On some nonlinear equations with critical exponents

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1. Introduction

The starting point of our investigations is the following problem tackled in [3], [4]

(1.1a) $\quad -\Delta u + f\left(\frac{|x|}{\lambda}\right)\frac{u}{\lambda^2} = u^5 \qquad \text{in } B_1,$

(1.1b) $\quad u > 0 \qquad \text{in } B_1,$

(1.1c) $\quad u = 0 \qquad \text{on } \partial B_1,$

where B_1 denotes the open unit ball in $\mathbb{R}^3$, $\lambda > 0$ and

$$f(r) = \frac{K}{(1+r^2)^2}$$

on $(0,\infty)$ for some $K > 0$.

One of the main results in [4] asserts that

(a) Problem (1.1) has no solution for $\lambda \geq 1$.

(This is an easy consequence of Pohozaev's identity.)

(b) Problem (1.1) admits a solution for $0 < \lambda < \lambda_0 = \lambda_0(K)$.

It was also suggested (but not proved) that

(c) There exists $0 < \lambda_1 = \lambda_1(K) < 1$ such that problem (1.1) has no radial solution for $\lambda_1 < \lambda < 1$.

On the other hand, it follows from remarkable results of A. Ambrosetti, A. Malchiodi and W.-M. Ni [1] that for every *fixed* $\lambda < 1$, problem (1.1) admits a radial solution for all K sufficiently large, i.e. $K > K_0(\lambda) > 0$. (Their condition (1.4) is satisfied since $(r^2 f(r))' < 0$ on $(1, \infty)$ and $(1/\lambda) > 1$).

It was also suggested in [4] (but not proved) that

(d) for every $\lambda < 1$ there exists $K_1 = K_1(\lambda) > 0$ such that problem (1.1) has no radial solution for $0 < K < K_1(\lambda)$.

The goal of our paper is to show that indeed (c) and (d) hold and that a similar phenomenon occurs for a general class of functions f.

Consider the problem

(1.2a) $\quad -\Delta u + f\left(\frac{|x|}{\lambda}\right)\frac{u}{\lambda^2} = |u|^{\frac{4}{N-2}}u \qquad$ in B_1,

(1.2b) $\quad u = 0 \qquad$ on ∂B_1,

where B_1 denotes the open unit ball in $\mathbb{R}^N$, $N \geq 3$, and $\lambda > 0$.

Assume that

(1.3) $\quad f \in L^\infty_{\text{loc}}([0, \infty))$ and $r^2 f(r)$ is non-decreasing on $[0, 1]$.

Note that (1.3) implies that $\lim_{r \uparrow 1} f(r) = f(1^-)$ exists.

Our assumptions are satisfied by the following examples :

(1.4) $\quad f(r) = \frac{K}{(1 + r^2)^2}$, $K > 0$,

(1.5) $\quad f(r) = K > 0$, $0 \leq r \leq 1$ and $f(r) = 0$, $r > 1$.

Theorem 1.1 *Assume that f satisfies (1.3). Then there exists $\lambda_1 \in (0, 1)$ (depending on f) such that for every $\lambda > \lambda_1$ the only radial solution of problem (1.2) is $u = 0$.*

Next we fix $\lambda = 1$. More precisely, consider the problem

(1.6a) $\quad -\Delta u + f(|x|)u = |u|^{\frac{4}{N-2}}u \qquad$ in B_1,

(1.6b) $\quad u = 0 \qquad$ on ∂B_1.

Assume

(1.7) $\quad f \in L^\infty(0, 1)$ and $r^2 f(r)$ is non-decreasing on $(0, \delta)$ for some $\delta \in (0, 1)$.

Note that (1.7) is satisfied for example if f is smooth on $[0, 1]$ and $f(0) > 0$.

Theorem 1.2 *For every $\delta \in (0,1)$, there exists $K_1 > 0$ (depending only on δ and N) such that, if f satisfies (1.7) and $||f||_\infty \leq K_1$, then the only radial solution of problem (1.6) is $u = 0$.*

When $N = 3$, we have a sharper conclusion.

Theorem 1.3 *Assume $N = 3$. There exists $K_1 > 0$ such that, if $f \in L^\infty(0,1)$ and $||f||_\infty \leq K_1$, then the only radial solution of problem (1.6) is $u = 0$.*

Remark 1.4 Theorem 1.3 is consistent with the result of [2] asserting that, when $N = 3$, the only radial solution of (1.6) with $f = -\lambda$ and $0 < \lambda < \pi^2/4$ is $u = 0$. However, when $N \geq 4$, (1.6) with $f = -\lambda$ has nontrivial radial solutions with $\lambda > 0$ arbitrarily small.

Remark 1.5 Under assumption 1.7, it follows from Theorem 1.2 that the only radial solution of

(1.8a) $\quad -\varepsilon^2 \Delta u + f(|x|)u = |u|^{\frac{4}{N-2}} u \qquad$ in B_1,

(1.8b) $\quad u = 0 \qquad$ on ∂B_1,

is $u = 0$ provided ε is sufficiently large.

Open problem 1.6 *Can one remove the word "radial" in Theorems 1.1, 1.2, 1.3 ? This question is open (and extremely interesting) in the framework of Theorem 1.3 even when f is a negative constant.*

Now we assume that

(1.9a) $\quad f : [0,\infty) \longrightarrow [0,\infty)$ is such that $f \neq 0$ on a set of positive measure and $f \in L^{N/2}_{\text{loc}}([0,\infty), s^{N-1}ds)$,

(1.9b) $$\lim_{\lambda \downarrow 0} \lambda^{N-2} \int_0^{1/\lambda} f(s) s^{N-1} ds = 0.$$

Note that (1.9b) is satisfied if $f \in L^{N/2}([0,\infty), s^{N-1}ds)$ or if

$$\lim_{s \to \infty} s^2 f(s) = 0.$$

Theorem 1.7 *Assume that f satisfies (1.9). Then there exists $\lambda_0 > 0$ such that problem (1.2) admits a positive radial solution for $0 < \lambda < \lambda_0$.*

Remark 1.8 Since $f \geq 0$, it is not possible to prove the existence of a positive solution of problem (1.2) by global minimization as in [2]. When $f \in L^{N/2}([0,\infty), s^{N-1}ds)$, the existence of a positive solution was proved by D. Passaseo in [5] using a clever constrained minimization problem. In section 4, we adapt his approach to assumption (1.9).

2. Proof of Theorem 1.1

Write $u(x) = u(r)$ with $r = |x|$. Problem (1.2) becomes

(2.1a) $$-u'' - \frac{N-1}{r}u' + f\left(\frac{r}{\lambda}\right)\frac{u}{\lambda^2} = |u|^{\frac{4}{N-2}}u, \quad 0 < r < 1,$$

(2.1b) $$u'(0) = u(1) = 0.$$

We use the classical Emden transformation :

$$u(r) = e^{\frac{N-2}{2}t}w(t), \quad t = -\log r.$$

Then problem (2.1) transforms to

(2.2a) $$-w'' + \frac{(N-2)^2}{4}w + \frac{e^{-2t}}{\lambda^2}f\left(\frac{e^{-t}}{\lambda}\right)w = |w|^{\frac{4}{N-2}}w,\ t > 0,$$

(2.2b) $$w(0) = 0,$$

(2.2c) $$|w(t)| \le e^{\frac{2-N}{2}t}||u||_\infty \text{ and } |w'(t)| \le e^{\frac{2-N}{2}t}\left(\frac{N-2}{2}||u||_\infty + e^{-t}||u'||_\infty\right).$$

Define

(2.3) $$t_\lambda = -\log\lambda \text{ and } F_\lambda(t) = \frac{e^{-2t}}{\lambda^2}f\left(\frac{e^{-t}}{\lambda}\right).$$

Lemma 2.1 *Let* $w : [0,\infty) \longrightarrow \mathbb{R}$ *be a solution of (2.2) where* f *satisfies assumption (1.3). Then, for* $0 < \lambda < 1$,

(2.4) $$w'(0)^2 \le -2\int_0^{t_\lambda} F_\lambda(t)w(t)w'(t)dt + f(1^-)w(t_\lambda)^2.$$

Proof. Multiply equation (2.2a) by w' and integrate over $(0,\infty)$. Using (2.2b) and (2.2c), we obtain

(2.5) $$\frac{1}{2}w'(0)^2 + \int_0^\infty F_\lambda(t)w(t)w'(t)dt = 0.$$

Next we write

(2.6) $$\int_0^\infty F_\lambda ww'dt = \int_0^{t_\lambda} F_\lambda ww'dt + \int_{t_\lambda}^\infty F_\lambda ww'dt$$

and observe that

(2.7) $$\int_{t_\lambda}^\infty F_\lambda ww'dt = -\frac{1}{2}F_\lambda({t_\lambda}^+)w(t_\lambda)^2 - \frac{1}{2}\int_{t_\lambda}^\infty w^2dF_\lambda.$$

By assumption (1.3) and by (2.3), $F_\lambda(t)$ is non-increasing on (t_λ,∞) and $F_\lambda({t_\lambda}^+) = f(1^-)$. Hence

(2.8) $$\int_{t_\lambda}^\infty F_\lambda ww'dt \ge -\frac{1}{2}f(1^-)w(t_\lambda)^2.$$

Combining (2.5), (2.6) and (2.8) yields (2.4). □

Lemma 2.2 *Let $A \geq 0$, $B > 0$, $L > 0$ and $w \in C^1([0, L])$ be such that $w(0) = 0$ and, for $0 \leq t \leq L$,*

$$w'(t)^2 \leq A^2 + 2B^2 \int_0^t |ww'| ds. \tag{2.9}$$

Then, for $0 \leq t \leq L$,

$$|w(t)| \leq \frac{A}{B}(e^{Bt} - 1) \tag{2.10}$$

and

$$|w'(t)| \leq Ae^{Bt}. \tag{2.11}$$

Proof. Define, on $[0, L]$, $W(t) = \int_0^t |w'(s)| ds$, so that $W' = |w'|$ and $|w| \leq W$. By assumption, we have

$$W'(t)^2 \leq A^2 + 2B^2 \int_0^t WW' ds = A^2 + B^2 W(t)^2. \tag{2.12}$$

Hence we obtain

$$W'(t) \leq A + BW(t)$$

or

$$\frac{d}{dt}(e^{-Bt} W(t)) \leq Ae^{-Bt}.$$

We find after integration

$$e^{-Bt} W(t) \leq \frac{A}{B}(1 - e^{-Bt})$$

or

$$|w(t)| \leq W(t) \leq \frac{A}{B}(e^{Bt} - 1).$$

Inserting this into (2.12) yields (2.11). □

Lemma 2.3 *Let $w : [0, \infty) \longrightarrow \mathbb{R}$ be a solution of (2.2) where f satisfies assumptions (1.3). Then, for $1/2 < \lambda < 1$ and $0 \leq t \leq t_\lambda$, we have*

$$|w(t)| \leq \frac{|w'(0)|}{c_0}(e^{c_0 t} - 1) \tag{2.13}$$

and

$$|w'(t)| \leq |w'(0)| e^{c_0 t}, \tag{2.14}$$

where $c_0 = \sup_{1<r<2} \left(\frac{(N-2)^2}{4} + r^2 |f(r)| \right)^{1/2}$.

Proof. It follows from equation (2.2a) that

$$\begin{aligned}\frac{w'(t)^2}{2} &= \frac{w'(0)^2}{2} + \int_0^t w'w''ds \\ &= \frac{w'(0)^2}{2} + \int_0^t \left[\frac{(N-2)^2}{4}ww' + \frac{e^{-2s}}{\lambda^2}f\left(\frac{e^{-s}}{\lambda}\right)ww' - |w|^{\frac{4}{N-2}}ww'\right]ds \\ &\leq \frac{w'(0)^2}{2} + \int_0^t \left[\frac{(N-2)^2}{4} + \frac{e^{-2s}}{\lambda^2}\left|f\left(\frac{e^{-s}}{\lambda}\right)\right|\right]|ww'|ds.\end{aligned}$$

For $1/2 < \lambda < 1$ and $0 \leq t \leq t_\lambda$, we obtain

$$w'(t)^2 \leq w'(0)^2 + 2c_0^2 \int_0^t |ww'|ds.$$

It suffices then to use Lemma 2.2 with $A = |w'(0)|$ and $B = c_0$. □

Proof of Theorem 1.1. If $\lambda \geq 1$, problem (1.2) has no nontrivial solution by the argument of Lemma 3.1 in [2].

We assume that $1/2 < \lambda < 1$. It follows from Lemma 2.1 that

$$(2.15) \qquad w'(0)^2 \leq 2\int_0^{t_\lambda} |F_\lambda(t)|\,|w(t)|\,|w'(t)|dt + f(1^-)w(t_\lambda)^2.$$

Inserting (2.13) and (2.14) into (2.15) gives

$$(2.16) \qquad w'(0)^2 \leq w'(0)^2\Phi(\lambda)$$

where

$$(2.17) \qquad \Phi(\lambda) = 2c_0\int_0^{t_\lambda}(e^{c_0t}-1)e^{c_0t}dt + f(1^-)\frac{1}{c_0^2}(e^{c_0t_\lambda}-1)^2.$$

Note that as $\lambda \uparrow 1$, $t_\lambda \downarrow 0$ and $\Phi(\lambda) \longrightarrow 0$. Hence there exists $\lambda_1 \in \left(\frac{1}{2}, 1\right)$ such that $\Phi(\lambda) < 1$ for $\lambda_1 < \lambda < 1$. It follows from (2.16) that $w'(0) = 0$ when $\lambda_1 < \lambda < 1$. By the uniqueness of the Cauchy problem, we complete the proof of Theorem 1.1.

3. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2. We follow the proof of Theorem 1.1 with minor modifications. Using the same transformation as in Section 2, problem (1.6) becomes

$$(3.1a) \qquad -w'' + \frac{(N-2)^2}{4}w + e^{-2t}f(e^{-t})w = |w|^{\frac{4}{N-2}}w, \qquad t > 0$$

$$(3.1b) \qquad w(0) = 0.$$

Letting $F(t) = e^{-2t}f(e^{-t})$ and $T_\delta = -\log\delta$ we have (as in Lemma 2.1)

(3.2) $$w'(0)^2 \le 2\int_0^{T_\delta} |F(t)|\,|w(t)|\,|w'(t)|dt + f(\delta^-)\delta^2 w(T_\delta)^2.$$

Moreover (2.13) and (2.14) still hold on $(0, T_\delta)$ with

$$c_0 = \left(\frac{(N-2)^2}{4} + ||f||_\infty\right)^{1/2}.$$

Hence we have

(3.3) $$w'(0)^2 \le w'(0)^2||f||_\infty\left[\frac{2}{c_0}\int_0^{T_\delta}(e^{c_0 t}-1)e^{c_0 t}dt + \frac{\delta^2}{c_0^2}(e^{c_0 T_\delta}-1)^2\right]$$

and the desired conclusions is derived when $||f||_\infty$ is sufficiently small. □

Proof of Theorem 1.3. We write

$$w'(0)^2 \le 2\int_0^{\infty} |F(t)|\,|w(t)|\,|w'(t)|dt.$$

Choosing $||f||_\infty$ small, we can assume that $c_0 \in (1/2, 1)$. Then we have

$$\begin{aligned} |F(t)| &\le ||f||_\infty e^{-2t}, \\ |w(t)| &\le \frac{|w'(0)|}{c_0}e^{c_0 t}, \\ |w'(t)| &\le |w'(0)|e^{c_0 t}. \end{aligned}$$

Hence we obtain

$$w'(0)^2 \le \frac{2}{c_0}|w'(0)|^2||f||_\infty \int_0^\infty e^{2(c_0-1)t}dt.$$

We conclude as before, since $c_0 < 1$. □

4. Proof of Theorem 1.7

We first define the manifolds

$$\begin{aligned} V(B_1) &= \{u \in H_0^1(B_1) : u \text{ is radial, } ||u||_{2^*} = 1\}, \\ V(\mathbb{R}^N) &= \{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : u \text{ is radial, } ||u||_{2^*} = 1\}, \end{aligned}$$

where $2^* = 2N/(N-2)$, and the functionals, for any $\lambda > 0$,

$$\begin{aligned} \varphi_\lambda(u) &= \int_{\mathbb{R}^N} |\nabla u|^2 + f\left(\frac{|x|}{\lambda}\right)\frac{u^2}{\lambda^2}\,dx, \\ \psi_\lambda(u) &= \int_{\mathbb{R}^N} \frac{|x|}{\lambda+|x|}|u|^{2^*}dx. \end{aligned}$$

Under assumption (1.9) the functional φ_λ is well defined but not necessarily finite on $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

In order to prove that

$$c(\lambda) = \inf\{\varphi_\lambda(u) : u \in V(B_1), \psi_\lambda(u) \geq 1/2\}$$

is a critical value of $\varphi_\lambda\Big|_{V(B_1)}$, we shall estimate

$$d(\lambda) = \inf\{\varphi_\lambda(u) : u \in V(B_1), \psi_\lambda(u) = 1/2\}$$

and

$$d = \inf\{\varphi_1(u) : u \in V(\mathbb{R}^N), \psi_1(u) = 1/2\}.$$

Let us recall that the best Sobolev constant S is defined by

$$S = S(N) = \min_{u \in V(\mathbb{R}^N)} ||\nabla u||^2_{L^2}.$$

Lemma 4.1 *Under assumption (1.9), for every $\lambda > 0$, we have that $S < d \leq d(\lambda)$.*

Proof. It is clear that $S \leq d$. Suppose by contradiction that $S = d$. Then there exists $(u_n) \subset V(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + f(|x|)u_n^2 dx \longrightarrow S, \quad \int_{\mathbb{R}^N} \frac{|x|}{1+|x|}|u_n|^{2^*} dx = 1/2, \quad ||u_n||_{2^*} = 1.$$

By definition of S and by the positivity of f, we have that

(4.1) $\quad ||u_n||_{2^*} = 1, \quad \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \longrightarrow S.$

Going if necessary to a subsequence, we can assume that

(4.2a) $\quad u_n \rightharpoonup u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$,

(4.2b) $\quad |\nabla(u_n - u)|^2 \rightharpoonup \mu$ in $[\mathcal{C}_0(\mathbb{R}^N)]^* = \mathcal{M}(\mathbb{R}^N)$,

(4.2c) $\quad |u_n - u|^{2^*} \rightharpoonup \nu$ in $[\mathcal{C}_0(\mathbb{R}^N)]^* = \mathcal{M}(\mathbb{R}^N)$,

(4.2d) $\quad u_n \longrightarrow u$ a.e. on $\mathbb{R}^N$.

Lemma 1.40 of [7] implies that

(4.3a) $\quad S = ||\nabla u||_2^2 + ||\mu|| + \mu_\infty$,

(4.3b) $\quad 1 = ||u||_{2^*}^{2^*} + ||\nu|| + \nu_\infty$,

(4.3c) $\quad ||\nu||^{2/2^*} \leq S^{-1}||\mu||$,

(4.3d) $\quad \nu_\infty^{2/2^*} \leq S^{-1}\mu_\infty$, where

$$\mu_\infty = \lim_{R\longrightarrow\infty} \overline{\lim_{n\longrightarrow\infty}} \int_{|x|>R} |\nabla u_n|^2 dx, \quad \nu_\infty = \lim_{R\longrightarrow\infty} \overline{\lim_{n\longrightarrow\infty}} \int_{|x|>R} |u_n|^{2^*} dx.$$

It follows from (4.3a,c,d) and from Sobolev inequality that

$$S\left[(||u||_{2^*}^{2^*})^{2/2^*} + ||\nu||^{2/2^*} + \nu_\infty^{2/2^*}\right] \le S.$$

By (4.3b), the only possible values for $||u||_{2^*}^{2^*}$, $||\nu||$ and ν_∞ are 0 or 1.

If $\nu_\infty = 1$, we obtain a contradiction :

$$1/2 = \int_{\mathbb{R}^N} \frac{|x|}{1+|x|} |u_n|^{2^*} dx \longrightarrow 1.$$

If $||\nu|| = 1$, then $u = 0$ and $\nu_\infty = 0$. It follows from an inequality due to Strauss (see [7] p. 56) that

$$u_n \longrightarrow 0 \text{ in } L^{2^*}_{\text{loc}}(\mathbb{R}^N\backslash\{0\}).$$

Thus ν is the Dirac measure at 0 and we obtain also a contradiction :

$$1/2 = \int_{\mathbb{R}^N} \frac{|x|}{1+|x|} |u_n|^{2^*} dx \longrightarrow 0.$$

If $||u||_{2^*} = 1$, then, by (4.1), $\int_{\mathbb{R}^N} |\nabla u|^2 dx = S$. In particular $u > 0$ on $\mathbb{R}^N$ since u is the instanton (see e.g. [2]). It follows then from Fatou's Lemma that

$$S < \int_{\mathbb{R}^N} |\nabla u|^2 + f(|x|)u^2 dx \le S.$$

Let $u \in V(B_1)$ be such that $\psi_\lambda(u) = 1/2$ and define $v_\lambda(y) = \lambda^{\frac{N-2}{2}} u(\lambda y)$ when $|y| \le 1/\lambda$, $v_\lambda(y) = 0$ when $|y| > 1/\lambda$. It is easy to verify that

$$\psi_1(v_\lambda) = \psi_\lambda(u) = 1/2, \quad \varphi_1(v_\lambda) = \varphi_\lambda(u), \quad ||v_\lambda||_{2^*} = ||u||_{2^*} = 1.$$

Hence we obtain $d \le d(\lambda)$. □

Lemma 4.2 *Under assumption (1.9), for every $\lambda > 0$, we have that $S < c(\lambda)$ and*

$$\lim_{\lambda\downarrow 0} c(\lambda) = S.$$

Proof. As in Lemma 4.1, it is easy to verify, by contradiction, that $S < c(\lambda)$. If $S = c(\lambda)$, there exists $(u_n) \subset V(B_1)$ such that

$$\int_{B_1} |\nabla u_n|^2 + f(\frac{|x|}{\lambda})\frac{u_n^2}{\lambda^2} dx \longrightarrow S, \quad 1/2 \le \int_{B_1} \frac{|x|}{\lambda + |x|} |u_n|^{2^*} dx, \quad ||u_n||_{2^*} = 1.$$

Going if necessary to a subsequence, we can assume that (4.2) is satisfied. It is clear that $\nu_\infty = \mu_\infty = 0$ since B_1 is bounded.

If $||\nu|| = 1$, we obtain, as in the preceding Lemma, that

$$1/2 \leq \int_{B_1} \frac{|x|}{\lambda + |x|} |u_n|^{2^*} dx \longrightarrow 0.$$

If $||u||_{2^*} = 1$, then $\int_{\mathbb{R}^N} |\nabla u|^2 dx = S$. But this is impossible since $u \in H_0^1(B_1)$ (see e.g. [2]). Hence we have proved that $S < c(\lambda)$.

Let $\varepsilon > 0$ and $u \in V(B_1) \cap \mathcal{D}(B_1)$ be such that

$$\int_{B_1} |\nabla u|^2 dx \leq S + \varepsilon.$$

By (1.10) we have

$$\lim_{\lambda \downarrow 0} \int_{B_1} f\left(\frac{|x|}{\lambda}\right) \frac{u^2}{\lambda^2} dx = 0.$$

Hence we obtain

$$\lim_{\lambda \downarrow 0} \varphi_\lambda(u) = \int_{B_1} |\nabla u|^2 dx \leq S + \varepsilon.$$

Since $\lim_{\lambda \downarrow 0} \psi_\lambda(u) = 1$, there exists $\delta > 0$ such that, for $0 < \lambda < \delta$,

$$S < c(\lambda) < S + \varepsilon. \qquad \square$$

Proof of Theorem 1.7. By Lemma's 4.1 and 4.2, there exists $\delta > 0$ such that, for $0 < \lambda < \delta$,

$$S < c(\lambda) < \min\{d, 2^{2/N} S\} \leq d(\lambda).$$

Since $c(\lambda) < d(\lambda)$, Ekeland variational principle implies the existence of a Palais-Smale sequence for $\varphi_\lambda\Big|_{V(B_1)}$ at the level $c(\lambda)$.

Hence there exists a sequence $(\alpha_n) \subset \mathbb{R}$ and a sequence $(u_n) \subset V(B_1)$ such that

$$\varphi_\lambda(u_n) \longrightarrow c(\lambda), -\Delta u_n + f\left(\frac{|x|}{\lambda}\right) \frac{u_n}{\lambda^2} - \alpha_n |u_n|^{\frac{4}{N-2}} u_n \longrightarrow 0$$

in $H^{-1}(B_1)$. Therefore, since $||u_n||_{2^*} = 1$, $\varphi_\lambda(u_n) - \alpha_n \longrightarrow 0$ and $\alpha_n \longrightarrow c(\lambda)$.

If we define $v_n = \alpha_n^{(N-2)/4} u_n$, we obtain

$$\Phi(v_n) = \int_{B_1} \left[\frac{|\nabla v_n|^2}{2} + f\left(\frac{|x|}{\lambda}\right) \frac{v_n^2}{2\lambda^2} - \frac{|v_n|^{2^*}}{2^*}\right] dx \longrightarrow c(\lambda)^{N/2}/N,$$

$$-\Delta v_n + f\left(\frac{|x|}{\lambda}\right) \frac{v_n}{\lambda^2} - |v_n|^{\frac{4}{N-2}} v_n \longrightarrow 0$$

in $H^{-1}(B_1)$. But $S < c(\lambda) < 2^{2/N} S$, so that

(4.4) $$\frac{S^{N/2}}{N} < \frac{c(\lambda)^{N/2}}{N} < 2\frac{S^{N/2}}{N}.$$

Passing to a subsequence we may assume that $v_n \longrightarrow v$ weakly in H_0^1 and v satisfies

(4.5) $$-\Delta v + f\left(\frac{|x|}{\lambda}\right)\frac{v}{\lambda^2} = |v|^{\frac{4}{N-2}} v \text{ in } B_1.$$

On the other hand a decomposition theorem (see Struwe [6], or [7] Theorem 8.13) implies that

$$\Phi(v_n) = \Phi(v) + \sum_{i=1}^{k} \frac{1}{N} \int_{\mathbb{R}^N} |w_i|^{2^*} + o(1),$$

where each $w_i \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ satisfies

(4.6) $$-\Delta w_i = |w_i|^{\frac{4}{N-2}} w_i \text{ in } \mathbb{R}^N.$$

Multiplying (4.6) by w_i^+ and w_i^- respectively we see that one of the following conditions holds for each i :

(4.7a) $$\int_{\mathbb{R}^N} |w_i|^{2^*} = 0,$$

(4.7b) $$\int_{\mathbb{R}^N} |w_i|^{2^*} = S^{N/2},$$

(4.7c) $$\int_{\mathbb{R}^N} |w_i|^{2^*} \geq 2S^{N/2}.$$

Similarly, one of the following holds

(4.8a) $v = 0$ and $\Phi(v) = 0$,

(4.8b) v has a constant sign and $\Phi(v) \geq \frac{1}{N} S^{N/2}$,

(4.8c) v changes sign and $\Phi(v) \geq \frac{2}{N} S^{N/2}$.

Since

$$\Phi(v) + \sum_{i=1}^{k} \frac{1}{N} \int_{\mathbb{R}^N} |w_i|^{2^*} = \frac{1}{N} c(\lambda)^{N/2}$$

we conclude, using (4.4), that the only possibility is (4.8b) together with (4.7a). □

Acknowledgment

The second author (M.W.) would like to thank the members of the Laboratoire Jacques-Louis Lions, where this work was done, for their invitation and hospitality.

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