# A good theory of ideals in regular multi-pointed categories

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**Abstract**

By a *multi-pointed category* we mean a category $\mathbb{C}$ equipped with a so called ideal of null morphisms, i.e. a class $\mathcal{N}$ of morphisms satisfying $f \in \mathcal{N} \ \vee \ g \in \mathcal{N} \ \Rightarrow fg \in \mathcal{N}$ for any composable pair $f, g$ of morphisms. Such categories are precisely the categories enriched in the category of pairs $X = (X, N)$ where $X$ is a set and $N$ is a subset of $X$, whereas a pointed category has the same enrichment, but restricted to those pairs $X = (X, N)$ where $N$ is a singleton. We extend the notion of an "ideal" from regular pointed categories to regular multi-pointed categories, and having "a good theory of ideals" will mean that there is a bijection between ideals and kernel pairs, which in the pointed case is the main property of ideal determined categories. The study of general categories with a good theory of ideals allows in fact a simultaneous treatment of ideal determined and Barr exact Goursat categories: we prove that in the case when all morphisms are chosen as null morphisms, the presence of a good theory of ideals becomes precisely the property for a regular category to be a Barr exact Goursat category. Among other things, this allows to obtain a unified proof of the fact that lattices of effective equivalence relations are modular both in the case of Barr exact Goursat categories and in the case of ideal determined categories.

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## Introduction

In this paper we propose so-called *multi-pointed categories* as a setting where pointed contexts in universal / categorical algebra and non-pointed ones could be treated simultaneously.

A multi-pointed category is a category enriched in a suitable monoidal category of *multi-pointed sets*, i.e. sets equipped with a subset of "base points". When the subset is a singleton we get a pointed category, whereas an arbitrary

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category can also be regarded as a multi-pointed category in which each hom-set is a multi-pointed set where all elements are base points.

Multi-pointed categories can be equivalently defined as categories equipped with an *ideal* $\mathcal{N}$ in the sense of R. Lavendhomme [23], i.e. a class of morphisms such that $fg \in \mathcal{N}$ as soon as $f \in \mathcal{N}$ or $g \in \mathcal{N}$. Thus, our context is very similar to the one used by M. Grandis for his "categorical foundations of homological and homotopical algebra" presented in [12]. In particular, we also assume the existence of kernels with respect to $\mathcal{N}$; however, in the "total context" (i.e. when $\mathcal{N}$ is the class of all morphisms) kernels degenerate to isomorphisms and so instead we use a new notion, a notion of a *kernel star*, as the fundamental tool for our purposes. In the total context it gives the notion of a kernel pair, while in the pointed context (i.e. when $\mathcal{N}$ is the class of zero morphisms in a pointed category) it becomes the notion of a kernel.

The first part of the title of the present paper is similar to the title of [26], where having "a good theory of ideals" meant a property of a variety of universal algebras, which is expressed by saying that the passage from congruences to suitably defined ideals is a bijection. Ideals were defined relatively to a constant 0 in the variety, and in the case of a pointed variety (i.e. when 0 is the unique constant), the notion of an ideal has been extended to (finitely cocomplete) pointed regular categories in [17]. In [18] it was observed that ideals can be characterized as regular images of kernels (i.e. normal monomorphisms). With this new insight, having a good theory of ideals becomes equivalent to every regular epimorphism being a cokernel of its kernel, and kernels being stable under regular images. There is a certain similarity with the characterization of Goursat categories obtained in [6], which states that a regular category $\mathbb{C}$ is a Goursat category (in the sense of [7]) if and only if equivalence relations in $\mathbb{C}$ are stable under regular images. Recall that a variety of universal algebras is a Goursat category precisely when it is a 3-permutable variety in the sense of universal algebra.

In a regular multi-pointed category $\mathbb{C}$ we define an *ideal* to be a special type of an internal relation, and namely, the one that arises as a regular image of a kernel star. We say that $\mathbb{C}$ has *a good theory of ideals* when the canonical map from the class of kernel pairs to the class of ideals is a bijection. In the pointed context our ideals coincide with those introduced in [17], and having a good theory of ideals obtains its former meaning. In particular, pointed categories with a good theory of ideals are precisely the ideal determined categories in the sense of [16] (the term "ideal determined" was used for the first time in [13] to mean "having a good theory of ideals" in the sense of [26]). In the total context, ideals are regular images of kernel pairs, and in particular, this includes all internal equivalence relations. Then, having a good theory of ideals becomes equivalent to the category being a Barr exact Goursat category.

The fact that in a categorical setting an ideal should be defined as a relation, rather than a subobject, can be justified by the fact that the usual ideals of unitary rings do not form subrings in a ring. Instead, an ideal $I \subseteq R$ in a unitary ring $R$ can be uniquely represented by the relation $\bigcup_{n\in\mathbb{Z}}\{n\} \times (n+I) \subseteq R \times R$, which is a subalgebra of $R \times R$ and hence "lives" in the category of unitary rings. Notice that in such a relation the image of its first projection is the smallest subring of $R$. This suggests another context where our theory could be applied, and namely the one where $\mathcal{N}$ is the class of those morphisms $f : X \to Y$ whose regular image is the least subobject of $Y$. We call it the *proto-pointed context*.

The varietal proto-pointed context has been treated in detail in [29], and in some sense, the present paper proposes a categorical approach to the theme of [29], which is more general than the one already suggested in [29].

The main goal of the present paper is to show that the study of general categories with a good theory of ideals allows a simultaneous treatment of ideal determined and Barr exact Goursat categories. In particular, we obtain unified formulas for the join of two ideals in both contexts which in turn gives a unified proof of the fact that in both cases kernel pairs form modular lattices. Thus, most of our results are known in the context of Barr exact Goursat categories and (pointed) ideal determined varieties, which have been studied systematically in [6] and [13], respectively. For ideal determined categories, however, many of these results are new.

The paper is organized in three sections, where in Section 1 we introduce multi-pointed categories, define kernels stars and other related notions and study some of their basic properties. In Section 2 we define *star-regular categories* which in the total context become regular categories [1], whereas in the pointed varietal context they become the same as 0-regular varieties in the sense of universal algebra. In Section 3 we define ideals and introduce the concept of having a good theory of ideals. We also remark that if in this concept ideals are traded with kernel stars, then we obtain star-regular categories. Then a category with a good theory of ideals is the same as a star-regular category where ideals coincide with kernel stars. We also obtain characterizations of categories with a good theory of ideals via properties of pushouts of regular epimorphisms. We then show how these properties of pushouts can be used to obtain the formula

$$\iota \vee \kappa = \bar{\iota} \circ \kappa \circ \iota$$

for the join of two ideals in a category with a good theory of ideals (where $\bar{\iota}$ denotes the kernel pair associated to the ideal $\iota$). In the total context of a Barr exact Goursat category, ideals coincide with kernel pairs and so the above formula becomes the well known one: $\iota \vee \kappa = \iota \circ \kappa \circ \iota$. In the pointed context the formula simplifies to $\iota \vee \kappa = \bar{\iota} \circ \kappa$, and so it becomes the known formula obtained for ideal determined varieties in [13]. Finally, we use the above formula to show that ideals in a category with a good theory of ideals always form a modular lattice (which implies that kernel pairs form modular lattices since the ordered set of kernel pairs is isomorphic to the ordered set of ideals in any category with a good theory of ideals).

## 1. Multi-pointed categories

Recall that a *pointed set* is a pair $(X, x)$ where $X$ is a set and $x$ is a fixed element of $x$, called the *base point*. A morphism $(X, x) \to (Y, y)$ of pointed sets is defined as a map $f : X \to Y$ which carries the base point $x$ of $X$ to the base point $y$ of $Y$. Notice thus that the category of pointed sets is a full subcategory of the category of what we call *multi-pointed sets* (which in the literature are known as *pairs of sets* — e.g. see [12]): a *multi-pointed set* is a pair $(X, X')$ where $X$ is a set and $X'$ is any subset of $X$, whose elements are called the *base points* of the multi-pointed set; a morphism $(X, X') \to (Y, Y')$ is defined as a map $f : X \to Y$ which carries every base point of $X$ to a base point of $Y$. This defines a category, under the usual composition of maps, and this category is in fact equivalent to the category of monomorphisms in **Set**, where objects are monomorphisms $x : X' \to X$ in **Set** and morphisms are commutative squares

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X' \arrow[u, "x'"] \arrow[r, "f'"'] & Y' \arrow[u, "y'"'] \end{tikzcd}$$

Multi-pointed sets form a full subcategory of this category, consisting of those monomorphisms which are inclusions of sets.

The category of pointed sets is usually denoted by $\mathrm{Set}_*$, and we will write $\mathrm{Set}_\circ$ for the category of multi-pointed sets. $\mathrm{Set}_\circ$ is a (symmetric) closed monoidal category where for any two multi-pointed sets $(X, X')$ and $(Y, Y')$, the exponential $(Y, Y')^{(X,X')}$ is defined as a multi-pointed set $(Z, Z')$ where $Z$ is the set of all morphisms $f : (X, X') \to (Y, Y')$ in $\mathrm{Set}_\circ$, while $Z'$ is its subset consisting of those morphisms $f$ for which $\mathrm{Im}(f) \subseteq Y'$.

The usual closed monoidal structure on $\mathrm{Set}_*$ is inherited from the above closed monoidal structure of $\mathrm{Set}_\circ$. A *pointed category* can be defined as a category which is enriched in $\mathrm{Set}_*$, and similarly, we define a *multi-pointed category* to be a category enriched in $\mathrm{Set}_\circ$. Specifying a multi-pointed category amounts to specifying a category $\mathbb{C}$ and a class $\mathcal{N}$ of morphisms in $\mathbb{C}$ such that for any composite $fg$ in $\mathbb{C}$, if either $f \in \mathcal{N}$ or $g \in \mathcal{N}$ then $fg \in \mathcal{N}$. Such class $\mathcal{N}$ was called an *ideal* by R. Lavendhomme in [23]. Categories equipped with an ideal (i.e. what we call multi-pointed categories) were used by M. Grandis as a basis for his categorical foundations of homological and homotopical algebra (see [12] and the references therein).

Morphisms in the class $\mathcal{N}$ of a multi-pointed category are called *null morphisms*. Note that for any two objects $X$ and $Y$ in a multi-pointed category $\mathbb{C}$, the set of null morphisms from $X$ to $Y$, written as $\mathcal{N}(X, Y)$, is precisely the set of base points of the "multi-pointed hom-set" of $\mathbb{C}$. A pointed category is precisely a multi-pointed category where for any two objects $X$ and $Y$, there is exactly one null morphism $X \to Y$.

Unlike enrichment in $\mathrm{Set}_*$, which does not always exist but once it exists is unique, enrichment in $\mathrm{Set}_\circ$ always exists and is essentially never unique. Firstly, we can always take $\mathcal{N}$ to be an empty set of morphisms, however, this case is not interesting for our purposes. The interesting instance of an enrichment in $\mathrm{Set}_\circ$ which exists for any category $\mathbb{C}$ is when $\mathcal{N}$ is the class of all morphisms in $\mathbb{C}$. We call this the *total context*, whereas when null morphisms are zero morphisms of a pointed category, we call it the *pointed context*.

We often represent a multi-pointed category $\mathbb{C}$ as a pair $\mathbb{C} = (\mathbb{C}, \mathcal{N})$ where the same $\mathbb{C}$ is used to denote the ground category and $\mathcal{N}$ denotes the class of null morphisms in the multi-pointed category.

For a multi-pointed category $\mathbb{C}$, the *category of stars*, written as $\mathrm{Stars}(\mathbb{C})$, is the category whose objects are objects of $\mathbb{C}$, and a morphism $\sigma$ from $X$ to $Y$ (called a *star* from $X$ to $Y$), displayed as $\sigma : X \rightrightarrows Y$, is a pair of parallel morphisms

$$\begin{tikzcd} X \arrow[r, shift left, "\sigma_1"] \arrow[r, shift right, "\sigma_2"'] & Y \end{tikzcd}$$

in $\mathbb{C}$, with $\sigma_1 \in \mathcal{N}$; we also write such pair as $\sigma = [\sigma_1, \sigma_2]$ and often display it as a double arrow

$$X \overset{\sigma}{\rightrightarrows} Y$$

Composition of stars is defined in a natural way, via component-wise composition of pairs of morphisms. Stars($\mathbb{C}$) itself is a multi-pointed category where null morphisms (which we call *null stars*) are those stars $\sigma$ for which $\sigma_1 = \sigma_2$. We will not distinguish between null stars and the corresponding null morphisms in $\mathbb{C}$. Notice that in the total context stars are simply pairs of parallel morphisms, while in the pointed context a star is uniquely determined by its second component and so we can always think of a star in the pointed context as a single morphism.

*Remark* 1.1. In $\mathbf{Set}_*$, when $\mathcal{N}$ is the class of zero morphisms, a pair $[\sigma_1, \sigma_2]$ of parallel morphisms $X \rightrightarrows Y$ with $\sigma_1$ the zero morphism, can be seen as a directed graph whose all arrows go out from the base point — this suggests to use the term "star" for such pairs. Our usage of this word is similar to that of R. Brown [5], who calls the set of all arrows in a groupoid with fixed domain $x$, the *star of* $x$. The term "star" is also used in a similar way in [14].

We write $\mathrm{stars}_{\mathbb{C}}$ for the functor $\mathrm{stars}_{\mathbb{C}} : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Set}$ defined as follows: it associates to a pair $X, Y$ of objects the set $\mathrm{stars}_{\mathbb{C}}(X, Y)$ of all stars from $X$ to $Y$; for any two morphisms $f : W \to X$ and $g : Y \to Z$, and any star $\sigma = [\sigma_1, \sigma_2] : X \rightrightarrows Y$, we have

$$\mathrm{stars}_{\mathbb{C}}(f, g)(\sigma) = [g\sigma_1 f, g\sigma_2 f].$$

In other words, we have a way of composing stars with morphisms from $\mathbb{C}$; such composition is associative in the obvious sense; in particular, we can view the star $[g\sigma_1 f, g\sigma_2 f]$ above as the composite $(g\sigma)f = g(\sigma f)$. This allows us to talk about commutativity of a mixed diagram of stars and morphisms in $\mathbb{C}$. In such a diagram, composites of stars with morphisms in $\mathbb{C}$ yield stars. In the pointed context stars become usual morphisms and hence such diagrams become usual diagrams.

Now, we extend some basic categorical notions from the pointed context to a general framework of stars:

A *monic star* is a star $\kappa : K \rightrightarrows X$ having the property that for any two morphisms $a, b : A \to K$, if $\kappa a = \kappa b$ then $a = b$. In other words, a star $\kappa$ is monic when its components $\kappa_1, \kappa_2$ are jointly monomorphic. In the pointed context this gives the notion of a monomorphism.

A *kernel star* of a morphism $f : X \to Y$, is a star $\kappa : K \rightrightarrows X$ such that the composite $f\kappa$ is a null star and for any other star $\lambda : L \rightrightarrows X$ with the property that the composite $f\lambda$ is null, there exists a unique morphism $u : L \to K$ such that $\kappa u = \lambda$:

$$\begin{tikzcd} K \arrow[r, shift left, "\kappa"] \arrow[r, shift right] & X \arrow[r, "f"] & Y \\ L \arrow[u, "u"] \arrow[ur, shift left] \arrow[ur, shift right, "\lambda"'] & & \end{tikzcd}$$

A kernel star is always monic. In the total context, the notion of a kernel star of a morphism becomes the notion of a kernel pair of a morphism, while in the pointed context it becomes the notion of a kernel of a morphism.

A *star-pullback* is a commutative diagram

$$
\begin{tikzcd}
P \arrow[r, shift left, "\pi"] \arrow[r, shift right] \arrow[d, "p"'] & X \arrow[d, "f"] \\
K \arrow[r, shift left] \arrow[r, shift right, "\kappa"'] & Y
\end{tikzcd}
\tag{1}
$$

such that for any other such commutative diagram

$$
\begin{tikzcd}
P' \arrow[r, shift left, "\pi'"] \arrow[r, shift right] \arrow[d, "p'"'] & X \arrow[d, "f"] \\
K \arrow[r, shift left] \arrow[r, shift right, "\kappa"'] & Y
\end{tikzcd}
$$

there exists a unique morphism $u : P' \to P$ making the following diagram commute:

$$
\begin{tikzcd}
P' \arrow[dr, "u"] \arrow[drr, bend left, shift left, "\pi'"] \arrow[drr, bend left, shift right] \arrow[ddr, bend right, "p'"'] & & \\
& P \arrow[r, shift left, "\pi"] \arrow[r, shift right] \arrow[d, "p"] & X \arrow[d, "f"] \\
& K \arrow[r, shift left] \arrow[r, shift right, "\kappa"'] & Y
\end{tikzcd}
$$

A *star-pushout* is defined dually (i.e. by reversing all arrows in the above diagram). In the pointed context star-pullbacks and star-pushouts become usual pullbacks and pushouts, respectively. In the total context they can still be defined via particular types of limits and colimits, respectively.

*Remark* 1.2. Our *star-pullbacks* are the same as *imaginary pullbacks*, in a category $\mathbb{C}$ equipped with a functor $E : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Set}$, introduced in [4]; in particular, in our case $E = \mathrm{stars}_{\mathbb{C}}$. In the future, it might be interesting to attempt to generalize our results to a context where the functor $\mathrm{stars}_{\mathbb{C}}$ is replaced with an abstract functor $E$.

As with usual pullbacks, we have:

**Lemma 1.3.** *In a star-pullback (1), if $\kappa$ is monic then also $\pi$ is monic.*

A morphism $f : X \to Y$ is a coequalizer of a star $\lambda : L \rightrightarrows X$, i.e. $f$ is a coequalizer of the components $\lambda_1, \lambda_2$ of $\lambda = [\lambda_1, \lambda_2]$, if and only if the composite $f\lambda$ is a null star, and for any morphism $g : X \to Z$ with $g\lambda$ a null star, there exists a unique morphism $v : Y \to Z$ such that $vf = g$:

$$
\begin{tikzcd}
L \arrow[r, shift left, "\lambda"] \arrow[r, shift right] & X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, "v"] \\
& & Z
\end{tikzcd}
$$

A coequalizer of a star, being a coequalizer, is in particular an epimorphism. In the total context, coequalizers of stars are the same as a regular epimorphism, while in the pointed context they are the same as a normal epimorphism.

The following lemma is well known independently in the total and pointed contexts:

**Lemma 1.4.** *Consider a commutative diagram*

$$\begin{tikzcd} P \arrow[r, shift left, "\lambda"] \arrow[r, shift right] \arrow[d, "e"'] & X \arrow[r, "c"] \arrow[d, "f"] & C \arrow[d, "m"] \\ K \arrow[r, shift left] \arrow[r, shift right, "\kappa"'] & Y \arrow[r, "d"'] & Q \end{tikzcd}$$

*of morphisms and stars.*

(a) *Suppose $\kappa$ is a kernel star of $d$. Then, the left square being a star-pullback implies that $\lambda$ is a kernel star of $c$. The converse implication also holds when $m$ is a monomorphism.*

(b) *Suppose $c$ is a coequalizer of $\lambda$. Then, the right square being a pushout implies that $d$ is a coequalizer of $\kappa$. The converse implication also holds when $e$ is an epimorphism.*

*Proof.* The proof is easy and it can be obtained using straightforward diagram arguments. □

For a multi-pointed category $(\mathbb{C}, \mathcal{N})$, we define a *star-comma category* at an object $X$, written as $(\mathbb{C} \downdownarrows_{\mathcal{N}} X)$, to be a category where objects are all stars $\lambda : L \rightrightarrows X$ with codomain $X$ and morphisms are commutative triangles

$$\begin{tikzcd} L \arrow[rr, "l"] \arrow[dr, shift left] \arrow[dr, shift right, "\lambda"'] & & L' \arrow[dl, shift left, "\lambda'"] \arrow[dl, shift right] \\ & X & \end{tikzcd}$$

When every morphism has a kernel star and every star has a coequalizer, we get an adjunction

$$(\mathbb{C} \downdownarrows_{\mathcal{N}} X) \rightleftarrows (X \downarrow \mathbb{C})$$

where the left adjoint maps each star $\sigma : S \rightrightarrows X$ to its coequalizer, while the right adjoint maps a morphism $f : X \to Y$ to its kernel star. As in the total and pointed contexts, this adjunction restricts to an equivalence

$$\mathrm{KerStars}(X) \cong \mathrm{StarCoeq}(X) \tag{2}$$

between all kernel stars with codomain $X$ and all coequalizers of stars, with domain $X$. This is due to the following:

**Lemma 1.5.** *Consider a star $\sigma : S \rightrightarrows X$ and a morphism $f : X \to Y$.*

(a) *If $f$ is a coequalizer of $\sigma$, then $\sigma$ is a kernel star of some morphism if and only if it is the kernel star of $f$.*

(b) *If $\sigma$ is a kernel star of $f$, then $f$ is a coequalizer of a star if and only if it is the coequalizer of $\sigma$.*

## 2. Star-regular categories

By a *regular multi-pointed category* we mean a multi-pointed category $(\mathbb{C}, \mathcal{N})$ where $\mathbb{C}$ is a regular category [1].

**Definition 2.1.** A *star-regular category* is a regular multi-pointed category in which any regular epimorphism is a coequalizer of a star, and any morphism $f : X \to Y$ has an $\mathcal{N}$*-kernel* — a morphism $k : K \to X$ such that $fk \in \mathcal{N}$ and $k$ is universal with this property, i.e. if $fk' \in \mathcal{N}$ for some morphism $k'$, then $k' = ku$ for a unique morphism $u$ (see Remark 2.3 below).

Notice that in the total context, a star-regular category is simply a regular category, where the $\mathcal{N}$-kernel of each morphism $f : X \to Y$ is $1_X$. In the pointed context, $\mathcal{N}$-kernels are the usual kernels and so a star-regular category is a regular category where every regular epimorphism is a normal epimorphism.

*Remark* 2.2. Regular categories where every regular epimorphism is a normal epimorphism were called *normal categories* in [20]. In universal algebra, these give precisely the pointed 0-regular varieties, which were called *varieties with ideals* by K. Fichtner in [9]. In the varietal proto-pointed context, where the set $E$ of all constants is non-empty, *star-regularity* is precisely $E$*-regularity* in the sense of [29].

*Remark* 2.3. Kernels with respect to an ideal $\mathcal{N}$ were defined by R. Lavendhomme in [23]. They play a central role in the "categorical foundation of homological algebra" of M. Grandis (see [12] and the references there).

In the case when $\mathcal{N}$ is the class of all morphisms, we omit the subscript $\mathcal{N}$ in $(\mathbb{C} \downdownarrows_{\mathcal{N}} X)$ and write it as $(\mathbb{C} \downdownarrows X)$. In this case, we call stars simply *pairs*. This agrees with the terminology "kernel star", which, in the case when $\mathcal{N}$ is the class of all morphisms is "kernel pair" in the usual sense.

For arbitrary $\mathcal{N}$, each $(\mathbb{C} \downdownarrows_{\mathcal{N}} X)$ is a full subcategory of $(\mathbb{C} \downdownarrows X)$, and the full subcategory inclusions $(\mathbb{C} \downdownarrows_{\mathcal{N}} X) \to (\mathbb{C} \downdownarrows X)$ have right adjoints if and only if $\mathcal{N}$-kernels exist. In particular, when $\mathcal{N}$-kernels exist, the right adjoints can be computed as follows: for a pair $[p_1, p_2]$, the corresponding star is $[p_1k, p_2k]$ where $k$ is an $\mathcal{N}$-kernel of $p_1$. Then the triangle

$$\begin{tikzcd} K \arrow[rr, "k"] \arrow[dr, shift left, "{[p_1k,p_2k]}"'] \arrow[dr, shift right] & & P \arrow[dl, shift left, "{[p_1,p_2]}"] \arrow[dl, shift right] \\ & X & \end{tikzcd}$$

is the corresponding component of the counit of the adjunction. We call $[p_1k, p_2k]$ *the star of the pair* $[p_1, p_2]$ and we write $[p_1k, p_2k] = [p_1, p_2]^*$.

When $\mathbb{C}$ is finitely complete, the presence of $\mathcal{N}$-kernels insures the existence of kernel stars and star-pullbacks. In particular, the kernel star of a morphism $f : X \to Y$ can be constructed by taking the star of the kernel pair $p_1, p_2$ of $f$ — i.e. pair of morphisms obtained via the usual pullback

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

while a star-pullback

$$\begin{tikzcd} P \arrow[r, Rightarrow, "\pi"] \arrow[d, "p"'] & X \arrow[d, "f"] \\ K \arrow[r, Rightarrow, "\kappa"'] & Y \end{tikzcd}$$

can be obtained by first forming the pullback

$$\begin{tikzcd} K' \arrow[r, "(\kappa_1',\kappa_2')"] \arrow[d, "f'"'] & X \times X \arrow[d, "f\times f"] \\ K \arrow[r, "(\kappa_1,\kappa_2)"'] & Y \times Y \end{tikzcd}$$

and then taking $\pi = [\kappa_1', \kappa_2']^*$ (then the morphism $p$ in the previous diagram is $p = f'k$ where $k$ is the $\mathcal{N}$-kernel of $\kappa_1'$).

Notice that an $\mathcal{N}$-kernel is always a monomorphism. A morphism $f : X \to Y$ is in $\mathcal{N}$ if and only if $1_X$ is an $\mathcal{N}$-kernel of $f$, which is also equivalent to $f$ having an isomorphism as an $\mathcal{N}$-kernel.

**Lemma 2.4.** *Under the presence of $\mathcal{N}$-kernels, if a composite $gf$ belongs to $\mathcal{N}$ and $f$ is a regular epimorphism, then $g$ belongs to $\mathcal{N}$.*

*Proof.* Indeed, let $k$ be the $\mathcal{N}$-kernel of $g$. Then $f$ factors through $k$. This implies that $k$ is an isomorphism, and hence $g \in \mathcal{N}$. □

In the total context the above lemma becomes trivial, while in the pointed context it follows from a more general fact that if a composite $gf$ is null and $f$ is an epimorphism then $g$ is null (and this does not require existence of kernels).

When $\mathbb{C}$ is a regular category with $\mathcal{N}$-kernels, any star $\varphi : X \rightrightarrows Y$ factorizes as a regular epimorphism followed by a monic star:

$$\begin{tikzcd} & M \arrow[dr, Rightarrow, "\mu"] & \\ X \arrow[ur, "e"] \arrow[rr, Rightarrow, "\varphi"'] & & Y \end{tikzcd}$$

In particular, such factorization is obtained from the usual factorization

$$\begin{tikzcd} & M \arrow[dr, "(\mu_1,\mu_2)"] & \\ X \arrow[ur, "e"] \arrow[rr, "(\varphi_1,\varphi_2)"'] & & Y \times Y \end{tikzcd}$$

of $(\varphi_1, \varphi_2)$ into a regular epimorphism followed by a monomorphism. Applying Lemma 2.4 we can conclude that the pair $\mu = [\mu_1, \mu_2]$ is indeed a star. As in the case of usual factorizations, the above factorization of a star is unique up to a canonical isomorphism.

A *star-relation* on an object $X$ is an equivalence class of monic stars $\varrho : R \rightrightarrows X$, under the equivalence relation which identifies two monic stars that are isomorphic as objects in $(\mathbb{C} \downdownarrows_{\mathcal{N}} X)$. In other words, a star-relation on $X$ is simply an internal relation on $X$ which is represented by a pair $r_1, r_2$ of jointly monomorphic morphisms, with $r_1 \in \mathcal{N}$ (the existence of one such representation implies that all representations are such). The following lemma is rather obvious, and we will often use it:

**Lemma 2.5.** *A subrelation of a star-relation is a star-relation. In particular, this implies that relational meet of two star-relations is always a star-relation.*

Notice that for a monic pair $\varrho = [\varrho_1, \varrho_2] : R \rightrightarrows X$, the associated star $\varrho^*$ is still monic. This allows to define the star of the relation $\varrho$ as the star-relation $\varrho^*$. In the pointed varietal context, when $\varrho$ is a congruence, this construction gives precisely the equivalence class of the unique constant. The following is then a generalization of an obvious familiar fact:

**Lemma 2.6.** *For any two relations $\varrho$ and $\sigma$ in a multi-pointed category with meets of relations and $\mathcal{N}$-kernels, we have:*

$$(\varrho \wedge \sigma)^* = \varrho^* \wedge \sigma^*$$

*Proof.* That $(\varrho \wedge \sigma)^*$ is a subrelation of $\varrho^* \wedge \sigma^*$ is trivial. Conversely, $\varrho^* \wedge \sigma^*$ is a subrelation of $\varrho \wedge \sigma$, which implies that $(\varrho^* \wedge \sigma^*)^*$ is a subrelation of $(\varrho \wedge \sigma)^*$. But $(\varrho^* \wedge \sigma^*)^* = \varrho^* \wedge \sigma^*$ since $\varrho^* \wedge \sigma^*$ is a star by Lemma 2.5. □

By a coequalizer of a relation we mean a coequalizer of any representation of the relation (the coequalizer is unique up to an isomorphism, and does not depend on the choice of the representation).

In the future, by a *kernel star* we mean a star-relation that arises as a kernel star of some morphism. Similarly, by a *kernel pair* we mean a relation that arises as a kernel pair of some morphism (thus by a kernel pair we mean a *kernel relation*, which in the literature is also called an *effective equivalence relation* or a *congruence*).

When $c : X \to Y$ is a coequalizer of a relation $\varrho : R \rightrightarrows X$, we write $\overline{\varrho}$ to denote the kernel pair of $c$ (considered as a relation).

**Lemma 2.7.** *If the relational meet of two kernel stars $\kappa : K \rightrightarrows X$ and $\lambda : L \rightrightarrows X$ of morphisms $f : X \to Y$ and $g : X \to Z$, respectively, exists, then it is the kernel star of the morphism $(f, g) : X \to Y \times Z$ (provided the product $Y \times Z$ exists).*

*Proof.* The proof is straightforward and hence we omit it. □

**Theorem 2.8.** *If $\mathbb{C}$ is finitely complete, then kernel stars of any object $X$ form a meet semi-lattice, where meet of two kernel stars is their relational meet. When furthermore $\mathbb{C}$ is star-regular, we have:*

(a) *any kernel star $\kappa : K \rightrightarrows X$ has a coequalizer $c : X \to X/\kappa$ and $\kappa$ is the star of the kernel pair of $c$, i.e. $\kappa = (\overline{\kappa})^*$.*
(b) *for any two kernel stars $\kappa$ and $\lambda$ we have: $\overline{\kappa \wedge \lambda} = \overline{\kappa} \wedge \overline{\lambda}$.*

*Proof.* The first claim in the theorem follows from Lemma 2.7. Let $\kappa$ be a kernel star of a morphism $f : X \to Y$. Then $\kappa$ is also a kernel star of the regular epimorphism $e$ arising in the decomposition $f = i \circ e$ of $f$ as a regular epimorphism $e \colon X \to I$ followed by a monomorphism $i \colon I \to Y$. Since the category is star-regular, $e$ is a coequalizer of its kernel star $\kappa$. Again, since the category is star-regular, to show $\overline{\kappa \wedge \lambda} = \overline{\kappa} \wedge \overline{\lambda}$ it suffices to show

$$(\overline{\kappa \wedge \lambda})^* = (\overline{\kappa} \wedge \overline{\lambda})^*$$

The left hand side is equal to $\kappa \wedge \lambda$ by what we have already shown applied for the kernel star $\kappa \wedge \lambda$. Then, thanks to Lemma 2.6 we observe that the right hand side is equal to $(\overline{\kappa} \wedge \overline{\lambda})^* = \overline{\kappa}^* \wedge \overline{\lambda}^* = \kappa \wedge \lambda$. $\square$

When $\mathbb{C}$ is a regular multi-pointed category, any two star-relations $\varrho : R \rightrightarrows X$ and $\sigma : S \rightrightarrows X$ can be composed $\varrho \circ \sigma$ by forming the pullback

$$\begin{tikzcd} & P \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ R \arrow[dr, "\varrho_1"'] & & S \arrow[dl, "\sigma_2"] \\ & X & \end{tikzcd} \tag{3}$$

and then taking $\varrho \circ \sigma$ to be the star-relation arising in the factorization

$$\begin{tikzcd} & M \arrow[dr, shift left, "\varrho\circ\sigma"] \arrow[dr, shift right] & \\ P \arrow[ur, "e"] \arrow[rr, shift left] \arrow[rr, shift right, "{[\sigma_1 p_1, \varrho_2 p_2]}"'] & & X \end{tikzcd}$$

of the pair $[\sigma_1 p_1, \varrho_2 p_2]$ via a regular epimorphism followed by a monic star. In other words, $\varrho \circ \sigma$ is the usual relational composite of $\varrho$ and $\sigma$, and so composition of star-relations is associative.

**Lemma 2.9.** *In the pointed context, for any two star-relations $\varrho$ and $\sigma$, we have $\varrho \circ \sigma = \varrho$.*

*Proof.* This follows from the fact that in the pointed context $\varrho_1$ is a zero morphism, and so the pullback (3) becomes the diamond

$$\begin{tikzcd} & \mathrm{Ker}(\sigma_1) \times R \arrow[dl, "\pi_2"'] \arrow[dr, "\ker(\sigma_2)\pi_1"] & \\ R \arrow[dr, "0"'] & & S \arrow[dl, "\sigma_2"] \\ & X & \end{tikzcd}$$

where $\pi_2$ is a regular epimorphism. $\square$

**Lemma 2.10.** *In a regular multi-pointed category, for any relation $\varrho : R \rightrightarrows X$ we have*

$$\varrho^* = \varrho \circ \delta_X^*$$

*where $\delta_X$ denotes the discrete relation $\delta_X = [1_X, 1_X] : X \rightrightarrows X$.*

*Proof.* Consider the pullback

$$\begin{tikzcd}
& P \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\
R \arrow[dr, "\varrho_1"'] & & K \arrow[dl, "k"] \\
& X &
\end{tikzcd}$$

where $k$ denotes the $\mathcal{N}$-kernel of $1_X$. Then $\varrho \circ \delta_X^* = [kp_1, \varrho_2 p_2]$. Now, complete the above diagram as follows:

$$\begin{tikzcd}
& & P \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\
& R \arrow[dl, "\varrho_1"'] \arrow[dr, "\varrho_1"] & & K \arrow[dl, "k"] \\
X \arrow[dr, "1_X"'] & & X \arrow[dl, "1_X"] & \\
& X & &
\end{tikzcd}$$

The fact that the upper diamond is a pullback and $k$ is an $\mathcal{N}$-kernel of $1_X$ implies that $p_2$ is an $\mathcal{N}$-kernel of $\varrho_1$. But then,

$$\varrho \circ \delta_X^* = [kp_1, \varrho_2 p_2] = [\varrho_1 p_2, \varrho_2 p_2] = \varrho^*.$$

□

The above lemma implies that for each object $X$, the star-relation $\delta_X^*$ is a right unit for composition of star-relations on $X$. In the total context, $\delta_X^* = \delta_X$ is the two-sided unit. The above lemma also has the following useful consequence:

**Corollary 2.11.** *In a regular multi-pointed category $\mathbb{C}$, for any two relations $\varrho$ and $\sigma$ on an object $X$,*

$$(\varrho \circ \sigma)^* = \varrho \circ \sigma^*.$$

From the construction of the composite of two relations above we easily get the following:

**Lemma 2.12.** *Consider a diagram*

$$\begin{tikzcd}
& T \arrow[d, shift left, "\tau=\varrho\circ\sigma"] \arrow[d, shift right] & \\
R \arrow[r, shift left, "\varrho"] \arrow[r, shift right] & X \arrow[d, "f"] & S \arrow[l, shift left] \arrow[l, shift right, "\sigma"'] \\
& Y &
\end{tikzcd}$$

*in a regular multi-pointed category, where $\varrho$ and $\sigma$ are star-relations and $\varrho \circ \sigma$ is their composite. If the composites $f\varrho$ and $f\sigma$ are null, then also the composite $f\tau$ is null.*

The ordered set of star-relations on $X$ will be denoted by $\mathrm{SRel}_{\mathcal{N}}(X)$. When $\mathbb{C}$ is a regular category with $\mathcal{N}$-kernels, each morphism $f : X \to Y$ defines an adjunction

$$\mathrm{SRel}_{\mathcal{N}}(X) \rightleftarrows \mathrm{SRel}_{\mathcal{N}}(Y),$$

$$\mathrm{SRel}_{\mathcal{N}}(X) \ni \varrho \mapsto f\langle\varrho\rangle \quad \dashv \quad \mathrm{SRel}_{\mathcal{N}}(Y) \ni \sigma \mapsto f^{-1}\langle\sigma\rangle^*$$

Here the star $f\langle\varrho\rangle$, called *the image of* $\varrho$ *under* $f$, is defined as the star-relation arising in the decomposition

$$\begin{tikzcd} R \arrow[r, "e"] \arrow[d, shift right, "\varrho"'] \arrow[d, shift left] & M \arrow[d, shift left, "f\langle\varrho\rangle"] \arrow[d, shift right] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

of the star $f\varrho$ into a regular epimorphism followed by a monic star. In other words, $f\langle\varrho\rangle$ is the usual image of the relation $\varrho$ under $f$. The star $f^{-1}\langle\sigma\rangle^*$, called the *star-inverse image of* $\sigma$ *under* $f$, is obtained via the star-pullback of $\sigma$ along $f$:

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, shift right, "f^{-1}\langle\sigma\rangle^*"'] \arrow[d, shift left] & S \arrow[d, shift left, "\sigma"] \arrow[d, shift right] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

In other words, $f^{-1}\langle\sigma\rangle^*$ is the star of the usual inverse image $f^{-1}\langle\sigma\rangle$ of the relation $\sigma$ under $f$ (i.e. $f^{-1}\langle\sigma\rangle^* = (f^{-1}\langle\sigma\rangle)^*$).

We now turn our attention to studying conditions on a commutative square

$$\begin{tikzcd} X \arrow[r, two heads, "f"] \arrow[d, two heads, "e"'] & Y \arrow[d, two heads, "g"] \\ W \arrow[r, two heads, "h"'] & Z \end{tikzcd} \tag{4}$$

of regular epimorphisms, which are equivalent to the square being a pushout. These conditions will be useful to characterize both star-regular categories and categories with a good theory of ideals introduced in the subsequent section.

**Proposition 2.13.** *In any regular multi-pointed category, consider a commutative square (4) of regular epimorphisms, and consider star-relations $\varepsilon : E \rightrightarrows X$, $\varphi : F \rightrightarrows X$ such that $f$ is a coequalizer of $\varphi$ and $e$ is a coequalizer of $\varepsilon$. Then the condition (a) below implies (b), and (b) is equivalent to (c):*

(a) *$gf = he$ is a coequalizer of $\varphi \circ \varepsilon$;*
(b) *$g$ is a coequalizer of $f\langle\varepsilon\rangle$;*
(c) *(4) is a pushout.*

*Moreover, under the presence of $\mathcal{N}$-kernels, if (4) is a pushout then the kernel star of $gf$ can be computed by the formula*

$$\kappa^*_{gf} = f^{-1}\left\langle \kappa^*_g \right\rangle^*$$

*Finally, if both $\varepsilon$ and $\varphi$ are subrelations of $\varphi \circ \varepsilon$, then all three conditions (a), (b), (c) above are equivalent.*

*Proof.* (a)⇒(b): Consider the commutative diagram

$$\begin{tikzcd}
 & E \arrow[r, two heads, "f'"] \arrow[d, shift right, "\varepsilon"'] \arrow[d, shift left] & E' \arrow[d, shift left, "f\langle\varepsilon\rangle"] \arrow[d, shift right] & \\
C \arrow[r, shift left, "\varphi\circ\varepsilon"] \arrow[r, shift right] & X \arrow[r, two heads, "f"] \arrow[d, "e"'] & Y \arrow[r, "a"'] \arrow[d, "g"] & A \\
 & W \arrow[r, two heads, "h"'] & Z &
\end{tikzcd}$$

where $a$ is any morphism such that the composite $af\langle\varepsilon\rangle$ is null. First, observe that the composite $g(f\langle\varepsilon\rangle)$ is null since composed with the regular epimorphism $f'$ it becomes null. To show that $g$ is a coequalizer of $f\langle\varepsilon\rangle$ we have to show that $a$ factors through $g$. Applying Lemma 2.12 we see that $af(\varphi\circ\varepsilon)$ is null. Now, if (a) holds, then $af$ factors through $gf$, and since $f$ is an epimorphism this implies that $a$ factors through $g$.

(b)⇔(c) by Lemma 1.4(b).

Next, we prove the second part of the proposition. Consider the commutative diagram, where the left part is given by the inverse image construction, and hence is a star-pullback:

$$\begin{tikzcd}
P \arrow[r, shift left, "f^{-1}\langle\kappa_g^*\rangle^*"] \arrow[r, shift right] \arrow[d, "e"'] & X \arrow[r, "gf"] \arrow[d, "f"] & Z \arrow[d, "1_Z"] \\
K \arrow[r, shift left] \arrow[r, shift right, "\kappa_g^*"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Then Lemma 1.4(a) gives the desired result.

Now, to prove the last part of the proposition, suppose (b) holds true. Using Lemma 2.12 we can deduce that $gf(\varphi\circ\varepsilon)$ is null. Take any morphism $b : X \to A$ such that $b(\varphi\circ\varepsilon)$ is null. If $\varphi$ is a subrelation of $\varphi\circ\varepsilon$, then $b\varphi$ is null and hence we can present $b\varphi$ as a composite $af$ for some morphism $a : Y \to A$. If $\varepsilon$ is a subrelation of $\varphi\circ\varepsilon$ then also $b\varepsilon$ is null, which implies that $a(f\langle\varepsilon\rangle)$ is null. This gives that $a$ factors through $g$ and hence $b = af$ factors through $gf$. Such factorization is unique since $gf$ is an epimorphism. This proves that $gf$ is a coequalizer of $\varphi\circ\varepsilon$. □

**Theorem 2.14.** *For a regular multi-pointed category $\mathbb{C}$ with $\mathcal{N}$-kernels the following conditions are equivalent:*

(a) *$\mathbb{C}$ is a star-regular category;*

(b) *A commutative square (4) of regular epimorphisms is a pushout if and only if $g$ is a coequalizer of $f\langle\kappa_e^*\rangle$, where $\kappa_e^*$ denotes the kernel star of $e$.*

*Proof.* (a)⇒(b): If every regular epimorphism is a coequalizer of a star, then $e$ is a coequalizer of $\varepsilon = \kappa_e^*$ and we get (b) by applying the equivalence (b)⇔(c) in Proposition 2.13.

(b)⇒(a): to show that a regular epimorphism $e : X \to W$ is a coequalizer of its star kernel, simply apply (b) to the pushout

$$\begin{tikzcd} X \arrow[r, two heads, "1_X"] \arrow[d, two heads, "e"'] & X \arrow[d, two heads, "e"] \\ W \arrow[r, two heads, "1_W"'] & W \end{tikzcd}$$

□

Consider the pointed varietal context, i.e. when $\mathbb{C}$ is a pointed variety of universal algebras (with unique constant 0), and $\mathcal{N}$ is the class of zero morphisms (= constant homomorphisms). In this case, Theorem 2.14 asserts that $\mathbb{C}$ is a 0-regular variety if and only if the following holds: a commutative square of surjective homomorphisms (4) is a pushout if and only if $g$ is the cokernel of the regular image $f(\mathrm{Ker}(e)) \to Y$ of $\mathrm{Ker}(e)$ along $f$.

## 3. Categories with a good theory of ideals

In this section we always work in a regular multi-pointed category $\mathbb{C}$ with $\mathcal{N}$-kernels.

**Definition 3.1.** Let $\mathcal{I}$ be a class of star-relations in $\mathbb{C}$. We say that $\mathbb{C}$ *has a good theory of* $\mathcal{I}$ when

(a) every star in $\mathcal{I}$ has a coequalizer;
(b) the assignment $\pi \mapsto \pi^*$ carries every kernel pair to a star-relation in $\mathcal{I}$ (in other words, $\mathcal{I}$ contains all kernel stars);
(c) moreover, the map

$$\mathrm{KernelPairs} \to \mathcal{I}, \quad \pi \mapsto \pi^*$$

is a bijection.

In the total context, take $\mathcal{I}$ to be the class of internal equivalence relations. Then the category has a good theory of $\mathcal{I}$ precisely when every equivalence relation is a kernel pair, i.e. when the category is a Barr exact category [1]. More generally, we have:

**Theorem 3.2.** *For any class $\mathcal{I}$ of star-relations, $\mathbb{C}$ has a good theory of $\mathcal{I}$ if and only if $\mathbb{C}$ is star-regular and $\mathcal{I}$ coincides with the class of kernel stars.*

*Proof.* Condition 3.1(b) simply says that $\mathcal{I}$ contains all kernel stars, and the assignment of 3.1(c) decomposes as $\mathrm{KernelPairs} \to \mathrm{KernelStars} \to \mathcal{I}$ where the left arrow is a surjective map and the right arrow is an inclusion and hence injective. To say that the composite is a bijection is the same as to say that the left arrow is injective and the right arrow is bijective. Thus, $\mathbb{C}$ has a good theory of $\mathcal{I}$ when 3.1(a) is satisfied and

(d) two kernel pairs having the same star are equal;

(e) $\mathcal{I}$ coincides with the class of kernel stars.

Now it remains to observe that when 3.1(a) is satisfied, (d) implies that every regular epimorphism is a coequalizer of a star, and conversely, when every regular epimorphism is a coequalizer of a star then all kernel stars have coequalizers and (d) is satisfied. □

Notice in particular that categories with *a good theory of kernel stars* are precisely the star-regular categories.

In the pointed context, having a good theory of *all* star-relations gives the classical notion of an abelian category:

**Theorem 3.3.** *In the pointed context, $\mathbb{C}$ has a good theory of $\mathcal{I}$, where $\mathcal{I}$ is the class of all star-relations, if and only if $\mathbb{C}$ is an abelian category.*

*Proof.* It is well known that a pointed category with finite products is abelian if and only if any morphism decomposes as a normal epimorphism followed by a normal monomorphism (see [10], for instance). In particular, a pointed regular category is abelian if and only if every regular epimorphism is normal and every monomorphism is normal. This is precisely to say that the category is star-regular and every star-relation is a kernel star. □

The following notion of an ideal gives a categorical counterpart of the notion of an ideal in the sense of [26], in the context of a (regular) multi-pointed category, and is based on a characterization of these ideals obtained in [18]. In particular, in the pointed context, our notion of an ideal is precisely the notion of an ideal in the sense of [18].

**Definition 3.4.** An *ideal* in $\mathbb{C}$ is an image $f\langle\kappa\rangle$ of a kernel star $\kappa$ under a regular epimorphism $f$. We say that $\mathbb{C}$ *has a good theory of ideals* when $\mathbb{C}$ has a good theory of $\mathcal{I}$ for the class $\mathcal{I}$ of all ideals.

The following is an immediate corollary of Theorem 3.2:

**Corollary 3.5.** *In the pointed context, categories having a good theory of ideals are precisely the ideal determined categories in the sense of [16] (under the presence of finite colimits which is required in [16]).*

*Proof.* In [16] an ideal determined category is defined as a pointed regular category (with finite colimits) where every regular epimorphism is a normal epimorphism and every ideal is a normal monomorphism. This is the same as to say, in our terminology, that the category is star-regular and every ideal is a kernel star. Hence we get the result by Theorem 3.2. □

Recall that a Goursat category [7] is a regular category with the property that the composition of equivalence relations on any object is 3-permutable:

$$\varrho \circ \sigma \circ \varrho = \sigma \circ \varrho \circ \sigma$$

for all equivalence relations $\varrho, \sigma$ on any object $X$. We have:

**Theorem 3.6.** *In the total context, categories having a good theory of ideals are precisely the Barr exact Goursat categories.*

*Proof.* In the total context, ideals are regular images of kernel pairs, and in particular this includes all internal equivalence relations. By Theorem 6.8 proved in [6], a regular category is a Goursat category if and only if equivalence relations are stable under regular images. From this result and Theorem 3.2 above it follows that regular categories having a good theory of ideals are precisely the Barr exact Goursat categories. □

Corollary 3.5 and Theorem 3.6 seem to suggest that the notion of a category with a good theory of ideals in some sense unifies the notion of an ideal determined category and the notion of a Barr exact Goursat category. In the varietal context these are the pointed ideal determined varieties and 3-permutable varieties, respectively. Notice also that in the proto-pointed varietal context, where the set $E$ of all constants is nonempty, having a good theory of ideals is equivalent to being $E$-ideal determined in the sense of [29].

*Remark* 3.7. The observations above naturally give rise to the following question: which other classes of categories studied in modern categorical algebra can be presented as categories having a good theory of $\mathcal{I}$, for a suitable class $\mathcal{I}$ (either in the pointed or the total contexts)? In the pointed context, it would be particularly interesting to find such a presentation for semi-abelian categories in the sense of [15], which in the varietal context are the same as pointed varieties with a "special good theory of ideals" [27].

**Theorem 3.8.** *For a star-regular category $\mathbb{C}$ where coequalizers of ideals exist, the following conditions are equivalent:*

(a) *$\mathbb{C}$ has a good theory of ideals.*
(b) *A commutative square*

$$\begin{tikzcd} X \arrow[r, two heads, "f"] \arrow[d, two heads, "e"'] & Y \arrow[d, two heads, "g"] \\ W \arrow[r, two heads, "h"'] & Z \end{tikzcd} \tag{5}$$

*of regular epimorphisms is a pushout if and only if*

$$\kappa_g^* = f\langle \kappa_e^* \rangle$$

*where $\kappa_g^*$ is the kernel star of $g$ and $\kappa_e^*$ is the kernel star of $e$.*

*Proof.* (a)⇒(b): Since $\mathbb{C}$ is star-regular, we can apply Theorem 2.14: then, all we have to show is that the equality $\kappa_g^* = f\langle \kappa_e^* \rangle$ is equivalent to $g$ being a coequalizer of $f\langle \kappa_e^* \rangle$. But this we have by the fact that $f\langle \kappa_e^* \rangle$ is a kernel star and $g$ is a coequalizer of a star (which itself follows from the fact that $\mathbb{C}$ has a good theory of ideals).

(b)⇒(a): All we have to show is that every ideal is a kernel star. By definition, any ideal is a star which arises as $f\langle \kappa_e^* \rangle$ for some regular epimorphism $f$ and a kernel star $\kappa_e^*$ of some morphism $e$. Since the category is regular, we can assume, without loss of generality, that $e$ is also a regular epimorphism. Now, taking $g$ to be the coequalizer of $f\langle \kappa_e^* \rangle$, we obtain a commutative square (5) where $h$ is a regular epimorphism since all other morphisms are. By Theorem 2.14, (5) is a pushout. Now, (b) implies that $f\langle \kappa_e^* \rangle$ is a kernel star. □

Theorems 2.14 and 3.8 together give:

**Corollary 3.9.** *A regular multi-pointed category, with $\mathcal{N}$-kernels and coequalizers of ideals, has a good theory of ideals if and only if the following holds: for any commutative square (5) of regular epimorphisms, the following conditions are equivalent:*

(a) *(5) is a pushout;*
(b) *$g$ is a coequalizer of $f\langle\kappa_e^*\rangle$;*
(c) *$\kappa_g^* = f\langle\kappa_e^*\rangle$.*

In a category with a good theory of ideals coequalizers of ideals exist, since ideals are the same as kernel stars, and in any star-regular category kernel stars have coequalizers. From Corollary 3.9 it follows that also pushouts of regular epimorphisms along regular epimorphisms exist. This in turn implies existence of joins of ideals: indeed, if $\iota : I \rightrightarrows X$ and $\kappa : K \rightrightarrows X$ are two ideals, then their join can be obtained as the kernel star of the diagonal $gf = he$ in the pushout

$$\begin{tikzcd} X \arrow[r, two heads, "f"] \arrow[d, two heads, "e"'] & X/\iota \arrow[d, two heads, "g"] \\ X/\kappa \arrow[r, two heads, "h"'] & Z \end{tikzcd} \tag{6}$$

Thus, in a category with a good theory of ideals, for each object $X$ the ideals of $X$ form a lattice. This lattice is isomorphic to the ordered set of kernel pairs at $X$, where the isomorphism is given by the mapping $\iota \mapsto \bar{\iota}$ which associates to each ideal $\iota$ the kernel pair $\bar{\iota}$ of the coequalizer of $\iota$. In what follows we obtain a convenient formula for the join of two ideals, which is later applied in showing that the lattice of ideals is a modular lattice.

**Theorem 3.10.** *Let $\iota : I \rightrightarrows X$ and $\kappa : K \rightrightarrows X$ be ideals in a category $\mathbb{C}$ with a good theory of ideals. Then the join $\iota \vee \kappa$ in the ordered set of all ideals of $X$ is given by the formula*

$$\iota \vee \kappa = \bar{\iota} \circ \kappa \circ \iota \tag{7}$$

*where $\bar{\iota}$ denotes the kernel pair of the coequalizer $f$ of $\iota$. Equivalently, the above formula can be rewritten as*

$$\iota \vee \kappa = f^{-1}\langle f\langle\kappa\rangle\rangle^*.$$

*Proof.* Consider the pushout (6). By Corollary 3.9, $f\langle\kappa\rangle$ is a kernel star of $g$, and from Lemma 1.4(a) it follows that $f^{-1}\langle f\langle\kappa\rangle\rangle^*$ is a kernel star of $gf$ which is the join $\iota \vee \kappa$. It suffices to show that $f^{-1}\langle f\langle\kappa\rangle\rangle^*$ is the composite $\bar{\iota} \circ \kappa \circ \iota$.

Note that any morphism $f : X \to Y$ gives rise to a relation

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

from $X$ to $Y$, whose opposite relation is

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "1_X"] & \\ Y & & X \end{tikzcd}$$

We denote these relations by $f^+$ and $f^-$, respectively. Then,

- the kernel pair of $f$ is the composite $f^- \circ f^+$,
- for any relation $\varrho : R \rightrightarrows X$, the image of $\varrho$ along $f$ is given by

$$f\langle\varrho\rangle = f^+ \circ \varrho \circ f^-,$$

- for any relation $\sigma : S \rightrightarrows X$, the inverse image of $\sigma$ along $f$ is given by

$$f^{-1}\langle\sigma\rangle = f^- \circ \sigma \circ f^+.$$

After this only a straightforward computation remains:

$$\begin{aligned} \bar{\iota} \circ \kappa \circ \iota &= f^- \circ f^+ \circ \kappa \circ (f^- \circ f^+)^* \\ &= (f^- \circ f^+ \circ \kappa \circ f^- \circ f^+)^* \\ &= f^{-1}\langle f\langle\kappa\rangle\rangle^* \end{aligned}$$

□

In the pointed context, in view of Lemma 2.9, formula (7) becomes

$$\iota \vee \kappa = \bar{\iota} \circ \kappa.$$

In the case of ideal determined varieties this formula gives the following well known description of the join of two ideals $I, K \subseteq X$ (where $\overline{I}$ is the congruence generated by the ideal $I$) — see [13]:

$$I \vee K = \{x \mid \exists_{k \in K} k \overline{I} x\}.$$

In the total context, $\mathbb{C}$ in the above theorem becomes a Barr exact Goursat category, the ideals $\iota$, $\kappa$ become simply kernel pairs, and we have $\bar{\iota} = \iota$, so the formula (7) gives back the known formula for the join of two kernel pairs in a Barr exact Goursat category. Equivalently, $\iota \vee \kappa = f^{-1}\langle f\langle\kappa\rangle\rangle$ where $f$ is the coequalizer of $\iota$.

**Theorem 3.11.** *The lattice of ideals of an object $X$ in a category with a good theory of ideals is a modular lattice, i.e. for any three ideals $\alpha$, $\beta$, $\gamma$, the equality*

$$((\alpha \wedge \beta) \vee \gamma) \wedge \beta = (\alpha \wedge \beta) \vee (\gamma \wedge \beta)$$

*holds true.*

*Proof.* The proof is based on deriving the above equality from the equality

$$((\overline{\alpha} \wedge \overline{\beta}) \circ_3 \gamma) \wedge \overline{\beta} = (\overline{\alpha} \wedge \overline{\beta}) \circ_3 (\gamma \wedge \overline{\beta}) \tag{8}$$

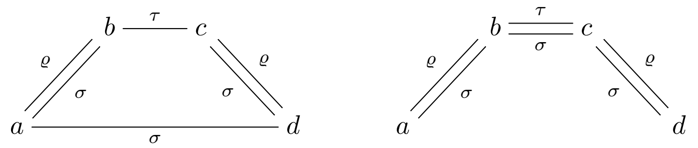
where for two relations $\varrho$ and $\sigma$ the "$n$-composite" is defined as

$$\varrho \circ_n \sigma = \underbrace{\varrho \circ \sigma \circ \varrho \circ \ldots}_{n-1 \text{ number of } \circ\text{'s}}.$$

Now, it is true in any regular category that the "$n$-modularity", i.e. the equality

$$((\varrho \wedge \sigma) \circ_n \tau) \wedge \sigma = (\varrho \wedge \sigma) \circ_n (\tau \wedge \sigma)$$

holds true when $n = 3$ for arbitrary relations $\varrho$, $\sigma$ and symmetric and transitive relation $\tau$. This can be proved by transferring via Yoneda the same result for sets, which can be readily exhibited by graphical representation of each side of the equality:




Note that for $n > 3$ the $n$-modularity would immediately fail (in some sense this explains why in universal algebra $n$-permutability of congruences implies congruence modularity only for $n \leqslant 3$).

Thus, (8) holds true and we now use it to prove modularity for ideals:

$$
\begin{aligned}
((\alpha \wedge \beta) \vee \gamma) \wedge \beta &= (\overline{\alpha \wedge \beta} \circ \gamma \circ (\alpha \wedge \beta)) \wedge \beta && \text{Theorem 3.10} \\
&= (\overline{\alpha \wedge \beta} \circ \gamma \circ (\overline{\alpha \wedge \beta})^*) \wedge (\overline{\beta})^* && \text{Theorem 2.8} \\
&= (\overline{\alpha \wedge \beta} \circ \gamma \circ \overline{\alpha \wedge \beta})^* \wedge (\overline{\beta})^* && \text{Corollary 2.11} \\
&= ((\overline{\alpha \wedge \beta} \circ \gamma \circ \overline{\alpha \wedge \beta}) \wedge \overline{\beta})^* && \text{Lemma 2.6} \\
&= (((\overline{\alpha} \wedge \overline{\beta}) \circ \gamma \circ (\overline{\alpha} \wedge \overline{\beta})) \wedge \overline{\beta})^* && \text{Theorem 2.8} \\
&= ((\overline{\alpha} \wedge \overline{\beta}) \circ (\gamma \wedge \overline{\beta}) \circ (\overline{\alpha} \wedge \overline{\beta}))^* && \text{by (8)} \\
&= ((\overline{\alpha \wedge \beta}) \circ (\gamma \wedge \overline{\beta}) \circ (\overline{\alpha \wedge \beta}))^* && \text{Theorem 2.8} \\
&= (\overline{\alpha \wedge \beta} \circ (\gamma \wedge \overline{\beta})^* \circ \overline{\alpha \wedge \beta})^* && \text{Lemma 2.5} \\
&= (\overline{\alpha \wedge \beta} \circ (\gamma^* \wedge (\overline{\beta})^*) \circ \overline{\alpha \wedge \beta})^* && \text{Lemma 2.6} \\
&= (\overline{\alpha \wedge \beta} \circ (\gamma \wedge \beta) \circ \overline{\alpha \wedge \beta})^* && \text{Theorem 2.8} \\
&= \overline{\alpha \wedge \beta} \circ (\gamma \wedge \beta) \circ (\overline{\alpha \wedge \beta})^* && \text{Corollary 2.11} \\
&= \overline{\alpha \wedge \beta} \circ (\gamma \wedge \beta) \circ (\alpha \wedge \beta) && \text{Theorem 2.8} \\
&= (\alpha \wedge \beta) \vee (\gamma \wedge \beta) && \text{Theorem 3.10}
\end{aligned}
$$

□

Since in a category with a good theory of ideals, the lattice of ideals is always isomorphic to the ordered set of kernel pairs, the above theorem includes, as two independent spacial cases, the fact that in a Barr exact Goursat category the ordered sets of kernel pairs are modular lattices (which was proved in [6]), and the fact that in an ideal determined category the ordered sets of kernel pairs are modular lattices. In particular, we obtain a unification of two independent known results from universal algebra: (a) any 3-permutable variety is congruence modular; (b) any (pointed) ideal determined variety is congruence modular.

*Remark* 3.12. It should be noted that there is no direct dependence between 3-permutable and ideal determined varieties — as shown in [22], in general ideal determined varieties are not 3-permutable, and conversely, there are 3-permutable varieties which are not ideal determined (e.g. the variety of pointed Mal'tsev algebras is 2-permutable and hence 3-permutable, but is not 0-regular and hence is not ideal determined either).

*Remark* 3.13. In the future, it would be interesting to revisit many of the topics from universal / categorical algebra in the context of multi-pointed categories, where certain parallel concepts / results from pointed and non-pointed settings could be unified. It can be readily seen that under such unification would fall

- a unification of Mal'tsev categories [7, 8] (which are a categorical counterpart of Mal'tsev varieties [24, 25]), also known as 2-permutable varieties in universal algebra) and subtractive categories [19] (which are a pointed categorical counterpart of subtractive varieties [28]);
- a unification of D. Bourn's non-pointed $3\times 3$ lemma [3] and the pointed one [2], which were shown to be equivalent to the Goursat property for regular categories in [11] (see also [21]), and subtractivity for normal categories in [20], respectively.

The present paper partly originated with a study of a connection between the notion of an ideal determined category and that of a subtractive normal category. This study has revealed a pointed counterpart of Barr exactness [1], and it was shown to be precisely the "missing link" between the two notions. These and other questions regarding subtractivity will be dealt with in a separate paper.

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