RELATIVE GOURSAT CATEGORIES

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Abstract. We define relative Goursat categories and prove relative versions of the equivalent conditions defining regular Goursat categories. These include 3-permutability of equivalence relations, preservation of equivalence relations under direct images, a condition on so-called Goursat pushouts, and the denormalised 3×3-Lemma. This extends recent work by Gran and Rodelo on a new characterisation of Goursat categories to a relative context.

Introduction

According to A. Carboni, G. M. Kelly and M. C. Pedicchio [2], a **Goursat category** can be defined as a regular category satisfying the 3-permutability of equivalence relations, that is, having $RSR = SRS$ for every two equivalence relations R and S on the same object. However, it is known that there are several other equivalent definitions and characterizations, including the following two, recently obtained by M. Gran and D. Rodelo [4]:

1. A regular category $\mathcal{A}$ is Goursat if and only if for every pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] & D \end{tikzcd}$$

   of regular epimorphisms in $\mathcal{A}$, where f and g are split epimorphisms, the induced morphism between the kernel pairs of f and g is a regular epimorphism; such pushouts are called **Goursat pushouts**.
2. A regular category $\mathcal{A}$ is Goursat if and only if it satisfies the so-called *denormalized 3×3-Lemma*.

This denormalised 3×3-Lemma was first introduced in a regular Mal'tsev context by D. Bourn in [1] and was proved to hold in regular Goursat categories by S. Lack in [10].

The purpose of the present paper is to extend these two results to characterize what we call **relative Goursat categories** by making the following replacements:

- The regular category $\mathcal{A}$ is replaced with a pair $(\mathcal{A}, \mathcal{E})$ where $\mathcal{E}$ is a class of regular epimorphisms in $\mathcal{A}$ satisfying suitable conditions; when $\mathcal{A}$ has all finite limits and coequalizers of kernel pairs and $\mathcal{E}$ is the class of all regular epimorphisms in $\mathcal{A}$, these conditions make $\mathcal{A}$ regular;
- the Goursat pushouts are required to have all their arrows in $\mathcal{E}$;
- the 3×3-Lemma is replaced by its *$\mathcal{E}$-relative version* (see Theorem 3.2 below).

In fact, we show in detail that our conditions on $(\mathcal{A}, \mathcal{E})$ allow us to repeat essentially all arguments of [10] and [4]. Our main tool here is the calculus of $\mathcal{E}$-relations developed in [6, 7], which, unlike previously known versions, does not require $\mathcal{E}$ to be part of a factorization system. Its original motivation was to introduce and study relative semi-abelian and relative homological categories in [7, 6, 5], and now we have the implications:

relative semi-abelian [7, 6] $\Rightarrow$ relative homological [5] $\Rightarrow$
relative Mal'tsev [3] $\Rightarrow$ relative Goursat $\Rightarrow$ relative regular.

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1. Relations in a relative setting

When working with relations, one usually uses the regular image factorisation in a regular category to obtain composition of relations (see e.g. [2]). In a relative setting, regular epimorphisms are replaced with a suitable class $\mathcal{E}$ of regular epimorphisms in the ground category $\mathcal{A}$, and a relative factorisation axiom is used instead of regular image factorisation to compose relations [7] (see also [6]).

In this paper, we consider a slightly more general setting for relations than the one in [7], namely, we do not require the existence of all pullbacks as in [7], but only ask for pullbacks of morphisms in $\mathcal{E}$ to exist. We introduce:

Definition 1.1. A **relative regular category** is a pair $(\mathcal{A}, \mathcal{E})$ where $\mathcal{A}$ is a category with finite products and $\mathcal{E}$ is a class of regular epimorphisms in $\mathcal{A}$ such that the following axioms hold:

- (E1) $\mathcal{E}$ contains all isomorphisms;
- (E2) pullbacks of morphisms in $\mathcal{E}$ exist in $\mathcal{A}$ and are in $\mathcal{E}$;
- (E3) $\mathcal{E}$ is closed under composition;
- (E4) if $f \in \mathcal{E}$ and $gf \in \mathcal{E}$ then $g \in \mathcal{E}$;
- (F) if a morphism f in $\mathcal{A}$ factors as $f = em$ with m a monomorphism and $e \in \mathcal{E}$, then it also factors (essentially uniquely) as $f = m'e'$ with m' a monomorphism and $e' \in \mathcal{E}$.

Note that if $\mathcal{A}$ is a category with products and all pullbacks and $\mathcal{E}$ is a class of regular epimorphisms in $\mathcal{A}$ containing all isomorphisms, then $(\mathcal{A}, \mathcal{E})$ is a relative regular category (i.e. it satisfies axioms (E2)–(F)) if and only if $(\mathcal{A}, \mathcal{E})$ satisfies Condition 1.1 of [7]. Note also that this context is not as general as the one considered in [8] where products are not required to exist. Therefore, the level of generality here is between those of [7] and [8].

Remark 1.2 (The "absolute case")**.** As easily follows from Definition 1.1, if $\mathcal{A}$ is a category with finite limits and has coequalizers of kernel pairs and $\mathcal{E}$ is the class of all regular epimorphisms in $\mathcal{A}$, then $(\mathcal{A}, \mathcal{E})$ is a relative regular category if and only if $\mathcal{A}$ is a regular category.

Relative regular categories provide a convenient setting for the calculus of *$\mathcal{E}$-relations* in the same way that regular categories do for the calculus of relations. The following definitions and properties of $\mathcal{E}$-relations are the relative versions of classical properties of relations. Their proofs easily follow those of the absolute version and appear in [6] (the absolute versions can be found for example in [2]).

Definition 1.3 ($\mathcal{E}$-relations)**.** Given two objects A and B in $\mathcal{A}$, an **$\mathcal{E}$-relation** R **from** A **to** B is a subobject $\langle r_1, r_2\rangle \colon R \to A \times B$ of $A \times B$ such that the morphisms $r_1 \colon R \to A$ and $r_2 \colon R \to B$ are in $\mathcal{E}$. We denote such an $\mathcal{E}$-relation by (R, r_1, r_2) or just by R, and its **opposite** (R, r_2, r_1) by R°; we will also write $R \colon A \to B$ for an $\mathcal{E}$-relation R from A to B. When $A = B$, we may also say that R is an **$\mathcal{E}$-relation on** A.

We can compose two $\mathcal{E}$-relations (R, r_1, r_2) from A to B and (S, s_1, s_2) from B to C by forming the pullback of r_2 and s_1 and then using the factorisation from Axiom (F) to obtain a monomorphism $SR \to A \times C$:

$$
\begin{tikzcd}
 & & P \arrow[dl] \arrow[dr] & & \\
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\
A & & B & & C
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, "\text{mono}"] \arrow[d, dotted, "\in\mathcal{E}"] & R \times S \arrow[d, "r_1 \times s_2"', "\in\mathcal{E}"] \\
SR \arrow[r, dotted, "\text{mono}"'] & A \times C
\end{tikzcd}
$$

Axioms (E2), (E3) and (E4) ensure that this composite is again an $\mathcal{E}$-relation. Moreover, the composition is associative (as we identify isomorphic relations) and we have:

- $(R^\circ)^\circ = R$,
- $(SR)^\circ = R^\circ S^\circ$,
- $R \leq R' \Rightarrow R^\circ \leq R'^\circ$ (ordering as subobjects),
- if $R \leq R'$ and $S \leq S'$ then $SR \leq S'R'$,

for all $\mathcal{E}$-relations $R \colon A \to B$, $R' \colon A \to B$, $S \colon B \to C$, and $S' \colon B \to C$ in $\mathcal{A}$.

Remark 1.4. Given a morphism $f \colon A \to B$ in $\mathcal{E}$, we can use (E1) to view f as an $\mathcal{E}$-relation $f = (A, 1_A, f)$; its opposite is $f^\circ = (A, f, 1_A)$. It is easy to see (cf. [2, 7]) that

- $f^\circ f$ is the kernel pair of f,
- $ff^\circ = 1_B$,

- $ff^\circ f = f$,
- $f^\circ f f^\circ = f^\circ$,
- for any $\mathcal{E}$-relation (R, r_1, r_2) we have $R = r_2 r_1^\circ$.

Definition 1.5. An $\mathcal{E}$-relation (R, r_1, r_2) on an object A in $\mathcal{A}$ is said to be

- **reflexive** if $1_A \leq R$,
- **symmetric** if $R^\circ \leq R$ (and thus $R^\circ = R$),
- **transitive** if $RR \leq R$,
- an **equivalence $\mathcal{E}$-relation** if it is reflexive, symmetric and transitive;
- an **$\mathcal{E}$-effective equivalence $\mathcal{E}$-relation** if it is a kernel pair of some morphism in $\mathcal{E}$.

As easily follows from Definition 1.5, an $\mathcal{E}$-relation $R\colon A \to A$ which is reflexive and transitive satisfies $RR = R$. Note also that the kernel pair of any morphism $f \in \mathcal{E}$ is an ($\mathcal{E}$-effective) equivalence $\mathcal{E}$-relation, by pullback-stability (E2).

This allows us to copy the $n = 3$ version of [2, Theorem 3.5] to a relative version of the same theorem. We give the proof for convenience.

Proposition 1.6. *Let $(\mathcal{A}, \mathcal{E})$ be a relative regular category. The following conditions are equivalent:*

(i) *for equivalence $\mathcal{E}$-relations R and S on an object A, we have $RSR = SRS$;*
(ii) *this 3-permutability $RSR = SRS$ holds when R and S are $\mathcal{E}$-effective equivalence $\mathcal{E}$-relations;*
(iii) *every $\mathcal{E}$-relation P from A to B satisfies $PP^\circ PP^\circ = PP^\circ$;*
(iv) *for every reflexive $\mathcal{E}$-relation E on an object A, the $\mathcal{E}$-relation EE° is an equivalence $\mathcal{E}$-relation;*
(v) *for every reflexive $\mathcal{E}$-relation E, the $\mathcal{E}$-relation EE° is transitive;*
(vi) *for every reflexive $\mathcal{E}$-relation E, we have $EE^\circ = E^\circ E$.*

Proof. Clearly (i) $\Rightarrow$ (ii). Given an $\mathcal{E}$-relation P from A to B we view it as $\langle p_1, p_2\rangle\colon P \to A \times B$ such that $P = p_2 p_1^\circ$. Then $p_1^\circ p_1$ and $p_2^\circ p_2$ are the kernel pairs of p_1 and p_2 respectively, and therefore $\mathcal{E}$-effective equivalence $\mathcal{E}$-relations. Hence, by (ii) and using Remark 1.4, we obtain:

$$PP^\circ PP^\circ = p_2 p_1^\circ p_1 p_2^\circ p_2 p_1^\circ p_1 p_2^\circ = p_2 p_2^\circ p_2 p_1^\circ p_1 p_2^\circ p_2 p_2^\circ = p_2 p_1^\circ p_1 p_2^\circ = PP^\circ,$$

proving (ii) $\Rightarrow$ (iii). Now given a reflexive $\mathcal{E}$-relation E on A as in (iv), the reflexivity $1_A \leq E$ and the induced $1_A \leq E^\circ$ imply $1 \leq EE^\circ$, giving reflexivity of EE°. Symmetry is automatic as $(EE^\circ)^\circ = EE^\circ$, and transitivity $EE^\circ EE^\circ = EE^\circ$ follows from (iii), therefore (iii) $\Rightarrow$ (iv). Clearly (iv) $\Rightarrow$ (v), and (v) $\Rightarrow$ (vi) since reflexivity of E gives

$$E^\circ E \leq EE^\circ EE^\circ \leq EE^\circ.$$

It remains to prove (vi) $\Rightarrow$ (i). Given two equivalence $\mathcal{E}$-relations R and S on an object A, we have $R = R^\circ$, $RR^\circ = R$ and the same for S; moreover, their composite $E = SR$ is clearly reflexive. Therefore we have

$$SRS = SRR^\circ S^\circ = R^\circ S^\circ SR = RSR,$$

which gives (i). $\square$

The case $n = 2$ of [2, Theorem 3.5] defines a regular Mal'tsev category and is stated in its relative version in [3].

In the absolute case of regular categories, it is possible to form the direct image of any endo-relation [2, 4]. In a similar way, we can form an $\mathcal{E}$-image of an endo-$\mathcal{E}$-relation in our relative setting.

Definition 1.7 ($\mathcal{E}$-image)**.** Let $(\mathcal{A}, \mathcal{E})$ be a relative regular category. Given an $\mathcal{E}$-relation (R, r_1, r_2) on an object A of $\mathcal{A}$ and a morphism $f\colon A \to B$ in $\mathcal{E}$, we define the **$\mathcal{E}$-image of R along f** to be the relation S on B which is induced by the ($\mathcal{E}$, mono)-factorisation $\langle s_1, s_2\rangle \circ \varphi$ of the morphism $(f \times f) \circ \langle r_1, r_2\rangle$

$$\begin{tikzcd}
R \arrow[r, dotted, "\varphi"] \arrow[d, "\langle r_1, r_2\rangle"'] & S \arrow[d, dotted, "\langle s_1, s_2\rangle"] \\
A \times A \arrow[r, "f \times f"'] & B \times B
\end{tikzcd}$$

which exists by axiom (F). We write $f(R) = S$, which is again an $\mathcal{E}$-relation by axiom (E4).

Remark 1.8. When R is a reflexive $\mathcal{E}$-relation, the essential uniqueness of ($\mathcal{E}$, mono)-factorisations implies that $f(R)$ is also reflexive, and when R is a symmetric $\mathcal{E}$-relation, φ being an epimorphism implies that $f(R)$ is symmetric. In the next section we will see under which conditions the $\mathcal{E}$-image $f(R)$ of an equivalence $\mathcal{E}$-relation R is again an equivalence $\mathcal{E}$-relation.

As in the absolute case [2, 4], we have an easy way to form the $\mathcal{E}$-image:

Lemma 1.9. *Let $(\mathcal{A},\mathcal{E})$ be a relative regular category. Given an $\mathcal{E}$-relation (R,r_1,r_2) on an object A in $\mathcal{A}$ and a morphism $f\colon A\to B$ in $\mathcal{E}$, the $\mathcal{E}$-image $f(R)$ can be formed as the composite $f(R)=fRf^\circ = fr_2r_1^\circ f^\circ$.* □

Furthermore, using Remark 1.4 and the definition of $\mathcal{E}$-image as well as Lemma 1.9, we easily see the following.

Corollary 1.10. *Let $(\mathcal{A},\mathcal{E})$ be a relative regular category. Given a commutative diagram*

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d, "g"'] & A \arrow[d, "f"] \\ S \arrow[r, shift left, "s_1"] \arrow[r, shift right, "s_2"'] & B \end{tikzcd}$$

where R and S are $\mathcal{E}$-relations and $f\in\mathcal{E}$, the morphism g is in $\mathcal{E}$ if and only if $S=f(R)$, or equivalently if and only if $s_2s_1^\circ = fr_2r_1^\circ f^\circ$. If (R,r_1,r_2) and (S,s_1,s_2) are kernel pairs with coequalizers r and s in $\mathcal{E}$, then the latter is also equivalent to $s^\circ s = fr^\circ rf^\circ$. □

For the main result in the next section, we need the following lemma from [3]:

Lemma 1.11 ([3, Lemma 3.3])**.** *Let $(\mathcal{A},\mathcal{E})$ be a relative regular category. Given a morphism of (downward) split epimorphisms*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, shift right, "f"'] & C \arrow[d, shift right, "g"'] \\ B \arrow[r, "k"'] \arrow[u, shift right] & D \arrow[u, shift right] \end{tikzcd}$$

with f in $\mathcal{E}$, the induced morphism between the kernel pairs of h and k is also in $\mathcal{E}$. □

2. The relative Goursat axiom

We now prove an equivalence of several conditions, which in the absolute case all characterise regular Goursat categories (see [2] and [4]).

Theorem 2.1. *Let $(\mathcal{A},\mathcal{E})$ be a relative regular category. Then the following conditions are equivalent:*

(i) *the **$\mathcal{E}$-Goursat axiom**: given a morphism of (downward) split epimorphisms*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, shift right, "f"'] & C \arrow[d, shift right, "g"'] \\ B \arrow[r, "k"'] \arrow[u, shift right] & D \arrow[u, shift right] \end{tikzcd} \tag{1}$$

in $\mathcal{A}$ with f, g, h and k in $\mathcal{E}$, the induced morphism between the kernel pairs of f and g is also in $\mathcal{E}$;

(ii) *the $\mathcal{E}$-image of an equivalence $\mathcal{E}$-relation is an equivalence $\mathcal{E}$-relation;*

(iii) *for every reflexive $\mathcal{E}$-relation E on an object A, the $\mathcal{E}$-relation EE° is an equivalence $\mathcal{E}$-relation;*

(iv) *for equivalence $\mathcal{E}$-relations R and S on an object A, we have $RSR=SRS$.*

Proof. The proof of (i) ⇒ (ii) is the same as its absolute version given in [4, Theorem 2.3]. We give it here for completeness. Let (R,r_1,r_2) be an equivalence $\mathcal{E}$-relation on A and let $f\colon A\to B$ be in $\mathcal{E}$. We want to show that the $\mathcal{E}$-image $f(R)=(S,s_1,s_2)$ of R along f is again an equivalence $\mathcal{E}$-relation. Since S is reflexive and symmetric by Remark 1.8, we only have to show that it is transitive, that is, $SS\le S$. However, since S is symmetric, it suffices to show the existance of a morphism $t_S\colon S_1\to S$, where (S_1,π_1,π_2) is the kernel pair of s_1, which makes the following diagram commute:

$$\begin{tikzcd} S_1 \arrow[r, dotted, "t_S"] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ S \arrow[r, "s_2"'] & B \end{tikzcd}$$

Since R is (symmetric and) transitive, there exists a morphism $t_R\colon R_1 \to R$, where R_1 is the kernel pair of r_1, making the corresponding diagram for R commute:

$$\begin{tikzcd} R_1 \arrow[r, "t_R"] \arrow[d, shift right] \arrow[d, shift left] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \\ R \arrow[r, "r_2"'] & A \end{tikzcd}$$

Using the morphisms e_R and e_S which define the reflexivity of R and S, we obtain a diagram

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, shift right, "r_1"'] & S \arrow[d, shift right, "s_1"'] \\ A \arrow[r, "f"'] \arrow[u, shift right, "e_R"'] & B \arrow[u, shift right, "e_S"'] \end{tikzcd}$$

of type (**1**), where φ is the $\mathcal{E}$-part of the ($\mathcal{E}$, mono)-factorisation $\langle s_1, s_2\rangle\circ\varphi$ of $(f\times f)\circ\langle r_1, r_2\rangle$. Therefore, by (i), the induced morphism $\overline{\varphi}\colon R_1 \to S_1$ between the kernel pairs of r_1 and s_1 is in $\mathcal{E}$. Since every morphism in $\mathcal{E}$ is a regular epimorphism and therefore a strong epimorphism and $\langle s_1, s_2\rangle$ is a monomorphism, we obtain a unique diagonal t_S in the square

$$\begin{tikzcd} R_1 \arrow[r, "\overline{\varphi}"] \arrow[d, "\varphi\circ t_R"'] & S_1 \arrow[d, "(s_2\times s_2)\circ\langle \pi_1, \pi_2\rangle"] \arrow[dl, dotted, "t_S"'] \\ S \arrow[r, "\langle -s_1, s_2\rangle"'] & B\times B \end{tikzcd}$$

making both triangles commute, which is the required morphism.

For (ii) $\Rightarrow$ (iii) it is easy to see that for a reflexive $\mathcal{E}$-relation (E, e_1, e_2) on an object A we have $EE^\circ = e_1(E_2)$, where E_2 is the kernel pair of e_2. Therefore EE° is an equivalence $\mathcal{E}$-relation as the $\mathcal{E}$-image of the equivalence $\mathcal{E}$-relation E_2. Conditions (iii) and (iv) are equivalent by Proposition 1.6. Finally, for (iv) $\Rightarrow$ (i) we again use the proof from [4, Theorem 2.3]. For convenience, we copy the proof and add our adapted justifications for the relative setting.

Given a diagram such as (**1**), Lemma 1.11 implies that the induced split epimorphism between the kernel pairs H of h and K of k is in $\mathcal{E}$. This means that $f(H) = K$. Now using Lemma 1.9 and the three-permutability from (iv) on the kernel pairs $H = h^\circ h$ and $F = f^\circ f$, we see that

$$\begin{aligned} h(F) &= hf^\circ fh^\circ && \text{(by Lemma 1.9)} \\ &= hh^\circ hf^\circ fh^\circ hh^\circ && \text{(since } hh^\circ h = h\text{)} \\ &= hf^\circ fh^\circ hf^\circ fh^\circ && \text{(by (iv))} \\ &= hf^\circ k^\circ kfh^\circ && \text{(since } f(H) = K\text{)} \\ &= hh^\circ g^\circ ghh^\circ && \text{(since } kf = gh\text{)} \\ &= g^\circ g && \text{(since } hh^\circ = 1\text{)} \\ &= G, \end{aligned}$$

where G is the kernel pair of g. By Corollary 1.10, this implies that the induced morphism between the kernel pairs $F \to G$ is in $\mathcal{E}$. $\square$

As all these conditions characterise Goursat categories in the absolute case [2, 4], we are now justified in the following definition.

Definition 2.2. A **relative Goursat category** is a relative regular category $(\mathcal{A}, \mathcal{E})$ in which moreover the following axiom holds:

(G) the **$\mathcal{E}$-Goursat axiom**: given a morphism of (downward) split epimorphisms

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, shift right, "f"'] & C \arrow[d, shift right, "g"'] \\ B \arrow[r, "k"'] \arrow[u, shift right] & D \arrow[u, shift right] \end{tikzcd}$$

in $\mathcal{A}$ with f, g, h and k in $\mathcal{E}$, the induced morphism between the kernel pairs of f and g is also in $\mathcal{E}$.

Compare this to the definition of relative Mal'tsev categories in [3]: the *$\mathcal{E}$-Mal'tsev axiom* (E5) given there says that for any morphism of split epimorphisms (**1**) with f, g, h and k in $\mathcal{E}$, the canonical morphism $\langle f,h\rangle$ to the pullback $B\times_D C$ is also in $\mathcal{E}$. As in the absolute case, this relative Mal'tsev axiom implies the $\mathcal{E}$-Goursat axiom (G) (see [3, Lemma 3.4]).

3. The relative 3×3-Lemma

In this section we prove the relative version of the so-called denormalized 3×3-Lemma in the context of relative Goursat categories. Furthermore, following the absolute case layed out in [4], we show that, in a relative regular category $(\mathcal{A},\mathcal{E})$, the relative 3×3-Lemma is in fact equivalent to the $\mathcal{E}$-Goursat axiom.

Definition 3.1. Let $(\mathcal{A},\mathcal{E})$ be a relative regular category. We will say that the diagram

$$\begin{tikzcd} F \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & A \arrow[r, "f"] & B \end{tikzcd} \qquad (\mathbf{2})$$

is **$\mathcal{E}$-exact** when (F,f_1,f_2) is the kernel pair of f and f is in $\mathcal{E}$.

Note that when (**2**) is $\mathcal{E}$-exact, the morphisms f_1 and f_2 are also in $\mathcal{E}$ by pullback-stability (E2).
In the proof of the relative 3×3-Lemma, we will need the following

Lemma 3.2 ([7, Theorem 2.10])**.** *Let $(\mathcal{A},\mathcal{E})$ be a relative regular category. Given a diagram*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

with the morphisms f, h, k, and g in $\mathcal{E}$, we have $hf^\circ = g^\circ k$ if and only if $kf = gh$ and the canonical morphism $\langle f,h\rangle\colon A\to B\times_D C$ is in $\mathcal{E}$. □

We first show that the relative 3×3-Lemma does indeed hold in any relative Goursat category.

Theorem 3.3 (The relative 3×3-Lemma)**.** *Let $(\mathcal{A},\mathcal{E})$ be a relative Goursat category. Given a commutative diagram*

$$\begin{tikzcd} \overline{F} \arrow[r, shift left, "\overline{h}_1"] \arrow[r, shift right, "\overline{h}_2"'] \arrow[d, shift right, "\overline{f}_2"'] \arrow[d, shift left, "\overline{f}_1"] & F \arrow[r, "\overline{h}"] \arrow[d, shift right, "f_2"'] \arrow[d, shift left, "f_1"] & G \arrow[d, shift right, "g_2"'] \arrow[d, shift left, "g_1"] \\ H \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] \arrow[d, "\overline{f}"'] & A \arrow[r, "h"] \arrow[d, "f"] & C \arrow[d, "g"] \\ K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & B \arrow[r, "k"] & D \end{tikzcd} \qquad (\mathbf{3})$$

with $\mathcal{E}$-exact columns and middle row, the first row is $\mathcal{E}$-exact if and only if the third row is $\mathcal{E}$-exact.

Proof. The proof is the same as in the absolute case [10]; we repeat it here with the appropriate justifications for the relative case.

Suppose the third row is $\mathcal{E}$-exact. Since k_1 and k_2 are jointly monic, an easy diagram chase proves that $(\overline{F},\overline{h_1},\overline{h_2})$ is the kernel pair of $\overline{h}$. Therefore, it remains to show that $\overline{h}$ is in $\mathcal{E}$. Note that since $\overline{f}$ is in $\mathcal{E}$, Corollary 1.10 implies that the $\mathcal{E}$-image $f(H) = fHf^\circ$ is equal to K. We have:

$$\begin{aligned} g^\circ g &= hh^\circ g^\circ ghh^\circ && \text{(since } hh^\circ = 1\text{)} \\ &= hf^\circ k^\circ kfh^\circ && \text{(since } gh = kf\text{)} \\ &= hf^\circ fh^\circ hf^\circ fh^\circ && \text{(since } f(H) = K\text{)} \\ &= hh^\circ hf^\circ fh^\circ hh^\circ && \text{(by Theorem 2.1(iv))} \\ &= hf^\circ fh^\circ && \text{(since } hh^\circ h = h\text{)} \end{aligned}$$

Therefore $G = h(F)$ and, by Corollary 1.10, $\overline{h}$ is in $\mathcal{E}$.

Conversely, suppose the first row is $\mathcal{E}$-exact. By Axiom (E4) the morphisms k_1, k_2 and k are in $\mathcal{E}$. We have:

$$\begin{aligned}
k^\circ k &= ff^\circ k^\circ kff^\circ && \text{(since } ff^\circ = 1\text{)}\\
&= fh^\circ g^\circ ghf^\circ && \text{(since } gh = kf\text{)}\\
&= fh^\circ hf^\circ fh^\circ hf^\circ && \text{(since } h(F) = G\text{)}\\
&= ff^\circ fh^\circ hf^\circ ff^\circ && \text{(by Theorem 2.1(iv))}\\
&= fh^\circ hf^\circ && \text{(since } ff^\circ f = f\text{)}\\
&= fh_2h_1^\circ f^\circ && \text{(since } H = h^\circ h = h_2h_1^\circ\text{)}\\
&= k_1\overline{f}\,\overline{f}^\circ k_2^\circ && \text{(since } fh_i = k_i\overline{f}\text{)}\\
&= k_1k_2^\circ && \text{(since } \overline{f}\,\overline{f}^\circ = 1\text{)}
\end{aligned}$$

Therefore, since $k^\circ k = k_1k_2^\circ$ and the morphisms k_1, k_2, and k are in $\mathcal{E}$, Lemma 3.2 implies that $k_1'e = k_1$ and $k_2'e = k_2$ for some $e \in \mathcal{E}$, where (k_1', k_2') is the kernel pair of k. It remains to prove that e is an isomorphism. For this, we replace (k_1, k_2) by (k_1', k_2') and $\overline{f}$ by $e\overline{f}$ in the 3×3-diagram (**3**). Then the three rows, the second column and the third column are $\mathcal{E}$-exact; therefore, by the first part of the proof, the first column is also $\mathcal{E}$-exact. Thus $\overline{f}$ and $e\overline{f}$ are both coequalizers of $\overline{f}_1$ and $\overline{f}_2$, yielding that e is an isomorphism. □

Now we show that the relative 3×3-Lemma is equivalent to the $\mathcal{E}$-Goursat axiom in a relative regular category.

Theorem 3.4. *Let $(\mathcal{A}, \mathcal{E})$ be a relative regular category. The following conditions are equivalent:*

(i) *$(\mathcal{A}, \mathcal{E})$ is a relative Goursat category, that is, the axiom (G) holds in $(\mathcal{A}, \mathcal{E})$;*
(ii) *the relative 3×3-Lemma holds in $(\mathcal{A}, \mathcal{E})$;*
(iii) *in a diagram such as* (**3**)*, if the first row is $\mathcal{E}$-exact then the third row is also $\mathcal{E}$-exact;*
(iv) *in a diagram such as* (**3**)*, if the third row is $\mathcal{E}$-exact then the first row is also $\mathcal{E}$-exact.*

Proof. The proof is the same as in the absolute case [4]; we again repeat it for convenience. The part (i) ⇒ (ii) is Theorem 3.3, and the implications (ii) ⇒ (iii) and (ii) ⇒ (iv) are obvious.

(iv) ⇒ (i): Consider the commutative diagram (**1**) with the assumptions of axiom (G). Let (H, h_1, h_2) and (K, k_1, k_2) be the kernel pairs of h and k respectively, and let $\overline{f}\colon H \to K$ be the induced morphism; since f is in $\mathcal{E}$, Lemma 1.11 implies that $\overline{f}$ is also in $\mathcal{E}$. Taking the kernel pairs of $\overline{f}$, f and g and the induced morphisms between them, we obtain the diagram (**3**) with the three columns and the second and the third rows $\mathcal{E}$-exact. Then (iv) implies that the first row is also $\mathcal{E}$-exact, and therefore the induced morphism between the kernel pairs of f and g is in $\mathcal{E}$, proving (i).

(iii) ⇒ (i): Again consider the commutative diagram (**1**). Let (F, f_1, f_2) be the kernel pair of f and let (T, t_1, t_2) be the $\mathcal{E}$-image of F along h. To prove (i) it suffices to show that (T, t_1, t_2) is the kernel pair of g. For this, consider the commutative diagram

$$\begin{tikzcd}
E \arrow[r, shift left, "\eta_1"] \arrow[r, shift right, "\eta_2"'] \arrow[d, shift right, "\overline{f}_2"'] \arrow[d, shift left, "\overline{f}_1"] & F \arrow[r, "\eta"] \arrow[d, shift right, "f_2"'] \arrow[d, shift left, "f_1"] & T \arrow[d, shift right, "t_2"'] \arrow[d, shift left, "t_1"] \\
H \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] \arrow[d, shift right, "\overline{f}"'] & A \arrow[r, "h"] \arrow[d, shift right, "f"'] & C \arrow[d, shift right, "g"'] \\
K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] \arrow[u, shift right] & B \arrow[r, "k"] \arrow[u, shift right] & D \arrow[u, shift right]
\end{tikzcd} \tag{4}$$

in which (E, η_1, η_2), (H, h_1, h_2), and (K, k_1, k_2) are the kernel pairs of η, h, and k respectively, and $\overline{f}_1, \overline{f}_2\colon E \to H$ are the induced morphisms. Since f is in $\mathcal{E}$, Lemma 1.11 implies that $\overline{f}$ is also in $\mathcal{E}$. Moreover, since t_1 and t_2 are jointly monic, it follows that $(E, \overline{f}_1, \overline{f}_2)$ is the kernel pair of $\overline{f}$, making the first column $\mathcal{E}$-exact. Then, applying (iii) to Diagram (**4**) with the role of columns and rows interchanged, we obtain that the third column is also $\mathcal{E}$-exact. Therefore, (T, t_1, t_2) is the kernel pair of g, as desired. □

Remark 3.5. Theorem 3.4 is in fact the relative version of Proposition 3.2 of [4]; in particular, our (iii) ⇒ (iv) corresponds to 3.2(c) ⇒ 3.2(d) in [4], which itself can be considered as a non-pointed version of 5.4(b) ⇒ 5.4(c) in [9].

As mentioned in the introduction, we have a chain of implications of various relative categories: any relative Goursat category is by definition relatively regular, and any relative Mal'tsev category is relatively Goursat, as proved in [3, Lemma 3.4]. Furthermore, every relative homological category is relatively Mal'tsev by [7, Theorem 2.14] or [3, Proposition 4.5], and any relative semi-abelian category is relatively homological by definition, see [7, Definition 3.2]. This is the same chain of implications as in the absolute case.

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