ B_\infty & \longrightarrow & S_\infty \end{array}$$

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[6]Manufactured with soap of course! With this regard one can read the advices of J. Plateau reported by E. Lamarle in [13, Deuxième partie, §25, second footnote] about mixing Marseille soap with French glycerol from Lamoureux. Alternatively one can use nowadays K. Brakke's *surface evolver* [1] to experiment with (virtual) soap films. The figures of the present paper were designed with that software.

Observe that L is contained in the algebraic boundary of S'_j because $S_j \subseteq S'_j$, $j = 1, 2, \ldots$. We now consider the following commutative diagram.

$$\begin{array}{ccc}
\check{H}_1(B;\mathbb{Z}) & \xrightarrow{\check{H}_1(i_{B,S})} & \check{H}_1(S;\mathbb{Z}) \\
{\scriptstyle \check{H}_1(f_B)}\big\downarrow & & {\scriptstyle \check{H}_1(f_S)}\big\downarrow \\
\check{H}_1(B_\infty;\mathbb{Z}) & \xrightarrow{\check{H}_1(i_{B_\infty,S_\infty})} & \check{H}_1(S_\infty;\mathbb{Z}) \\
{\scriptstyle l_1(\mathbf{B})}\big\downarrow & & {\scriptstyle l_1(\mathbf{S}')}\big\downarrow \\
\lim\limits_{\leftarrow} \check{H}_1(B;\mathbb{Z}) & \xrightarrow{\lim\limits_{\leftarrow} \check{H}_1(i_{B,S'_j})} & \lim\limits_{\leftarrow} \check{H}_1(S'_j;\mathbb{Z})
\end{array}$$

The vertical arrows in the first row are isomorphisms because f_B and f_S are homeomorphisms, whereas the vertical arrows of the second row are isomorphisms according to the continuity property of Čech homology with coefficients in $\mathbb{Z}$, [8, Chap. X, Theorem 3.1]. It follows from the definition of $\lim\limits_{\leftarrow} \check{H}_1(i_{B,S'_j})$, $l_1(\mathbf{B})$ and $\check{H}_1(f_B)$ that the composition of these three homomorphisms maps any element of L to 0. The conclusion immediately follows. □

Next we introduce the integral currents homology groups and a sufficient criterion for their coincidence with the Čech homology groups with coefficients in $\mathbb{Z}$. Given a set $X \subseteq \mathbb{R}^3$ we let $\mathbf{I}_q(X)$, $q = 0, 1, 2, 3$, denote the group of integral currents $T \in \mathbf{I}_q(\mathbb{R}^3)$ such that $\operatorname{spt} T \subseteq X$. The homology groups corresponding to the following chain complex are denoted $\overline{\mathbf{H}}_q(X)$.

$$\mathbf{I}_3(X) \xrightarrow{\ \partial\ } \mathbf{I}_2(X) \xrightarrow{\ \partial\ } \mathbf{I}_1(X) \xrightarrow{\ \partial\ } \mathbf{I}_0(X) \xrightarrow{\ \alpha\ } \mathbb{Z}$$

(where α is an augmentation map defined by $\alpha(T) = \langle T, \mathbb{1} \rangle$ where $\mathbb{1}$ denotes any test function which equals 1 in a neighborhood of $\operatorname{spt} T$). A functor $\overline{\mathbf{H}}_q$ is then defined on the category of subsets of some Euclidean space and their locally Lipschitzian maps. It satisfies the axioms of Eilenberg-Steenrod. For details see [5] where the following result is proved as well.

3.2. **Definition.** We say $X \subseteq \mathbb{R}^3$ is $(\overline{\mathbf{H}}, 1)$ *locally connected* if the following holds. For every $x \in X$ and every open set $U \subseteq \mathbb{R}^3$ containing x there exists an open set $U' \subseteq U$ containing x such that the homomorphisms induced in homology by inclusion are trivial, $\overline{\mathbf{H}}_q(X \cap U') \to \overline{\mathbf{H}}_q(X \cap U)$, $q = 0, 1, 2$.

3.3. **Proposition.** *Let $\mathscr{A}^{LC,\overline{\mathbf{H}},1}$ denote the category whose objects are the $(\overline{\mathbf{H}}, 1)$ locally connected subsets of some Euclidean space $\mathbb{R}^n$, together with their locally Lipschitzian maps, and let $\mathsf{Mod}_{\mathbb{Z}}$ denote the category of modules over $\mathbb{Z}$. The functors*

$$\overline{\mathbf{H}}_1 : \mathscr{A}^{LC,\overline{\mathbf{H}},1} \longrightarrow \mathsf{Mod}_{\mathbb{Z}} \qquad \textit{(integral currents homology)}$$

and

$$\check{H}_1(\cdot;\mathbb{Z}) : \mathscr{A}^{LC,\overline{\mathbf{H}},1} \longrightarrow \mathsf{Mod}_{\mathbb{Z}} \qquad \textit{(Čech homology with coefficients in } \mathbb{Z}\textit{)}$$

are naturally equivalent.

In the remainder of this paper we will denote by $\nu : \overline{\mathbf{H}}_1 \to \check{H}_1(\cdot;\mathbb{Z})$ the natural equivalence whose existence is asserted above.

3.4. *Remark.* With the vocabulary at hand it is now easy to relate B_0 and L rigorously. Indeed we let L_0 be the subgroup of $\overline{\mathbf{H}}_1(B)$ generated by the class $[B_0]$, and $L = \nu_B(L_0)$. This is well-defined since B is $(\overline{\mathbf{H}}, 1)$ locally connected.

The following is nearly trivial and very useful. It justifies the introduction of the $(\overline{\mathbf{H}}, 1)$ locally connected spaces.

3.5. **Proposition.** *Assume that $S \subseteq \mathbb{R}^3$ contains B and is $(\overline{\mathbf{H}}, 1)$ locally connected. The following conditions are equivalent.*

(A) *L is contained in the algebraic boundary of S;*
(B) *There exists $T \in \mathscr{R}_2(\mathbb{R}^3)$ such that $\operatorname{spt} T \subseteq S$ and $\partial T = B_0$.*

Proof. Since both B and S are objects of the category $\mathscr{A}^{LC,\overline{\mathbf{H}},1}$, Proposition 3.3 implies the existence of the following commutative diagram whose vertical arrows are isomorphisms.

$$\begin{array}{ccc} \check{H}_1(B;\mathbb{Z}) & \xrightarrow{\check{H}_1(i_{B,S})} & \check{H}_1(S;\mathbb{Z}) \\ {\scriptstyle \nu_B}\big\uparrow & & {\scriptstyle \nu_S}\big\uparrow \\ \overline{\mathbf{H}}_1(B) & \xrightarrow{\overline{\mathbf{H}}_1(i_{B,S})} & \overline{\mathbf{H}}_1(S) \end{array}$$

Assuming (A) holds we infer that $\overline{\mathbf{H}}_1(i_{B,S})([B_0]) = [i_{B,S\#}B_0] = [B_0]$ vanishes in $\overline{\mathbf{H}}_1(S)$. The definition of $\overline{\mathbf{H}}_1(S)$ then readily implies (B). That (B) implies (A) follows from the same observation. $\square$

In view of the previous result the next proposition is useful for replacing any set with a set of nearly the same size and verifying the required boundary condition in *both* settings. For a proof consult [3].

3.6. **Proposition.** *Assume that*

(A) *$X \subseteq \mathbb{R}^3$ is compact and $(\mathscr{H}^2, 2)$ rectifiable;*
(B) *$B \subseteq X$ is a compact 1 dimensional submanifold of $\mathbb{R}^3$ of class C^3 (without boundary);*
(C) $\varepsilon > 0$.

There then exists a Lipschitzian map $f : \mathbb{R}^3 \to \mathbb{R}^3$ and an $(\mathscr{H}^2, 2)$ rectifiable set $Y \subseteq \mathbb{R}^3$ verifying the following properties:

(D) *$f(X) \subseteq Y$;*
(E) *$\operatorname{dist}_{\mathscr{H}}(X, Y) < \varepsilon$;*
(F) *$|\mathscr{H}^2(X) - \mathscr{H}^2(Y)| < \varepsilon$;*
(G) *$\mathscr{H}^2(\operatorname{Clos}(Y) \setminus Y) = 0$;*
(H) *Y is $(\overline{\mathbf{H}}, 1)$ locally connected;*
(I) *$f(B) = B$ and $f \restriction B$ is homotopic in the Lipschitzian category to id_B.*

When applying this result the following will be handy.

3.7. **Proposition.** *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a Lipschitzian map such that $f(B) = B$ and $f \restriction B$ is homotopic to the identity of B in the Lipschitzian category. The following hold.*

(A) *If the algebraic boundary of some set $S \supseteq B$ contains L, then so does the algebraic boundary of $f(S)$;*
(B) *If $T \in \mathscr{R}_2(\mathbb{R}^3)$ and $\partial T = B_0$ then $\partial f_\# T = B_0$.*

Proof. Conclusion (A) follows from the fact that $\check{H}_1(f) \circ \check{H}_1(i_{B,S}) = \check{H}_1(f \circ i_{B,S}) = \check{H}_1(i_{B,f(S)})$, the last equality being a consequence of the Homotopy Axiom and the fact that $f \circ i_{B,S}$ and $i_{B,f(S)}$ are homotopic. In order to prove conclusion (B) we consider a Lipschitzian map $F : [0,1] \times B \to B$ such that $F(0,x) = x$ and $F(1,x) = f(x)$ for every $x \in B$, and we extend it to a Lipschitzian map $\hat{F} : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$. The homotopy formula for integral currents yields

$$\begin{aligned} \partial f_{\#} T - \partial T &= f_{\#} \partial T - \partial T \\ &= \partial \hat{F}_{\#}([\![0,1]\!] \times \partial T) \\ &= 0 \,. \end{aligned}$$

The latter is a consequence of the fact that $\hat{F}_{\#}([\![0,1]\!] \times \partial T) \in \mathscr{R}_2(\mathbb{R}^3)$ together with the inequality

$$\begin{aligned} \mathscr{H}^2 \left(\operatorname{spt} \hat{F}_{\#}([\![0,1]\!] \times \partial T) \right) &\leqslant \mathscr{H}^2 \left(\hat{F}([0,1] \times \operatorname{spt} \partial T) \right) \\ &\leqslant \mathscr{H}^2 (F([0,1] \times B)) \\ &= \mathscr{H}^2(B) \\ &= 0 \,. \end{aligned}$$

□

For a rectifiable current $T \in \mathscr{R}_2(\mathbb{R}^3)$ we let $\|T\|$ denote the measure $|\theta| \mathscr{H}^2 \llcorner M$ (see also [9, 4.1.7]). For a Radon measure ϕ in an open set $U \subseteq \mathbb{R}^3$ we define

$$\Theta^2(\phi, x) = \lim_{r \downarrow 0} \frac{\phi(\mathbf{B}(x,r))}{\pi r^2}$$

$x \in U$ (whenever the limit exists), and

$$\operatorname{set}_2(\phi) = U \cap \{x : 0 < \Theta^2(\phi, x) < \infty\} \,.$$

Size can then be defined as $\mathbf{S}(T) = \mathscr{H}^2(\operatorname{set}_2(\|T\|))$, $T \in \mathscr{R}_2(\mathbb{R}^3)$. It is lower semicontinuous with respect to integral flat convergence (see for instance [7, Lemma 3.2.14]). Thus for every $\varepsilon > 0$ the compactness Theorem 1.1 implies that the following variational problem admits a minimizer.

$$(\mathscr{P}^{\varepsilon}_{FF,\mathbf{S},B_0}) \begin{cases} \text{minimize } \mathbf{S}(T) + \varepsilon \mathbf{M}(T) \\ \text{among } T \in \mathscr{R}_2(\mathbb{R}^3) \text{ such that } \partial T = B_0 \end{cases}$$

Simple considerations in [7, Lemma 2.1.1] show that

$$\lim_{\varepsilon \to 0} \inf(\mathscr{P}^{\varepsilon}_{FF,\mathbf{S},B_0}) = \inf(\mathscr{P}_{FF,\mathbf{S},B_0}) \,. \tag{2}$$

The support of a minimizing current relative to the problem above enjoys some weak regularity property called monotonicity which we now describe.

3.8. **Definition.** A Radon measure ϕ defined in an open set $U \subseteq \mathbb{R}^3$ is called *2-monotonic* whenever the following holds. For every $x \in U$ the function

$$(0, \operatorname{dist}(x, \operatorname{Bdry} U)) \to \mathbb{R} : r \mapsto \frac{\phi(\mathbf{B}(x,r))}{\pi r^2}$$

is nondecreasing. If also $\Theta^2(\phi, x) \geqslant 1$ for every $x \in \operatorname{spt} \phi$ then we say that ϕ is *2-concentrated*.

The following can be proved for instance as in [6, Proposition 3.4.5].

3.9. **Proposition.** *If $T \in \mathscr{R}_2(\mathbb{R}^3)$ is a minimizing current relative to $(\mathscr{P}^{\varepsilon}_{FF,\mathbf{S},B_0})$ then the measure*

$$\phi = \big(1 + \varepsilon\Theta^2(\|T\|,\cdot)\big)\,\mathscr{H}^2 \llcorner \operatorname{set}_2(\|T\|)$$

is 2-monotonic and 2-concentrated in $U = \mathbb{R}^3 \setminus B$.

We will need two properties related to monotonicity. For a proof see [4, Proposition 4.3 and Corollary 6.13].

3.10. **Proposition.** *Let $\phi_1, \phi_2, \dots$ be a sequence of 2-concentrated 2-monotonic measures in an open set $U \subseteq \mathbb{R}^3$ and assume that $\sup_j \phi_j(U) < \infty$. There then exists a subsequence $\phi_{j(1)}, \phi_{j(2)}, \dots$ and a Radon measure ϕ in U such that*

(A) *$\phi_{j(k)} \to \phi$ weakly* as $k \to \infty$;*
(B) *ϕ is 2-concentrated and 2-monotonic;*
(C) *For every compact $C \subseteq U$,*

$$\lim_{k\to\infty} \inf \Big\{ \delta > 0 : \operatorname{spt}(\phi) \cap C \subseteq \mathbf{B}(\operatorname{spt}(\phi_{j(k)}), \delta) \\ \text{and } \operatorname{spt}(\phi_{j(k)}) \cap C \subseteq \mathbf{B}(\operatorname{spt}(\phi), \delta) \Big\} = 0\,.$$

3.11. **Proposition.** *Let ϕ be a 2-concentrated 2-monotonic measure in an open set $U \subseteq \mathbb{R}^3$ with $\phi(U) < \infty$. Then $\operatorname{spt}\phi = \operatorname{set}_2(\phi)$ is $(\mathscr{H}^2, 2)$ rectifiable.*

4. End of proof

We are now in a position to give the proof of Theorem 2.1. Before starting we notice that the collection of competitors is nonempty (for both formulations of the problem) as follows from a cone construction.

First we show that

$$\inf(\mathscr{P}_{FF,\mathbf{S},B_0}) \leqslant \inf(\mathscr{P}_{R,\mathbb{Z},L})\,. \tag{3}$$

Let X be a competitor for $(\mathscr{P}_{R,\mathbb{Z},L})$ and $\varepsilon > 0$. Let Y be associated with X and ε in Proposition 3.6. It follows from Proposition 3.6(I) together with Proposition 3.7 that the algebraic boundary of Y contains L, and in turn we infer from Proposition 3.6(H) and Proposition 3.5 that there exists $T \in \mathscr{R}_2(\mathbb{R}^3)$ with $\partial T = B_0$ and $\operatorname{spt} T \subseteq Y$. Since

$$\mathbf{S}(T) = \mathscr{H}^2(\operatorname{set}_2(\|T\|)) \leqslant \mathscr{H}^2(\operatorname{spt} T) \leqslant \mathscr{H}^2(Y) \leqslant \varepsilon + \mathscr{H}^2(X)\,,$$

inequality (3) follows from the arbitrariness of X and ε.

Next we choose $\varepsilon_j \downarrow 0$ as $j \to \infty$ and we let T_j, $j = 1, 2, \dots$, denote a minimizer of $(\mathscr{P}^{\varepsilon_j}_{FF,\mathbf{S},B_0})$. We denote by ϕ_j, $j = 1, 2, \dots$, the 2-monotonic 2-concentrated measures associated with T_j in Proposition 3.9. Passing to a subsequence (still denoted ϕ_j) if necessary we may assume that the conclusions of Proposition 3.10 are satisfied. Letting $X_j = \operatorname{spt}(T_j) = \operatorname{spt}(\phi_j) \cup B$ we infer from Proposition 3.10(C) that $\operatorname{dist}_{\mathscr{H}}(X_j, \operatorname{spt}(\phi) \cup B) \to 0$ as $j \to \infty$ (Hausdorff distance). Next we let Y_j and f_j be associated with X_j and ε_j in Proposition 3.6. Since $f_j \restriction B$ is homotopic to the identity of B in the Lipschitzian category we infer from Proposition 3.7(B) that $\partial f_{j\,\#} T_j = B_0$, and since $\operatorname{spt} f_{j\,\#} T_j \subseteq f_j(\operatorname{spt} T_j) \subseteq Y_j$ (according to Proposition 3.6(D)) we infer from Proposition 3.5 that the algebraic boundary of $\operatorname{Clos} Y_j$ contains L. Proposition 3.6(E) says that $\operatorname{dist}_{\mathscr{H}}(X_j, \operatorname{Clos} Y_j) < \varepsilon_j$. Therefore $\operatorname{dist}_{\mathscr{H}}(\operatorname{Clos} Y_j, \operatorname{spt}(\phi) \cup B) \to 0$ as $j \to \infty$. Now Proposition 3.1 implies that the algebraic boundary of $S = \operatorname{spt}(\phi) \cup B$ contains L. Furthermore S is $(\mathscr{H}^2, 2)$

rectifiable according to Proposition 3.11 so that it is an admissible competitor for $(\mathscr{P}_{R,\mathbb{Z},L})$. Finally,

$$\begin{aligned}
\mathscr{H}^2(S) &= \mathscr{H}^2(\operatorname{spt}(\phi)) \\
&\leqslant \phi(U) \qquad\qquad \text{(because } \phi \text{ is 2-concentrated)} \\
&= \lim_j \phi_j(U) \\
&= \lim_j \inf(\mathscr{P}^{\varepsilon_j}_{FF,\mathbf{S},B_0}) \\
&= \inf(\mathscr{P}_{FF,\mathbf{S},B_0}) \,,
\end{aligned}$$

and both conclusions are proved at once.

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