HIGHER CENTRAL EXTENSIONS AND COHOMOLOGY

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Abstract. We establish a Galois-theoretic interpretation of cohomology in semi-abelian categories: cohomology with trivial coefficients classifies central extensions, also in arbitrarily high degrees. Thus we obtain a duality between homology and cohomology. These results depend on a geometric viewpoint of the concept of higher central extension, as well as the algebraic one in terms of commutators.

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Introduction

This article exposes a hidden duality between homology and cohomology: we prove, in a very general context, that cohomology with trivial coefficients classifies (higher) central extensions. In line with the work in low dimensions and with several closely related results in homology theory, this reveals the first few layers of a deep connection between Galois theory and cohomology, and a close correspondence with homology which has been invisible so far.

The context in which we work is sufficiently general to cover cohomology of, say, groups, Lie algebras and non-unitary rings, as well as the Yoneda Ext groups in the abelian case, and new examples can easily be added to the list. In fact, almost any semi-abelian category would do, as long as it satisfies a certain commutator condition which occurs naturally in this setting—see below.

This interpretation of cohomology is part of a bigger programme which intends to understand homological algebra in a non-abelian environment from the viewpoint of (categorical) Galois theory. Related results include, for instance, higher Hopf

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formulae for homology in semi-abelian categories [34], higher-dimensional torsion theories [32], a theory of satellites for homology without projectives [40], and higher-dimensional commutator theory based on a notion of higher centrality [36, 38].

Higher centrality. The key novelty in the present approach to (co)homology of non-abelian algebraic objects is the concept of *higher centrality.* It allows us to express in an abstract, but simple, way the (sometimes rather cumbersome) commutativity relations which we have to deal with.

Following the ideas of Janelidze [49, 50], the formal theory of *higher extensions*—these are the objects which may or may not be central—was first developed in [34] in order to provide a general setting for the Brown–Ellis–Hopf formulae [21, 27]. Since *centrality* itself comes from categorical Galois theory [4, 47, 52], these higher extensions were introduced alongside a tower of compatible Galois structures.

Let us make this explicit with a concrete example. Consider the category Gp of all groups and its (reflective) subcategory Nil_2 determined by all groups of nilpotency class at most 2. The induced reflector $\mathsf{nil}_2\colon \mathsf{Gp} \to \mathsf{Nil}_2$, left adjoint to the inclusion functor, takes a group G and sends it to its 2-nilpotent quotient $G/[[G,G],G]$. This situation—Gp being a variety of algebras over Set, and Nil_2 a subvariety of it—admits a canonical homology theory: Barr–Beck comonadic homology [2] with coefficients in the reflector nil_2. Now for any group Z, the induced third homology group $\mathrm{H}_3(Z,\mathsf{nil}_2)$ of Z may be expressed by a Hopf formula, namely the quotient [34, Theorem 9.3]

$$\frac{K_0 \cap K_1 \cap [[X,X],X]}{[[K_0\cap K_1,X],X][[K_0,K_1],X][[K_0,X],K_1][[X,K_0],K_1][[X,X],K_0\cap K_1]}.$$

Here the objects $K_0 = \mathrm{K}[c]$ and $K_1 = \mathrm{K}[d]$ are the kernels of c and d, for any *double presentation*

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "d"'] & C \arrow[d] \\ D \arrow[r] & Z \end{tikzcd} \qquad (\mathbf{A})$$

of Z. This means that the objects C, D and X are projective (= free) groups, and furthermore this commutative square is a *double extension* of Z: all its arrows, as well as the induced arrow to the pullback $\langle d,c\rangle\colon X \to D\times_Z C$, are surjections. The denominator in the formula is a generalised commutator: a double extension of groups such as (**A**) is central (with respect to Nil_2) precisely when this denominator is zero.

It is not hard to construct a double presentation of an object, certainly not in the varietal case, since a truncation of any simplicial projective resolution will do. As is apparent from the formula, the main difficulty lies in characterising the (double) central extensions corresponding to the functor which is being derived (in this case, nil_2). Higher central extensions are defined by induction; let us explain how this is done for lowest degrees (more details can be found in the following sections and in the articles [30, 31, 34], amongst others).

A **semi-abelian** category [53, 3] is pointed, Barr exact and Bourn protomodular with binary sums. Let $\mathcal{A}$ be a semi-abelian category and $\mathcal{B}$ a **Birkhoff subcategory** of $\mathcal{A}$—full, reflective and closed under subobjects and regular quotients, so that a Birkhoff subcategory of a variety is nothing but a subvariety. Let

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "I"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, shift left, "\supset"] \end{tikzcd} \qquad (\mathbf{B})$$

denote the induced adjunction, with I the reflector and $\eta\colon 1_{\mathcal{A}} \Rightarrow I$ the unit.

An **extension** $f\colon A \to B$ in $\mathcal{A}$ is a regular epimorphism. Together with the classes of extensions in $\mathcal{A}$ and in $\mathcal{B}$, the adjunction (**B**) forms a *Galois structure* in the sense of Janelidze [4, 47]. Central extensions are defined with respect to such a Galois structure, as follows. An extension f is called **trivial** when the induced naturality square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\ IA \arrow[r, "If"'] & IB \end{tikzcd}$$

is a pullback; $f\colon A \to B$ is **central** when either of the kernel pair projections pr_0, $\mathrm{pr}_1\colon \mathrm{R}[f] \to A$ is trivial [52].

It turns out that the central extensions relative to $\mathcal{B}$ determine a reflective subcategory $\mathsf{CExt}_{\mathcal{B}}\mathcal{A}$ of the category $\mathsf{Ext}\mathcal{A}$ of extensions in $\mathcal{A}$, so we have an adjunction

$$\begin{tikzcd} \mathsf{Ext}\mathcal{A} \arrow[r, shift left, "I_1"] \arrow[r, phantom, "\perp"] & \mathsf{CExt}_{\mathcal{B}}\mathcal{A}. \arrow[l, shift left, "\supset"] \end{tikzcd}$$

Together with classes of double extensions, defined as in the case of groups above, this adjunction forms a Galois structure, and thus we acquire the notion of *relative double central extension* with respect to $\mathcal{B}$. This construction may be repeated ad infinitum, so that notions of *n-fold extension* (special n-dimensional cubes in $\mathcal{A}$) and *n-fold central extension* are obtained.

Of course, whether or not a (higher) extension is central with respect to some chosen Birkhoff subcategory depends on this subcategory more than anything else. In many cases (like, for instance, the case of groups vs. 2-nilpotent groups) there are explicit descriptions of the central extensions in some, or in all, degrees (see, for instance, [25, 32, 33, 37]). Knowing, in a given case, what the central extensions are, gives a complete description of the corresponding homology objects as higher Hopf formulae: this is the content of Theorem 8.1 in [34]. In this article we only consider higher extensions which are central with respect to the Birkhoff subcategory $\mathcal{B} = \mathsf{Ab}\mathcal{A}$ of all *abelian* objects in $\mathcal{A}$, the objects which admit an internal abelian group structure; that is to say, they are central with respect to abelianisation. In this situation, the Hopf formulae take the following shape [34]: for any n-presentation F of Z,

$$\mathrm{H}_{n+1}(Z, \mathsf{Ab}\mathcal{A}) \cong \frac{\bigcap_{i\in n} \mathrm{K}[f_i] \cap \langle F_n\rangle}{L_n[F]}. \tag{C}$$

Here F_n is the initial object of the n-fold extension F and the f_i are the initial arrows (see the solid part of Figure 1 for a picture in degree three). The brackets $\langle - \rangle$ in the formula give the *zero-dimensional commutator* of F_n determined by its abelianisation: for any object X of $\mathcal{A}$ there is a short exact sequence

$$0 \longrightarrow \langle X\rangle \rightarrowtail X \xrightarrow{\eta_X} \mathsf{ab}X \longrightarrow 0,$$

so $\langle X\rangle = [X,X]$, the Huq commutator [15, 45] of X with itself. The object in the denominator of the Hopf formula is the smallest normal subobject of F_n which, when divided out, makes F central; in other words, an n-fold extension F is central if and only if $L_n[F] = 0$. In many cases (see Section 5) this "abstract higher-dimensional commutator" may be computed as a join of binary Huq commutators.

On the other hand, the use of higher central extensions is not at all limited to homology and Hopf formulae. The concept of higher extension is quite interesting in its own right [31] while centrality may, for instance, be used to model more exotic commutator theories [36, 38]. The present article is meant to clarify the connection with cohomology and to extend the low-dimensional work which has been done in this context to higher degrees [7, 18, 43, 67].

Cohomology and centrality. Everything starts with the long-established interpretation of the second cohomology group $\mathrm{H}^2(Z,A)$ of a group Z with coefficients in an abelian group A in terms of central extensions of Z by A (see for instance [57]). Such a central extension f of Z by A corresponds to a short exact sequence of groups

$$0 \longrightarrow A \overset{\ker f}{\rightarrowtail} X \overset{f}{\twoheadrightarrow} Z \longrightarrow 0$$

such that the commutator $[A,X]$ is trivial, i.e., $axa^{-1}x^{-1} = 1$ for all $a \in A$ and $x \in X$. Two extensions $f\colon X \to Z$ and $f'\colon X' \to Z$ of Z by A are equivalent if and only if there exists a group (iso)morphism $x\colon X \to X'$ satisfying $f' \circ x = f$ and $x \circ \ker f = \ker f'$. The induced equivalence classes, together with the so-called Baer sum, form an abelian group $\mathrm{Centr}^1(Z,A)$, and this group is isomorphic to $\mathrm{H}^2(Z,A)$.

In this article we generalise this isomorphism in two ways: first of all, we replace the context of groups by the much larger setting of semi-abelian categories; secondly, we also consider higher cohomology groups.

It was proved in [43], see also [12] and [18], that this interpretation of cohomology via central extensions may be extended categorically from the context of groups to semi-abelian categories. Here the concept of centrality is the one coming from Galois theory, using the Birkhoff subcategory of all abelian objects [13], that is, we use centrality relative to abelianisation. Thus the well-known similar results for Lie algebras over a field, commutative algebras, non-unitary rings, (pre)crossed modules, etc. could be included in a general theory, and new examples could be studied.

The next step, an interpretation of the third cohomology group in similar terms, turned out to be quite hard to take. The reason is that one needs a theory of higher central extensions for this—which until recently was unavailable. (Of course there are many other interpretations of cohomology!) The problem was finally solved in [67], where the characterisation of double central extensions given in [49, 42] is extended to semi-abelian categories and an isomorphism

$$\mathrm{H}^3(Z,A) \cong \mathrm{Centr}^2(Z,A)$$

is constructed. (It must be mentioned that the cohomology theory used in [67]—the *directions approach*, using internal n-fold groupoids, introduced by Bourn in [7, 9, 12] and further worked out by Bourn and Rodelo in [20, 66]—is less classical than the one we shall be using here, or at least is not obviously related to it in higher degrees.) The abelian group $\mathrm{Centr}^2(Z,A)$ consists of equivalence classes of double central extensions of an object Z by an abelian object A, equipped with a canonical addition induced by the internal group structure of A.

A key ingredient here is the concept of *direction* of a higher (central) extension. The **direction** of a one-fold extension $f\colon X \to Z$ is its kernel $A = \mathrm{K}[f]$, while for a double extension F such as (**A**) it is the intersection of the kernels $\mathrm{K}[d] \cap \mathrm{K}[c]$, which is isomorphic to the kernel of the kernel of F, so we write it as $\mathrm{K}^2[F]$. In higher degrees a similar (inductive) definition makes sense: an n-fold extension F has direction $\mathrm{K}^n[F] = \bigcap_{i \in n} \mathrm{K}[f_i]$, an abelian object of $\mathcal{A}$ when F is central. (Compare with the Hopf formula (**C**).)

Figure 1 gives a picture in degree 3. The different but equivalent ways in which the direction may be computed come from the several ways in which a three-fold

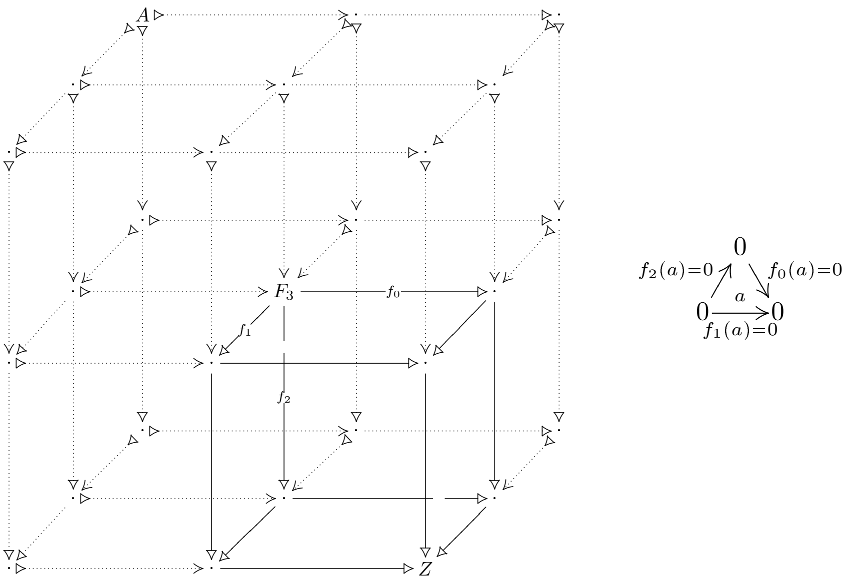


FIGURE 1. The direction A of a 3-fold (central) extension F of Z

extension may be considered as an arrow between double extensions, etc. An element of F_3 should be viewed as a triangle with faces given by f_0, f_1 and f_2, and such a triangle a is in the direction A if and only if all its faces are zero.

Thus for every $n \geqslant 1$ and every Z in $\mathcal{A}$ there is a functor

$$\mathsf{D}_{(n,Z)} \colon \mathsf{CExt}^n_Z\mathcal{A} \to \mathsf{Ab}\mathcal{A}$$

that sends an n-fold central extension F of Z to its direction A, which is an abelian object. Given any object Z in $\mathcal{A}$ and any abelian object A, an **n-fold central extension of Z by A** is an n-fold central extension F of Z with direction A, i.e., an object of the fibre $\mathsf{D}^{-1}_{(n,Z)}A$. Taking connected components gives us the set

$$\mathrm{Centr}^n(Z, A) = \pi_0(\mathsf{D}^{-1}_{(n,Z)}A)$$

of which we prove in Corollary 2.20 that it admits a canonical abelian group structure.

Now the question remains whether these groups have any cohomological meaning. The main body of this article explains that, indeed, they have: we shall prove that, under certain mild conditions, they agree with the interpretation of comonadic cohomology in terms of higher torsors.

Cohomology via higher torsors. One could say that Duskin and Glenn's *higher torsors* [28, 29, 39] are to central extensions what truncations of simplicial resolutions are to extensions, or what groupoids are to pregroupoids:

$$\frac{\text{torsor}}{\text{central extension}} = \frac{\text{truncation of simplicial resolution}}{\text{extension}} = \frac{\text{groupoid}}{\text{pregroupoid}}.$$

In a groupoid

$$G_1 \mathrel{\substack{\xrightarrow{\partial_1} \\ \xleftarrow{\sigma_0} \\ \xrightarrow[\partial_0]{}}} G_0$$

there are identities (given by σ_0) and a composition m

$$\begin{tikzcd}[sep=small] & \cdot \arrow[dr, "\beta"] & \\ \cdot \arrow[ur, "\alpha"] \arrow[rr, dotted, "\gamma"'] & & \cdot \end{tikzcd} \qquad\qquad m(\beta,\alpha)=\gamma$$

which is associative, admits inverses and is compatible with the identities; there is only one object of objects, G_0. On the other hand, a pregroupoid

$$\begin{tikzcd} & G_1 \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & \\ G_0 & & G_0' \end{tikzcd}$$

has two objects of objects, G_0 and G_0'. Consequently, it has no identities, and instead of a composition, there is a Mal'tsev operation p

$$\begin{tikzcd}[sep=small] & \cdot & \\ \cdot \arrow[ur, "\gamma"] \arrow[dr, dotted, "\delta"'] & & \cdot \arrow[ul, "\beta"'] \arrow[dl, "\alpha"] \\ & \cdot & \end{tikzcd} \qquad\qquad p(\alpha,\beta,\gamma)=\delta$$

satisfying $p(\alpha,\alpha,\gamma)=\gamma$ and $p(\alpha,\gamma,\gamma)=\alpha$. Groupoids (and torsors) live in the *simplicial* world, whereas pregroupoids belong to the *cubical* world of higher (central) extensions. (Here we do not mean that they actually form *cubical objects*; rather, higher central extensions are cubical in an obvious sense—after all, higher extensions are n-dimensional cubes—and also in a more subtle geometrical sense we shall come back to later.)

Given an object Z and an abelian object A in a semi-abelian category $\mathcal{A}$, we consider the augmented simplicial object $\mathbb{K}(Z,A,n)$ determined by

$$\begin{tikzcd}[column sep=large] {\scriptstyle n+1} & {\scriptstyle n} & {\scriptstyle n-1} & {\scriptstyle n-2} & \cdots & {\scriptstyle 0} & {\scriptstyle -1} \\ A^{n+1}\times Z \arrow[r, shift left=3, "\partial_{n+1}\times 1_Z"] \arrow[r, shift left=1, "\mathrm{pr}_n\times 1_Z"'] \arrow[r, phantom, shift right=1, "\vdots"] \arrow[r, shift right=3, "\mathrm{pr}_0\times 1_Z"'] & A\times Z \arrow[r, shift left=2, "\mathrm{pr}_Z"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\mathrm{pr}_Z"'] & Z \arrow[r, equal, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, equal, shift right=2] & Z & \cdots & Z \arrow[r, equal] & Z \end{tikzcd}$$

with $\partial_{n+1} = (-1)^n\sum_{i=0}^{n}(-1)^i \operatorname{pr}_i$. An **$n$-torsor of Z by A** is an augmented simplicial object $\mathbb{T}$ equipped with a simplicial morphism $\mathbb{t}\colon \mathbb{T}\to\mathbb{K}(Z,A,n)$ such that

(T1) $\mathbb{t}$ is a fibration which is exact from degree n on;
(T2) $\mathbb{T}\cong \mathsf{Cosk}_{n-1}\mathbb{T}$, the $(n-1)$-coskeleton of $\mathbb{T}$;
(T3) $\mathbb{T}$ is a resolution.

Axiom (T2) means that $\mathbb{T}$ does not contain information above level $n-1$, which together with (T3) amounts to the $(n-1)$-truncation T of $\mathbb{T}$, considered as an n-cube, being an n-fold extension. The fibration property in (T1) is (almost) automatic, while the exactness tells us that, for all i,

$$\triangle(\mathbb{T},n)\cong A\times \wedge^i(\mathbb{T},n). \tag{D}$$

Here $A=\bigcap_{i\in n}\mathrm{K}[\partial_i]$ is the direction of T, the object $\triangle(\mathbb{T},n)$ consists of all n-cycles in $\mathbb{T}$ and $\wedge^i(\mathbb{T},n)$ is the object of (n,i)-horns in $\mathbb{T}$. In degree two, for instance, we obtain the following picture:

$$\begin{array}{ccccc} \triangle(\mathbb{T},2) & \cong & A & \times & \wedge^1(\mathbb{T},2) \\ \begin{tikzcd}[sep=small] & \cdot \arrow[dr, "\beta"] & \\ \cdot \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & & \cdot \end{tikzcd} & & \begin{tikzcd} 0 \arrow[r, "a"'] & 0 \end{tikzcd} & & \begin{tikzcd}[sep=small] & \cdot \arrow[dr, "\beta"] & \\ \cdot \arrow[ur, "\alpha"] & & \cdot \end{tikzcd} \end{array} \tag{E}$$

Given a, α and β, there is a unique arrow $\gamma = \mu^1(a, \beta, \alpha)$ such that the projection $\mathrm{pr}_A(\beta, \gamma, \alpha)$ on A gives back a. In some sense $a = 0$ if and only if the triangle on the left "commutes", and taking $\gamma = \mu^1(0, \beta, \alpha) = m^1(\beta, \alpha)$ as a composite of β and α really does define a groupoid structure m^1 on T.

Let $\mathsf{S}^+\mathcal{A}$ denote the category of augmented simplicial objects in $\mathcal{A}$. The full subcategory of the slice $\mathsf{S}^+\mathcal{A}/\mathbb{K}(Z, A, n)$ determined by the n-torsors of Z by A is written $\mathrm{Tors}^n(Z, A)$. Taking connected components we obtain the set

$$\mathrm{Tors}^n[Z, A] = \pi_0 \, \mathrm{Tors}^n(Z, A)$$

of equivalence classes of n-torsors of Z by A. It is, in fact, an abelian group [29].

Duskin explains in [28, 29] that the group $\mathrm{Tors}^n[Z, A]$ may be considered as a cohomology group $\mathrm{H}^{n+1}(Z, A)$ of Z with coefficients in the trivial module A, and that under certain conditions this cohomology coincides with other known cohomology theories. For instance, when $\mathcal{A}$ is a semi-abelian category which is monadic over Set, we obtain Barr–Beck cohomology.

If $\mathbb{G}$ is the comonad induced by the forgetful/free adjunction of $\mathcal{A}$ to Set, if Z is an object of $\mathcal{A}$ and A an abelian object, then for any integer n,

$$\mathrm{H}^{n+1}(Z, A)_{\mathbb{G}} = \mathrm{H}^n \, \mathrm{hom}(\mathsf{ab}\mathbb{G}Z, A)$$

is the $(n{+}1)$**-st cohomology group of** Z **with coefficients in** A **(relative to the comonad** $\mathbb{G}$**)** [2]. This defines a functor $\mathrm{H}^{n+1}(-, A) \colon \mathcal{A} \to \mathsf{Ab}$, for any $n \geqslant 0$. The equivalence of cohomology theories mentioned above amounts to an isomorphism

$$\mathrm{H}^{n+1}(Z, A)_{\mathbb{G}} \cong \mathrm{H}^{n+1}(Z, A) = \mathrm{Tors}^n[Z, A].$$

The geometry of higher central extensions. In Section 3 we analyse higher central extensions from a geometrical point of view so that we can compare them with higher torsors. We work towards Proposition 4.12 which says that an augmented simplicial object carries a structure of n-torsor as soon as its underlying n-fold arrow is an n-fold central extension. This result is based on Theorem 3.8 which gives a new characterisation of higher central extensions: an n-fold extension F, of which the direction A is abelian, is central if and only if

$$\mathop{\Box}_{i \in n} \mathrm{R}[f_i] \cong A \times \mathop{\boxdot}_{i \in n}^{I} \mathrm{R}[f_i] \qquad (\mathbf{F})$$

for any (hence, all) $I \subseteq n$. (Compare with the isomorphism (**D**).) The precise definition of the objects $\Box_{i \in n} \mathrm{R}[f_i]$ and $\boxdot^I_{i \in n} \mathrm{R}[f_i]$ will be presented in Section 3, but we can already explain the meaning of this characterisation in some low-dimensional cases and give the main idea.

When $n = 1$ this isomorphism reduces to the well-known result (see [13, 14]) that an extension $f \colon X \to Z$ is central if and only if $\mathrm{R}[f] \cong A \times X$, where the kernel A of f (its direction) is abelian.

When F is the double extension (**A**) the isomorphism becomes

$$\mathrm{R}[d] \Box \mathrm{R}[c] \cong A \times (\mathrm{R}[d] \times_X \mathrm{R}[c]),$$

where now $A = \mathrm{K}[d] \cap \mathrm{K}[c]$ is the direction of F. The isomorphism can be obtained as a consequence of the analysis of double extensions carried out in [67]. It is indeed explained there that F is central if and only if the morphism

$$\langle \pi \rangle \colon \langle \mathrm{R}[d] \Box \mathrm{R}[c] \rangle \to \langle \mathrm{R}[d] \times_X \mathrm{R}[c] \rangle$$

is an isomorphism. (The kernel of $\langle \pi \rangle$ is the denominator $L_3[F]$ of the Hopf formula (**C**) for $\mathrm{H}_3(Z, \mathsf{Ab}\mathcal{A})$.) Recall [54, 3] that $\mathrm{R}[d] \Box \mathrm{R}[c]$ contains *diamonds* (as on the left)

$$\begin{tikzcd}[column sep=small, row sep=small]
& \cdot & \\
\cdot \arrow[ur, "\gamma"] \arrow[dr, "\delta"'] & & \cdot \arrow[ul, "\beta"'] \arrow[dl, "\alpha"] \\
& \cdot &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small, row sep=small]
& \cdot & \\
\cdot \arrow[ur, "\gamma"] & & \cdot \arrow[ul, "\beta"'] \arrow[dl, "\alpha"] \\
& \cdot &
\end{tikzcd}$$

in X, so that the object $\mathrm{R}[d] \times_X \mathrm{R}[c]$, which is an instance of the pullback (**H**) below, contains *diamonds with one face missing* (as on the right above) and π is the projection which forgets δ. The analogy with (**E**) is clear and not accidental: the missing δ corresponds to a unique element a of A; on the other hand, given any diamond (including δ), the corresponding element a of A measures how far the diamond is from being "commutative" (in which case one may think of δ as a composite $\alpha\beta^{-1}\gamma$). Note that instead of forgetting δ, we could have chosen to forget α, β or γ; each of those choices determines a different pullback $\mathrm{R}[d] \times_X \mathrm{R}[c]$ which, for the sake of clarity, could be written as $\mathrm{R}[d] \boxdot^I \mathrm{R}[c]$ where the index $I \subseteq 2$ determines the chosen projection (indeed there are four options).

In general, given an n-fold extension F, the object $\square_{i\in n} \mathrm{R}[f_i]$ contains n-dimensional diamonds in F_n and $\boxdot^I_{i\in n} \mathrm{R}[f_i]$ contains n-dimensional diamonds with one face (determined by the index $I \subseteq n$) missing. The extension F is central when its direction A is abelian and the canonical projection

$$\pi^I \colon \mathop{\square}_{i\in n} \mathrm{R}[f_i] \to \mathop{\boxdot}_{i\in n}^{I} \mathrm{R}[f_i]$$

induces the isomorphism (**F**); this means that a missing face in any n-fold diamond in F_n is completely determined by an element in A. We also obtain an explicit formula for the splitting $\mathrm{pr}_A \colon \square_{i\in n} \mathrm{R}[f_i] \to A$ of the kernel of π^I, the projection on A which gives us a "measure of commutativity" for n-fold diamonds: Proposition 3.12 states that

$$\mathrm{pr}_A = \sum_{J\subseteq n} (-1)^{|J|} \eta_{F_n} \circ \mathrm{pr}_J,$$

where $\mathrm{pr}_J \colon \square_{i\in n} \mathrm{R}[f_i] \to F_n$ sends a diamond to its J-face.

Using this geometrical interpretation of centrality we can compare torsors and central extensions. Any n-cycle may be "completed" into an n-fold diamond by adding well-chosen degeneracies, and thus restricting the isomorphism (**F**) to an isomorphism (**D**) we may prove that any augmented simplicial object of which the underlying n-fold arrow is a central extension is in fact an n-torsor. The converse, however, needs more, since in general it is not clear how an isomorphism on the simplicial level may be extended to an isomorphism on the level of higher-dimensional diamonds. For this implication we pass via an interpretation of centrality in terms of commutators.

The commutator condition. In order to complete the equivalence between torsors and higher central extensions, we shall assume that centrality may be characterised in terms of binary Huq commutators. We call this assumption, which we believe is a new fundamental property (semi-abelian) categories may or may not have, the **commutator condition (CC)** on higher central extensions: it holds when, for all $n \geqslant 1$, an n-fold extension F is central if and only if

$$\Big[\bigcap_{i\in I} \mathrm{K}[f_i], \bigcap_{i\in n\setminus I} \mathrm{K}[f_i]\Big] = 0$$

for all $I \subseteq n$. Alternatively, following [41], one could call an n-fold extension which satisfies this commutator condition **algebraically central** and name the concept of centrality coming from Galois theory **categorical centrality**; then (CC) says that *algebraically central and categorically central extensions are the same.*

The condition (CC) amounts to asking that the Hopf formula for higher homology (**C**) becomes a quotient of binary Huq commutators: its denominator $L_n[F]$ is then equal to the join $\bigcup_{I\subseteq n}\big[\bigcap_{i\in I}\mathrm{K}[f_i],\bigcap_{i\in n\setminus I}\mathrm{K}[f_i]\big]$, so that

$$\mathrm{H}_{n+1}(Z,\mathsf{Ab}\mathcal{A})\cong\frac{\bigcap_{i\in n}\mathrm{K}[f_i]\cap[F_n,F_n]}{\bigcup_{I\subseteq n}\big[\bigcap_{i\in I}\mathrm{K}[f_i],\bigcap_{i\in n\setminus I}\mathrm{K}[f_i]\big]}$$

for any n-fold presentation F of any object Z and for any $n\geqslant 1$. We shall, however, focus on the cohomological meaning of this condition rather than on Hopf formulae.

It is certain that many categories satisfy (CC), although thus far no explicit characterisation is known; in the article [34] the categories of groups, Lie algebras and non-unitary rings are given as examples, and it is not difficult to add new examples to the list by using the technique explained there. A wide range of (generally non-trivial) examples are those semi-abelian categories with a protoadditive abelianisation functor [33, 32], of which two extreme special cases are all semi-abelian arithmetical categories, such as the categories of von Neumann regular rings, Boolean rings and Heyting semilattices (where the cohomology theory becomes trivial) on the one hand, and all abelian categories (where the theory gives us the Yoneda Ext groups) on the other. Two good candidates for conditions which *may* imply (CC) are *action accessibility* [19] (which would make all *categories of interest* [62] examples [61]) and *strong protomodularity* [15], but this should of course be further explored. In any case, every semi-abelian category satisfies (CC) for $n=1$ (see [41, 43]), and when $n=2$ the *Smith is Huq* condition considered in [59] is sufficient, as a consequence of the results in [67].

Proposition 5.8 now tells us that in a semi-abelian category with (CC), the n-fold extension underlying an n-torsor is always central, so that for a truncated augmented simplicial object the two concepts are equivalent (Theorem 5.9).

The main theorem. Since any n-fold central extension of an object Z by an abelian object A is connected to an n-fold central extension of Z by A which is a truncation of an augmented simplicial object (Proposition 6.3), any n-fold central extension is connected to an n-torsor, and we acquire an isomorphism

$$\mathrm{Tors}^n[Z,A]=\pi_0\,\mathrm{Tors}^n(Z,A)\cong\pi_0(\mathsf{D}^{-1}_{(n,Z)}A)=\mathrm{Centr}^n(Z,A).$$

Thus we obtain the main result of this article, Theorem 6.5: if Z is an object and A an abelian object in a semi-abelian category with the commutator condition, then for every $n\geqslant 1$ we have that

$$\mathrm{H}^{n+1}(Z,A)\cong\mathrm{Centr}^n(Z,A).$$

This establishes the result claimed in [67] and has several other interesting implications. For instance, from [28] it follows that there is a long exact sequence for $\mathrm{Centr}^n(Z,-)$.

Duality between homology and cohomology. Combined with the main result of the article [40], this interpretation of cohomology gives an answer to the following somewhat naive question:

In which sense are homology and cohomology dual to each other?

It is true that there is a kind of duality, or at least a strong symmetry, in the definitions of homology and cohomology when one uses, for instance, the comonadic Barr–Beck approach. Nevertheless, so far there was no meaningful connection at all between the *interpretations* of homology (using Hopf formulae, say) and cohomology (many different approaches here), at least not for non-abelian algebraic objects. Following [69], we claim that the hidden connection is the concept of *direction for higher central extensions* and the analysis of both homology (relative to abelianisation) and cohomology (with trivial coefficients) in these terms.

The theory of *satellites* [40, 46] makes it possible to replace Hopf formulae for homology with (possibly large) limits, so that homology objects may also be computed in contexts where not enough projective objects are available. The results in [40] are again based on higher central extensions in semi-abelian categories, and the article's Corollary 4.10 tells us that, for any Birkhoff subcategory $\mathcal{B}$ of a semi-abelian category $\mathcal{A}$, for any object Z of $\mathcal{A}$ and any integer $n \geqslant 1$, the homology object $\mathrm{H}_{n+1}(Z, \mathcal{B})$ is the limit of the diagram

$$\mathrm{K}^n[-]\colon \mathsf{CExt}^n_Z\mathcal{A} \to \mathcal{B}.$$

That is to say, in the case of abelianisation, all the homological and cohomological information on an object Z at a given level n is contained in one and the same functor

$$\mathsf{D}_{(n,Z)}\colon \mathsf{CExt}^n_Z\mathcal{A} \to \mathsf{Ab}\mathcal{A}\colon F \mapsto \mathsf{D}_{(n,Z)}F = \bigcap_{i\in n} \mathrm{K}[f_i] = \mathrm{K}^n[F]$$

in two "opposite" ways,

$$\mathrm{H}_{n+1}(Z, \mathsf{Ab}\mathcal{A}) = \lim \mathsf{D}_{(n,Z)} \qquad \text{and} \qquad \mathrm{H}^{n+1}(Z, A) = \pi_0(\mathsf{D}^{-1}_{(n,Z)}A);$$

homology is a limit of $\mathsf{D}_{(n,Z)}$ while cohomology consists of connected components of a fibre of $\mathsf{D}_{(n,Z)}$. So on the one end we have the limit of all possible directions and, on the other, all classes of all central extensions with one given and fixed direction—again, see Figure 1. We consider this duality (Theorem 6.6) to be a major point of the present article.

Structure of the text. In Section 1 we recall some basic definitions and results which we need later on: semi-abelian categories, simplicial objects, higher extensions and higher torsors. Section 2 contains all the theory needed to introduce the groups $\mathrm{Centr}^n(Z, A)$. In Section 3 we give a geometric interpretation of the concept of higher central extension (Theorem 3.8 and Proposition 3.12), used in the next section where we analyse torsors in terms of this geometry. The most important result here is Proposition 4.12 which says that a truncation of an augmented simplicial object, considered as a higher extension, is a torsor as soon as it is a central extension. The other implication in the equivalence between torsors and central extensions is obtained in Section 5 (Proposition 5.8 and Theorem 5.9). However, to make it work, we have to strengthen the context of semi-abelian categories with the additional commutator condition (CC). The short last Section 6 explains how to suitably transform an n-fold central extension (which need not be a truncation of anything simplicial) into an n-fold central extension underlying a torsor, so that we may conclude with Theorem 6.5, the isomorphism $\mathrm{H}^{n+1}(Z, A) \cong \mathrm{Centr}^n(Z, A)$ for all $n \geqslant 1$, and Theorem 6.6, the duality between homology and cohomology.

1. Preliminaries

We sketch the context in which we shall be working: homological and semi-abelian categories for all general results, with the approach to cohomology in Barr exact categories due to Duskin [28, 29] and Glenn [39]. We also recall the definition of higher extensions and the relation with simplicial resolutions from [34, 31].

1.1. **Barr exact, homological and semi-abelian categories.** For the sake of clarity, the results in this article will be presented in the context of semi-abelian categories. Although this is an extremely convenient environment to work in, it is probably not the most general context in which the theory may be developed. Nevertheless, we believe that in this first approach it is better not to cloud our results in technical subtleties concerning the surrounding category, but rather to focus on their intrinsic meaning and their correctness. The only disadvantage this added transparency may possibly have is the potential loss of some more elaborate examples; such examples can always be recovered later on.

We recall the main definitions and properties of Barr exact [1], homological [3] and semi-abelian categories [53].

Recall that a regular epimorphism is the coequaliser of some pair of morphisms. A finitely complete category endowed with a pullback-stable (regular epi, mono)-factorisation system is called **regular**. Regular categories provide a natural context for working with relations. We denote the kernel relation (= kernel pair) of a morphism f, the pullback of f along itself, by $(\mathrm{R}[f], \mathrm{pr}_0, \mathrm{pr}_1)$ or $(\mathrm{R}[f], f_0, f_1)$, depending on the situation. A regular category is said to be **Barr exact** when every equivalence relation is the kernel pair of some morphism [1].

A **pointed** category (i.e., with a **zero object**, an initial object that is also terminal) that admits pullbacks is **protomodular** [5] when the Split Short Five Lemma holds. Moreover, if the pointed category is regular, then protomodularity is equivalent to the (Regular) Short Five Lemma: given a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{K}[f'] \arrow[r, tail, "\ker f'"] \arrow[d, "k"'] & X' \arrow[r, two heads, "f'"] \arrow[d, "x"'] & Y' \arrow[r] \arrow[d, "y"] & 0 \\ 0 \arrow[r] & \mathrm{K}[f] \arrow[r, tail, "\ker f"'] & X \arrow[r, two heads, "f"'] & Y \arrow[r] & 0 \end{tikzcd} \tag{$\mathbf{G}$}$$

with regular epimorphisms f, f' and their kernels, if k and y are isomorphisms then also x is an isomorphism. We usually denote the kernel of a morphism f by $(\mathrm{K}[f], \ker f)$. A pointed, regular and protomodular category is called **homological** [3]. This is a context where many of the basic diagram lemmas of homological algebra hold. In particular, here the notion of *(short) exact sequence* has its full meaning: a regular (= normal) epimorphism with its kernel.

In order for commutator theory to work flawlessly, the context should be finitely cocomplete and Mal'tsev. A **Mal'tsev** category [23] is finitely complete and such that every reflexive relation is necessarily an equivalence relation. It is well known that any finitely complete protomodular category is necessarily a Mal'tsev category [6].

Joining all these conditions brings us to the notion of a **semi-abelian** category which can be defined as a pointed, Barr exact and protomodular category that admits binary coproducts. This definition unifies many older approaches towards a suitable categorical context for the study of homological properties of non-abelian categories such as the categories of groups, Lie algebras, etc. In the founding article [53] which introduces the concept, it is explained how this solves the problem of finding the right axioms to be added to Barr exactness in order that the resulting context is equivalent with the contexts obtained in terms of "old-style axioms" such as, for instance, the one introduced in [45].

Examples of semi-abelian categories include all varieties of Ω-groups [44], such as groups and non-unitary rings, precrossed and crossed modules, and categories of non-unitary algebras such as associative algebras and Leibniz and Lie n-algebras;

then there are non-unitary C^*-algebras and loops; also any abelian category is an example, as is the dual of the category of pointed objects in any elementary topos.

Lemma 1.2. [17, 8] *In a semi-abelian category, given a commutative diagram with short exact rows such as* (**G**) *above, k is an isomorphism if and only if the right-hand square is a pullback.* □

Lemma 1.3. [16, Theorem 4.9] *In a semi-abelian category, given a short exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{K}[f] \arrow[r, shift right, "\ker f"'] & X \arrow[l, shift right, "p"'] \arrow[r, "f"] & Y \arrow[r] & 0 \end{tikzcd}$$

in which the kernel of f is split by a morphism p, the object X is a product of which the projections are p and f.

Proof. Applying Lemma 1.2 to the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{K}[f] \arrow[r, "\ker f"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{K}[f] \arrow[r, equal] & \mathrm{K}[f] \arrow[r] & 0 \end{tikzcd}$$

shows that its right hand square is a pullback. □

1.4. **The Huq commutator and the Smith/Pedicchio commutator.** We work in a semi-abelian category $\mathcal{A}$. A coterminal pair

$$K \xrightarrow{k} X \xleftarrow{l} L$$

of morphisms in $\mathcal{A}$ **(Huq-)commutes** [15, 45] when there is a (necessarily unique) morphism φ such that the diagram

$$\begin{tikzcd} & K \arrow[dl, "\langle 1_K,0\rangle"'] \arrow[dr, "k"] & \\ K \times L \arrow[rr, dotted, "\varphi"] & & X \\ & L \arrow[ul, "\langle 0,1_L\rangle"] \arrow[ur, "l"'] & \end{tikzcd}$$

is commutative. We shall only consider the case where k and l are normal monomorphisms (i.e., kernels). The **Huq commutator** $[k,l]\colon [K,L] \to X$ **of** k **and** l is the smallest normal subobject of X which should be divided out to make k and l commute, so that they do commute if and only if $[K,L] = 0$. It may be obtained through the colimit Q of the outer square above, as the kernel of the (normal epi)morphism $X \to Q$. The commutator $[K,L]$ becomes the ordinary commutator of normal subgroups K and L in the case of groups, the ideal generated by $KL+LK$ in the case of non-unitary rings, the Lie bracket in the case of Lie algebras, and so on.

Consider a pair of equivalence relations (R,S) on a common object X

$$\begin{tikzcd} R \arrow[r, shift left=2, "r_0"] \arrow[r, shift right=2, "r_1"'] & X \arrow[l, "\langle 1_X,1_X\rangle" description] \arrow[r, "\langle 1_X,1_X\rangle" description] & S, \arrow[l, shift right=2, "s_1"'] \arrow[l, shift left=2, "s_0"] \end{tikzcd}$$

and consider the induced pullback of r_1 and s_0:

$$\begin{tikzcd} R \times_X S \arrow[r, "\pi_S"] \arrow[d, "\pi_R"'] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d, "s_0"] \\ R \arrow[r, "r_1"'] & X \end{tikzcd} \qquad (\mathbf{H})$$

The pair (R,S) **(Smith/Pedicchio-)commutes** [68, 63, 15] when there is a (necessarily unique) morphism θ such that the diagram

$$\begin{tikzcd} & R \arrow[dl, "\langle 1_R, \langle 1_X, 1_X\rangle r_1\rangle"'] \arrow[dr, "r_0"] & \\ R\times_X S \arrow[rr, dotted, "\theta"] & & X \\ & S \arrow[ul, "\langle\langle 1_X,1_X\rangle s_0, 1_S\rangle"] \arrow[ur, "s_1"'] & \end{tikzcd}$$

is commutative. As for the Huq commutator, the **Smith/Pedicchio commutator** is the smallest equivalence relation $[R,S]$ on X which, divided out of X, makes R and S commute. It can be obtained through a colimit, similarly to the situation above. Thus R and S commute if and only if $[R,S]=\Delta_X$, where Δ_X is the smallest equivalence relation on X. We say that R is a **central** equivalence relation when it commutes with ∇_X, the largest equivalence relation on X, so that $[R,\nabla_X]=\Delta_X$.

1.5. **Abelian objects, Beck modules.** In a semi-abelian category $\mathcal{A}$, an object A is said to be **abelian** when $[A,A]=0$. The abelian objects of $\mathcal{A}$ determine a full and reflective subcategory which is denoted $\mathsf{Ab}\mathcal{A}$. Given any object X of $\mathcal{A}$, we shall write $\langle X\rangle=[X,X]$, so that we obtain a short exact sequence

$$0 \longrightarrow \langle X\rangle \rightarrowtriangle X \xrightarrow{\eta_X} \mathsf{ab}X = X/[X,X] \longrightarrow 0$$

where η_X is the X-component of the unit η of the adjunction

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "\mathsf{ab}"] \arrow[r, phantom, "\perp"] & \mathsf{Ab}\mathcal{A}. \arrow[l, shift left, "\supset"] \end{tikzcd}$$

An object in a semi-abelian category is abelian precisely when it admits a (necessarily unique) internal abelian group structure. In fact, $\mathsf{Ab}\mathcal{A}$ may be viewed as the abelian category of internal abelian groups in $\mathcal{A}$. For instance, an abelian object in the category of groups is an abelian group, and an abelian associative algebra over a field is a vector space (equipped with a trivial multiplication).

Given an object Z of $\mathcal{A}$, a **Z-module** or **Beck module over** Z is an abelian group in the slice category $\mathcal{A}/Z$. Thus a Z-module (f,m,s) consists of a morphism $f\colon X\to Z$ in $\mathcal{A}$, equipped with a multiplication m and a unit s as in the diagrams

$$\begin{tikzcd} Z \arrow[r, "s"] \arrow[dr, equal] & X \arrow[d, "f"] \\ & Z \end{tikzcd} \qquad \begin{tikzcd} \mathrm{R}[f] \arrow[r, "m"] \arrow[dr] & X \arrow[d, "f"] \\ & Z \end{tikzcd}$$

satisfying the usual axioms. In particular we obtain a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, tail, "\ker f"] & X \arrow[r, shift right, "f"'] & Z \arrow[l, shift right, "s"'] \arrow[r] & 0 \end{tikzcd}$$

where A is an abelian object in $\mathcal{A}$ and f is split by s. Furthermore, the morphism f satisfies $[\mathrm{R}[f],\mathrm{R}[f]]=\Delta_X$. Conversely, given the splitting s of f, this latter condition makes it possible to recover the multiplication m. Hence, for split epimorphisms in a semi-abelian category, "being a Beck module" is a property; the entire module structure is contained in the splitting. Using the equivalence between split epimorphisms and internal actions [17], we can replace X with a semi-direct product $(A,\xi)\rtimes Z$. By the above, modules are "abelian actions". For simplicity, we denote a Z-module by its induced Z-algebra (A,ξ).

For us, the most important case arises when the Z-module structure on A is the trivial one, denoted (A,τ): then A is just an abelian object, the semidirect product $(A,\tau)\rtimes Z$ is $A\times Z$ and f is the product projection $\mathrm{pr}_Z\colon A\times Z\to Z$.

1.6. **Connected components.** In a category $\mathcal{A}$, two objects are **connected** when there exists a (finite) zigzag of morphisms between them. This defines an equivalence relation between the objects of $\mathcal{A}$, of which the equivalence classes form the set $\pi_0\mathcal{A}$ of **connected components** of $\mathcal{A}$.

In general $\pi_0\mathcal{A}$ may not be a small set, and even in the two situations where we shall use this construction (Subsection 1.22 and Definition 2.14) it will a priori not be clear whether or not the result is not a proper class. In fact, even when it *is* a proper class, this has no significant effect at all on the theory we develop, so we decided not to go into this question any further. Additionaly, in the monadic case the smallness of the cohomology groups follows from the interpretation in terms of Barr–Beck cohomology.

1.7. **A lemma on double split epimorphisms.** By a result in [6], a category is **naturally Mal'tsev** [55] when, given a split epimorphism of split epimorphisms as in

$$\begin{tikzcd} A_1 \arrow[r, shift right, "f_1"'] \arrow[d, shift right, "a"'] & B_1 \arrow[l, shift right, "\overline{f_1}"'] \arrow[d, shift right, "b"'] \\ A_0 \arrow[r, shift right, "f_0"'] \arrow[u, shift right, "\overline{a}"'] & B_0 \arrow[l, shift right, "\overline{f_0}"'] \arrow[u, shift right, "\overline{b}"'] \end{tikzcd} \qquad (\mathbf{I})$$

(all squares commute), if the square is a (down-right) pullback of split epimorphisms, then it is an (up-left) pushout of split monomorphisms. As a consequence we obtain the following lemma (see also [60] and [31]).

Lemma 1.8. *In a naturally Mal'tsev category, given a double split epimorphism* (**I**)*, the universally induced comparison morphism*

$$\langle a,f_1\rangle\colon A_1\to A_0\times_{B_0}B_1$$

to the pullback of f_0 and b is a split epimorphism. □

It is well known that every additive category is naturally Mal'tsev. In particular, for any semi-abelian category $\mathcal{A}$, the above lemma is valid in the abelian category $\mathsf{Ab}\mathcal{A}$.

1.9. **The von Neumann construction of the finite ordinals.** We shall write $0=\varnothing$ and $n=\{0,\dots,n-1\}$ for $n\geqslant 1$. We also write 2^n for the power-set of n, considered as a category of which an object is a subset of n, and an arrow $I\to J$ is an inclusion $I\subseteq J$.

1.10. **Higher arrows.** Let $\mathcal{A}$ be any category. The category $\mathsf{Arr}^n\mathcal{A}$ consists of **n-fold arrows** in $\mathcal{A}$: $\mathsf{Arr}^0\mathcal{A}=\mathcal{A}$, while $\mathsf{Arr}^1\mathcal{A}=\mathsf{Arr}\mathcal{A}$ is the category of arrows in $\mathcal{A}$ and $\mathsf{Arr}^{n+1}\mathcal{A}=\mathsf{Arr}\mathsf{Arr}^n\mathcal{A}$.

The category of arrows in $\mathcal{A}$ is the functor category $\mathsf{Fun}(2^{\mathrm{op}},\mathcal{A})=\mathcal{A}^{2^{\mathrm{op}}}$. Similarly, any n-fold arrow F in $\mathcal{A}$ may be viewed as an "n-fold cube with chosen directions", a functor $F\colon(2^n)^{\mathrm{op}}\to\mathcal{A}$, and any morphism of n-fold arrows as a natural transformation between such functors. If F is an n-fold arrow and I and J are subsets of n such that $I\subseteq J$, we shall write $F_I=F(I)$ for the value of F in I and $f_I^J\colon F_J\to F_I$ for the value of F in the morphism induced by the inclusion $I\subseteq J$. When $I=J\backslash\{i\}$ we write $f_i\colon F_J\to F_I$ for f_I^J.

An n-fold arrow given as a functor $F\colon (2^n)^{\mathrm{op}} \to \mathcal{A}$ can be seen as an arrow between $(n-1)$-fold arrows $F\colon \operatorname{dom} F \to \operatorname{cod} F$, where its domain $\operatorname{dom} F$ is determined by the restriction of F to all $I \subseteq n$ which contain $n-1$, and its codomain $\operatorname{cod} F$ by the restriction of F to all $I \subseteq n$ which do not contain $n-1$.

Given an n-fold arrow $F\colon (2^n)^{\mathrm{op}} \to \mathcal{A}$, we can always consider the restriction of this diagram to the subcategory $2^n \backslash \{n\}$; it is the n-fold cube F without its "initial object" F_n. When it exists, write $(\mathsf{L}F, (\mathrm{pr}_i)_{i\in n})$ for the limit of this diagram, and

$$l_F = \langle f_0, \ldots, f_{n-1} \rangle \colon F_n \to \mathsf{L}F$$

for the universally induced comparison morphism.

1.11. **Higher extensions.** Suppose that $\mathcal{A}$ is a semi-abelian category. A **zero-fold extension** in $\mathcal{A}$ is an object of $\mathcal{A}$ and a **one-fold extension** is a regular epimorphism in $\mathcal{A}$. For $n \geqslant 2$, an **n-fold extension** is an object (c, f) of $\mathsf{Arr}^n\mathcal{A}$ (or, equivalently, a morphism of $\mathsf{Arr}^{n-1}\mathcal{A}$)

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "d"'] & C \arrow[d, "g"] \\ D \arrow[r, "f"'] & Z \end{tikzcd} \tag{$\mathbf{J}$}$$

such that the morphisms c, d, f, g and the universally induced comparison morphism $\langle d, c\rangle\colon X \to D \times_Z C$ to the pullback of f with g are $(n-1)$-fold extensions. The n-fold extensions determine a full subcategory $\mathsf{Ext}^n\mathcal{A}$ of $\mathsf{Arr}^n\mathcal{A}$. A two-fold extension is also called a **double extension**, and $\mathsf{Ext}\mathcal{A} = \mathsf{Ext}^1\mathcal{A}$.

Proposition 1.12. [31] *Given any n-fold arrow F in a semi-abelian category, the following are equivalent:*

(i) *F is an n-fold extension;*
(ii) *for all $\varnothing \neq I \subseteq n$, the morphism $F_I \to \lim_{J \subsetneq I} F_J$ is a regular epimorphism.*

In particular, the induced comparison $l_F = \langle f_0, \ldots, f_{n-1}\rangle\colon F_n \to \mathsf{L}F$ is regular epimorphic. □

A double split epimorphism such as ($\mathbf{I}$) above is always a double extension. That is to say, the induced comparison morphism $\langle a, f_1\rangle$ may not be a split epimorphism as in Lemma 1.8, but it will certainly be a regular epimorphism. More generally, any split epimorphism between extensions is a double extension, as follows from [22, Theorem 5.7].

1.13. **Augmented simplicial objects.** Recall that the **augmented simplicial category Δ^+** has finite ordinals $n \geqslant 0$ for objects and order preserving functions for morphisms. The category $\mathsf{S}^+\mathcal{A}$ of **augmented simplicial objects** and augmented simplicial morphisms in a category $\mathcal{A}$ is the functor category $\mathsf{Fun}((\Delta^+)^{\mathrm{op}}, \mathcal{A})$. An augmented simplicial object $\mathbb{X}\colon (\Delta^+)^{\mathrm{op}} \to \mathcal{A}$ is usually considered as a sequence of objects $(\mathbb{X}_n)_{n\geqslant -1}$, with **face operators** $\partial_i\colon \mathbb{X}_n \to \mathbb{X}_{n-1}$ and **degeneracy operators** $\sigma_i\colon \mathbb{X}_n \to \mathbb{X}_{n+1}$ for $n \geqslant i \geqslant 0$, subject to the simplicial identities

$$\begin{aligned} \partial_i \circ \partial_j &= \partial_{j-1} \circ \partial_i \quad \text{if } i < j \\ \sigma_i \circ \sigma_j &= \sigma_{j+1} \circ \sigma_i \quad \text{if } i \leqslant j \end{aligned} \qquad \text{and} \qquad \partial_i \circ \sigma_j = \begin{cases} \sigma_{j-1} \circ \partial_i & \text{if } i < j \\ 1 & \text{if } i = j \text{ or } i = j+1 \\ \sigma_j \circ \partial_{i-1} & \text{if } i > j+1. \end{cases}$$

An augmented simplicial object $\mathbb{X}$ is **contractible** when there is a sequence of morphisms $(\sigma_n\colon \mathbb{X}_{n-1} \to \mathbb{X}_n)_{n\geqslant 0}$ such that

$$\partial_n \circ \sigma_n = 1_{\mathbb{X}_{n-1}} \qquad \text{and} \qquad \partial_i \circ \sigma_n = \sigma_{n-1} \circ \partial_i$$

for all $i \in n$.

Remark 1.14. All simplicial objects we shall be considering in this text will come equipped with some augmentation, even when we occasionally drop the word "augmented".

Remark 1.15. Beware of the shift in numbering, it will appear again when we consider a truncated simplicial object as a higher-dimensional arrow with extra structure. We denote the objects $\mathbb{X}(n)$ by $\mathbb{X}_{n-1}$, the image of the inclusion $n \to n+1$ which leaves out i by ∂_i, and the image of the function $n+1 \to n$ which sends both i and $i+1$ in $n+1$ to i in n by σ_i. When we need to make this explicit, we say that $n-1$ is the **simplicial degree** and n is the **absolute degree** when referring to $\mathbb{X}(n) = \mathbb{X}_{n-1}$.

1.16. **Truncations and coskeleta.** For $n \geqslant 0$, let Δ_n^+ denote the full subcategory of Δ^+ determined by the ordinals $i \leqslant n$. The functor category

$$\mathsf{S}^n\mathcal{A} = \mathsf{Fun}((\Delta_n^+)^{\mathrm{op}}, \mathcal{A})$$

is the category of $(n-1)$**-truncated simplicial objects** in $\mathcal{A}$. Indeed, as soon as $\mathcal{A}$ is finitely complete, there is the adjunction

$$\begin{tikzcd} \mathsf{S}^+\mathcal{A} \arrow[r, shift left=1.5, "\mathsf{tr}_{n-1}"] \arrow[r, phantom, "\perp"] & \mathsf{S}^n\mathcal{A}, \arrow[l, shift left=1.5, "\mathsf{cosk}_{n-1}"] \end{tikzcd}$$

where the truncation functor tr_{n-1} is given by composition of a simplicial object with the inclusion $\Delta_n^+ \subseteq \Delta^+$, and its right adjoint cosk_{n-1} by right Kan extension along this functor. More explicitly, a coskeleton of an $(n-1)$-truncated simplicial object may be computed using iterated simplicial kernels (see the next subsection).

Clearly, $\mathsf{tr}_{n-1}\mathsf{cosk}_{n-1} = 1_{\mathsf{S}^n\mathcal{A}}$. Conversely, a coskeleton of an $(n-1)$-truncated simplicial object contains no information above simplicial degree $n-1$; given any simplicial object $\mathbb{X}$, we can remove all higher-dimensional information by applying the functor $\mathsf{Cosk}_{n-1} = \mathsf{cosk}_{n-1}\mathsf{tr}_{n-1} \colon \mathsf{S}^+\mathcal{A} \to \mathsf{S}^+\mathcal{A}$ to it.

Any $(n-1)$-truncated simplicial object may be considered as an n-fold arrow, through composition with the functor

$$\mathsf{a}_n \colon 2^n \to \Delta_n^+$$

which maps a set $I \subseteq n$ to the associated ordinal $|I|$, and an inclusion $I \subseteq J$ to the corresponding order-preserving map $|I| \to |J|$. This defines a functor

$$\mathsf{arr}_n = \mathsf{Fun}(-, \mathsf{a}_n) \colon \mathsf{S}^n\mathcal{A} \to \mathsf{Arr}^n\mathcal{A}$$

which allows us to consider $\mathsf{S}^n\mathcal{A}$ as a (non-full) subcategory of $\mathsf{Arr}^n\mathcal{A}$. (An $(n-1)$-truncated simplicial object has the additional structure of the degeneracies: a morphism of n-fold arrows between two given $(n-1)$-truncated simplicial objects need not commute with the degeneracy operators, and furthermore its components at two given sets of the same size need not coincide.) Hence, if X denotes the n-fold arrow underlying the $(n-1)$-truncation of a simplicial object $\mathbb{X}$, then $X_I = \mathbb{X}(|I|) = \mathbb{X}_{|I|-1}$ and, in particular, $X_n = \mathbb{X}_{n-1}$, in accordance with Remark 1.15. Note how the difference in font style allows to distinguish between the absolute degree n and the simplicial degree $n-1$. Also note that truncated simplicial objects, considered as higher arrows, are special in that they are *symmetric*: the several ways in which the higher cube may be considered as an arrow of arrows coincide.

1.17. **Simplicial kernels.** Let

$$(f_i \colon X \to Y)_{i \in n}$$

be a sequence of n morphisms in a finitely complete category $\mathcal{A}$. A **simplicial kernel** of $(f_0, \ldots, f_{n-1})$ is a sequence

$$(k_i \colon K \to X)_{i \in n+1}$$

of $n+1$ morphisms in $\mathcal{A}$ satisfying $f_i k_j = f_{j-1} k_i$ for $0 \leqslant i < j \leqslant n$, which is universal with respect to this property. It may be computed as a limit in $\mathcal{A}$.

When $\mathbb{X}$ is a simplicial object and $n \geqslant 0$, we write

$$(\partial_i \colon \triangle(\mathbb{X}, n) \to \mathbb{X}_{n-1})_{i \in n+1}$$

for the simplicial kernel of the faces $(\partial_i \colon \mathbb{X}_{n-1} \to \mathbb{X}_{n-2})_{i \in n}$. The object $\triangle(\mathbb{X}, n)$ consists of **n-cycles** in $\mathbb{X}$. For instance, the object $\triangle(\mathbb{X}, 2)$ of 2-cycles in $\mathbb{X}$ contains empty triangles:

$$\begin{tikzcd} & \cdot \arrow[dr, "\beta"] & \\ \cdot \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & & \cdot \end{tikzcd}$$

Note that $\triangle(\mathbb{X}, n) = \mathsf{L}(\mathsf{tr}_n \mathbb{X})$. Clearly $\triangle(\mathbb{X}, 1) = \mathrm{R}[\partial_0]$; we also write $\triangle(\mathbb{X}, 0)$ for $\mathbb{X}_{-1}$.

As mentioned in Subsection 1.16, any $(n-1)$-truncated simplicial object X in $\mathcal{A}$ may be universally extended to an n-truncated simplicial object. Its initial object and morphisms, in (absolute!) degree $n+1$, are given by the simplicial kernel $(k_i \colon K \to X_n)_{i \in n+1}$ of the initial morphisms $(x_i \colon X_n \to X_{n-1})_{i \in n}$ of X. The degeneracies $(\sigma_j \colon X_n \to K)_{j \in n}$ are induced by the simplicial identities

$$k_i \circ \sigma_j = \begin{cases} \sigma_{j-1} \circ x_i & \text{if } i < j \\ 1_{X_n} & \text{if } i = j \text{ or } i = j+1 \\ \sigma_j \circ x_{i-1} & \text{if } i > j+1 \end{cases}$$

of X and the universal property of the simplicial kernel. Repeating this construction indefinitely gives the $(n-1)$-coskeleton of X.

1.18. **Resolutions.** An augmented simplicial object $\mathbb{X}$ in a regular category is called **acyclic** or **a resolution (of $\mathbb{X}_{-1}$)** when for every $n \geqslant 0$, the comparison morphism

$$\langle \partial_i \rangle_i \colon \mathbb{X}_n \to \triangle(\mathbb{X}, n)$$

is a regular epimorphism. (Every n-cycle is a boundary of an n-simplex.) As explained in [31], in a semi-abelian category this is the case precisely when all the truncations of $\mathbb{X}$, considered as higher arrows, are extensions. For this reason we may sometimes also call a truncated simplicial resolution an extension.

1.19. **The simplicial objects $\mathbb{K}(A, n)$ and $\mathbb{K}(Z, A, n)$.** Let A be an abelian group in a Barr exact category $\mathcal{A}$ and take $n \geqslant 1$. The augmented simplicial object $\mathbb{K}(A, n)$ is the coskeleton of the $(n+1)$-truncated simplicial object

$$\begin{tikzcd}[row sep=small] n+1 & n & n-1 & n-2 & \cdots & 0 & -1 \\ A^{n+1} \arrow[r, shift left=3, "\partial_{n+1}"] \arrow[r, shift left=1, "\mathrm{pr}_n" description] \arrow[r, phantom, shift right=1, "\vdots"] \arrow[r, shift right=3, "\mathrm{pr}_0"'] & A \arrow[r, shift left=2, "!"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "!"'] & 1 \arrow[r, equal, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, equal, shift right=2] & 1 & \cdots & 1 \arrow[r, equal] & 1 \end{tikzcd}$$

with the A in simplicial degree n (in absolute degree $n+1$), where the degeneracies $1 \to A$ are determined by the neutral element 0 of A and ∂_{n+1} is equal to

$$(-1)^n \sum_{i=0}^{n} (-1)^i \operatorname{pr}_i .$$

When the category is a slice $\mathcal{A}/Z$ over an object Z in a semi-abelian category $\mathcal{A}$ and (A,ξ) is a Z-module, the simplicial object $\mathbb{K}((A,\xi),n)$, considered as a diagram in $\mathcal{A}$, takes the following shape:

$$\begin{tikzcd}[row sep=small]
{\scriptstyle n+1} & {\scriptstyle n} & {\scriptstyle n-1} & {\scriptstyle n-2} & \cdots & {\scriptstyle 0} & {\scriptstyle -1} \\
(A,\xi)^{n+1} \rtimes Z \arrow[r, shift left=3, "\partial_{n+1}\rtimes 1_Z"] \arrow[r, shift left=1, "\operatorname{pr}_n \rtimes 1_Z"'] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "\operatorname{pr}_0 \rtimes 1_Z"'] & (A,\xi) \rtimes Z \arrow[r, shift left=2, "f"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "f"'] & Z \arrow[r, equal, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, equal, shift right=2] & Z & \cdots & Z \arrow[r, equal] & Z
\end{tikzcd}$$

In case ξ is the trivial module structure τ, we obtain

$$\begin{tikzcd}[row sep=small]
{\scriptstyle n+1} & {\scriptstyle n} & {\scriptstyle n-1} & {\scriptstyle n-2} & \cdots & {\scriptstyle 0} & {\scriptstyle -1} \\
A^{n+1} \times Z \arrow[r, shift left=3, "\partial_{n+1}\times 1_Z"] \arrow[r, shift left=1, "\operatorname{pr}_n \times 1_Z"'] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "\operatorname{pr}_0 \times 1_Z"'] & A \times Z \arrow[r, shift left=2, "\operatorname{pr}_Z"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\operatorname{pr}_Z"'] & Z \arrow[r, equal, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, equal, shift right=2] & Z & \cdots & Z \arrow[r, equal] & Z
\end{tikzcd}$$

with ∂_{n+1} as above and degeneracies $\langle 0, 1_Z\rangle \colon Z \to A \times Z$. Given any object Z and any abelian object A, we shall write $\mathbb{K}(Z,A,n)$ for this simplicial object in $\mathcal{A}$. In particular, $\mathbb{K}(0,A,n) = \mathbb{K}(A,n)$.

1.20. **(Exact) fibrations.** Let $\mathbb{X}$ be a simplicial object in a finitely complete category $\mathcal{A}$ and consider $n \geqslant 2$ and $0 \leqslant i \leqslant n$. The **object of (n,i)-horns in $\mathbb{X}$** is an object $\wedge^i(\mathbb{X},n)$ together with morphisms $x_j \colon \wedge^i(\mathbb{X},n) \to \mathbb{X}_{n-1}$ for $i \neq j \in n+1$ satisfying

$$\partial_j \circ x_k = \partial_{k-1} \circ x_j \text{ for all } j < k \text{ with } j,\, k \neq i$$

which is universal with respect to this property; also $\wedge^0(\mathbb{X},1) = \mathbb{X}_0 = \wedge^1(\mathbb{X},1)$.

For instance, the object $\wedge^1(\mathbb{X},2)$ of $(2,1)$-horns in $\mathbb{X}$

$$\begin{tikzcd}
& \cdot \arrow[dr, "\beta"] & \\
\cdot \arrow[ur, "\alpha"] & & \cdot
\end{tikzcd}$$

contains "composable pairs of arrows".

We write

$$\widehat{f_i} = \langle f_j \rangle_{i \neq j \in n+1} \colon W \to \wedge^i(\mathbb{X},n)$$

for the morphism induced by a family $(f_j \colon W \to \mathbb{X}_{n-1})_{i\neq j \in n+1}$, i.e., in which the morphism f_i is missing.

Now suppose that $\mathcal{A}$ is a regular category. A simplicial morphism $\mathbb{f} \colon \mathbb{X} \to \mathbb{Y}$ **satisfies the Kan condition** (resp. **satisfies the Kan condition exactly**) in degree n for i when the morphism

$$\langle \widehat{\partial}_i, \mathbb{f}_n \rangle \colon \mathbb{X}_n \to \wedge^i(\mathbb{X},n) \times_{\wedge^i(\mathbb{Y},n)} \mathbb{Y}_n$$

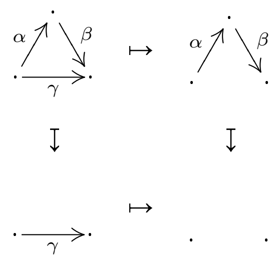
universally induced by the square

$$\begin{tikzcd}
\mathbb{X}_n \arrow[r, "\mathbb{f}_n"] \arrow[d, "\widehat{\partial}_i"'] & \mathbb{Y}_n \arrow[d, "\widehat{\partial}_i"] \\
\wedge^i(\mathbb{X},n) \arrow[r, "\wedge^i(\mathbb{f},n)"'] & \wedge^i(\mathbb{Y},n)
\end{tikzcd}$$

is a regular epimorphism (resp. an isomorphism). The morphism $\mathbb{f}$ is called a **fibration** when it satisfies the Kan condition for all $n \geqslant 1$ and all i. A fibration is **exact** in degrees larger than n when the Kan condition is satisfied exactly in simplicial degrees larger than n for all i.

A regular category is Mal'tsev if and only if every simplicial object is Kan, i.e., every morphism $\mathbb{X} \to \mathbb{1}$ is a fibration [22, Theorem 4.2]. Furthermore, a regular epimorphism of simplicial objects in a regular Mal'tsev category is always a fibration [35, Proposition 4.4]. The Kan property for simplicial objects may also be expressed in terms of higher extensions: in a semi-abelian category, a simplicial object $\mathbb{X}$ is Kan if and only if all of its truncations, considered as higher arrows in all possible directions, have a domain which is an extension [31].

The following technical lemma is easily seen to hold in any category where the needed limits exist; here is a picture in degree $n = 2$ for $i = 1$:



Lemma 1.21. *In a finitely complete category, given $n \geqslant 1$, $i \in n$, and an augmented simplicial object $\mathbb{X}$, the square*

$$\begin{tikzcd} \triangle(\mathbb{X}, n) \arrow[r, "\hat{\partial}_i"] \arrow[d, "\partial_i"'] & \wedge^i(\mathbb{X}, n) \arrow[d] \\ \mathbb{X}_{n-1} \arrow[r, "\langle \partial_i \rangle_i"'] & \triangle(\mathbb{X}, n-1) \end{tikzcd}$$

is a pullback. □

1.22. **Higher-dimensional torsors.** Let A be an abelian group in a Barr exact category $\mathcal{A}$ and consider $n \geqslant 1$. A $\mathbb{K}(A, n)$**-torsor** is an augmented simplicial object $\mathbb{T}$ equipped with a simplicial morphism $\mathbb{t}\colon \mathbb{T} \to \mathbb{K}(A, n)$ such that

(T1) $\mathbb{t}$ is a fibration which is exact from degree n on;
(T2) $\mathbb{T} \cong \mathsf{Cosk}_{n-1}\mathbb{T}$;
(T3) $\mathbb{T}$ is a resolution.

Let Z be an object of a semi-abelian category $\mathcal{A}$ and (A, ξ) a Z-module. An n**-torsor of** Z **by** (A, ξ) is a $\mathbb{K}((A, \xi), n)$-torsor in the category $\mathcal{A}/Z$. Morphisms of $\mathbb{K}(A, n)$-torsors are defined as in the slice over $\mathbb{K}(A, n)$, and thus we obtain the category $\mathrm{Tors}^n(\mathcal{A}, A)$ of $\mathbb{K}(A, n)$-torsors in $\mathcal{A}$ as a full subcategory of $\mathsf{S}^+\mathcal{A}/\mathbb{K}(A, n)$. When the action ξ is trivial, we call $(\mathbb{T}, \mathbb{t})$ an n**-torsor of** Z **by** A, and obtain the following picture:

$$\begin{tikzcd} \cdots & \triangle(\mathbb{T}, n+1) \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[d, "\langle\langle \varsigma \circ \partial_i \rangle_i, \partial_0^{n+2} \rangle"'] & \triangle(\mathbb{T}, n) \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[d, "\langle \varsigma, \partial_0^{n+1} \rangle"] & \mathbb{T}_{n-1} \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[d, "\partial_0^n"] & \mathbb{T}_{n-2} \arrow[d, "\partial_0^{n-1}"] & \cdots & \mathbb{T}_0 \arrow[r, "\partial_0"] \arrow[d, "\partial_0"] & \mathbb{T}_{-1} \arrow[d, equal] \\ \cdots & A^{n+1} \times Z \arrow[r, shift left=3, "\partial_{n+1} \times 1_Z"] \arrow[r, shift left=1, "\mathrm{pr}_n \times 1_Z"'] \arrow[r, phantom, shift right=1, "\vdots"] \arrow[r, shift right=3, "\mathrm{pr}_0 \times 1_Z"'] & A \times Z \arrow[r, shift left=2, "\mathrm{pr}_Z"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\mathrm{pr}_Z"'] & Z \arrow[r, equal, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, equal, shift right=2] & Z & \cdots & Z \arrow[r, equal] & Z \end{tikzcd}$$

When Z is an object of a semi-abelian category $\mathcal{A}$ and A is an abelian object in $\mathcal{A}$ considered as a trivial Z-module (A,τ), we write $\mathrm{Tors}^n(Z,A)$ for the category $\mathrm{Tors}^n(\mathcal{A}/Z,(A,\tau))$. Taking connected components we obtain the set

$$\mathrm{Tors}^n[Z,A] = \pi_0 \,\mathrm{Tors}^n(Z,A)$$

of equivalence classes of n-torsors of Z by A which is, in fact, an abelian group [29].

We shall further analyse the concept of torsor in Section 4; for now it suffices to understand their cohomological meaning.

1.23. **The $(n+1)$-st cohomology group.** It follows from the results in [29] that, when $\mathcal{A}$ is a Barr exact category and

$$\mathbb{G} = (G\colon \mathcal{A} \to \mathcal{A},\ \epsilon\colon G \Rightarrow 1_{\mathcal{A}},\ \delta\colon G \Rightarrow G^2)$$

is a comonad on $\mathcal{A}$ such that the $\mathbb{G}$-projectives coincide with the regular projectives in $\mathcal{A}$, then

$$\mathrm{H}^{n+1}(1,A)_{\mathbb{G}} \cong \pi_0 \,\mathrm{Tors}^n(\mathcal{A},A)$$

where A is an internal abelian group in $\mathcal{A}$ and 1 is the terminal object. If now Z is an object of $\mathcal{A}$ then $\mathbb{G}$ induces a comonad $\mathbb{G}/Z = (G^Z,\epsilon^Z,\delta^Z)$ on $\mathcal{A}/Z$ via

$$\epsilon^Z_f = \left(\begin{tikzcd} GX \arrow[r, "\epsilon_X"] \arrow[dr, "G^Z f = f\circ\epsilon_X"'] & X \arrow[d, "f"] \\ & Z \end{tikzcd}\right) \quad\text{and}\quad \delta^Z_f = \left(\begin{tikzcd} GX \arrow[r, "\delta_X"] \arrow[dr, "G^Z f = f\circ\epsilon_X"'] & GGX \arrow[d, "G^Z G^Z f = f\circ\epsilon_X\circ\epsilon_{GX}"] \\ & Z \end{tikzcd}\right)$$

for all $f\colon X \to Z$. Hence when, in a semi-abelian category $\mathcal{A}$, we consider an abelian object A as a trivial Z-module, we see that

$$\mathrm{H}^{n+1}(1_Z,(A,\tau))_{\mathbb{G}/Z} \cong \pi_0 \,\mathrm{Tors}^n(\mathcal{A}/Z,(A,\tau))$$

and

$$\mathrm{H}^{n+1}(Z,A)_{\mathbb{G}} \cong \mathrm{Tors}^n[Z,A].$$

For instance, $\mathcal{A}$ may be chosen to be a variety of algebras over Set, so that $\mathbb{G}$ is canonically induced by the forgetful/free adjunction. In any case, $\mathrm{Tors}^n[Z,A]$ does indeed carry an abelian group structure. Moreover, this defines an additive functor

$$\mathrm{Tors}^n[Z,-]\colon \mathsf{Ab}\mathcal{A} \to \mathsf{Ab}.$$

2. The groups of equivalence classes of higher central extensions

We work towards a definition of the group $\mathrm{Centr}^n(Z,A)$ of equivalence classes of n-fold central extensions of Z by A, extending the definition of $\mathrm{Centr}^2(Z,A)$ given in Section 4 of [67]. We start with some basic theory of (higher-dimensional) central extensions, first recalling known results and then proving some new ones.

2.1. **Central extensions.** We first consider some general definitions and results valid in a homological category with a chosen strongly Birkhoff subcategory. Here we follow [34].

A **Galois structure** [48] $\Gamma = (\mathcal{A},\mathcal{B},\mathcal{E},\mathcal{F},I,H)$ consists of categories $\mathcal{A}$ and $\mathcal{B}$, an adjunction

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "I"] \arrow[r, phantom, "\perp"] & \mathcal{B}, \arrow[l, shift left, "H"] \end{tikzcd}$$

and classes $\mathcal{E}$ and $\mathcal{F}$ of morphisms of $\mathcal{A}$ and $\mathcal{B}$ respectively, such that:

(i) $\mathcal{A}$ has pullbacks along morphisms in $\mathcal{E}$;
(ii) $\mathcal{E}$ and $\mathcal{F}$ contain all isomorphisms, are closed under composition and are pullback-stable;
(iii) $I(\mathcal{E}) \subseteq \mathcal{F}$;
(iv) $H(\mathcal{F}) \subseteq \mathcal{E}$.

An element of $\mathcal{E}$ is called an **$\mathcal{E}$-extension**.

We shall only consider Galois structures where $\mathcal{A}$ is (at least) a homological category, all $\mathcal{E}$-extensions are regular epimorphisms, and $\mathcal{B}$ is a full replete $\mathcal{E}$-reflective subcategory of $\mathcal{A}$. We shall never write its inclusion H. Such a subcategory is called **strongly $\mathcal{E}$-Birkhoff** when for every $\mathcal{E}$-extension $f\colon X \to Z$ the induced naturality square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Z \arrow[d, "\eta_Z"] \\ IX \arrow[r, "If"'] & IZ \end{tikzcd} \tag{$\mathbf{K}$}$$

is a double $\mathcal{E}$-extension. (The universally induced morphism to the pullback must be in $\mathcal{E}$.) From now on we shall always assume this to be the case.

If $\mathcal{A}$ is an exact Mal'tsev category and $\mathcal{E}$ consists of all regular epimorphisms, a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$ is precisely a **Birkhoff subcategory**: full, reflective and closed under subobjects and regular quotients in $\mathcal{A}$, see [52]. A Birkhoff subcategory of a variety of algebras is the same thing as a subvariety. Outside the exact Mal'tsev context, however, when $\mathcal{E}$ is the class of regular epimorphisms, the strong $\mathcal{E}$-Birkhoff property is generally stronger than the Birkhoff property, as not every pushout of extensions needs to be a double extension.

Example 2.2 (Abelianisation)**.** It is well known that, in any semi-abelian category $\mathcal{A}$, the full subcategory $\mathsf{Ab}\mathcal{A}$ determined by the abelian objects is Birkhoff. This is the situation which we shall be most interested in here, in particular from Subsection 2.13 on.

An $\mathcal{E}$-extension $f\colon X \to Z$ in $\mathcal{A}$ is **trivial** when the induced square ($\mathbf{K}$) is a pullback. Of course, if X and Z lie in $\mathcal{B}$ then f is a trivial $\mathcal{E}$-extension. The $\mathcal{E}$-extension f is said to be **normal** when both projections pr_0, pr_1 in the kernel pair $(\mathrm{R}[f], \mathrm{pr}_0, \mathrm{pr}_1)$ of f are trivial. Finally, f is **central** when there exists an $\mathcal{E}$-extension $g\colon Y \to Z$ such that the pullback of f along g is trivial.

It is clear that every trivial $\mathcal{E}$-extension is central. Moreover, every normal $\mathcal{E}$-extension is central; in the present context, also the converse holds (via Theorem 4.8 of [52] or Proposition 2.6 in [34]). Hence the concepts of normality and centrality coincide. It follows immediately from the definition that pullbacks of $\mathcal{E}$-extensions along $\mathcal{E}$-extensions reflect centrality. Furthermore, in the present context, Proposition 4.1 and 4.3 in [52] may be modified to prove that both the classes of trivial and of central $\mathcal{E}$-extensions are pullback-stable. It is also well known that a split epimorphic central $\mathcal{E}$-extension is always trivial.

The following important result (see [41, 34]) will be used in Section 3.

Lemma 2.3. *When $\mathcal{A}$ is a homological category and $\mathcal{B}$ is a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$, the reflector $I\colon \mathcal{A} \to \mathcal{B}$ preserves pullbacks of $\mathcal{E}$-extensions along split epimorphisms.* □

2.4. **The tower of Galois structures for higher central extensions.** Now we describe the Galois structures for centrality of higher extensions introduced in [34]. We start with a semi-abelian category $\mathcal{A}$ and a Birkhoff subcategory $\mathcal{B}$ of $\mathcal{A}$. Choosing $\mathcal{E}$ and $\mathcal{F}$ to be the classes of regular epimorphisms in $\mathcal{A}$ and $\mathcal{B}$, we obtain a Galois structure Γ as above—$\mathcal{B}$ is strongly $\mathcal{E}$-Birkhoff. We may now drop the prefix $\mathcal{E}$; the elements of this class are the one-fold extensions of Subsection 1.11.

Let us view the objects of $\mathcal{A}$ as zero-fold extensions, and the objects of $\mathcal{B}$ as zero-fold central extensions. With respect to the Galois structure $\Gamma_0 = \Gamma$, there is the notion of central extension, and it is such that the full subcategory $\mathsf{CExt}^1_{\mathcal{B}}\mathcal{A}$

of $\mathsf{Ext}^1\mathcal{A}$ determined by the central extensions is again reflective. Its reflector $I_1\colon \mathsf{Ext}^1\mathcal{A} \to \mathsf{CExt}^1_{\mathcal{B}}\mathcal{A}$, together with the classes $\mathcal{E}^1$ and $\mathcal{F}^1$ of extensions in $\mathsf{Ext}^1\mathcal{A}$ and in $\mathsf{CExt}^1_{\mathcal{B}}\mathcal{A}$ (which we choose to be double extensions in $\mathcal{A}$, and double extensions with central domain and codomain), in turn determines a Galois structure Γ_1. This Galois structure is again "nice" in that $\mathsf{CExt}^1_{\mathcal{B}}\mathcal{A}$ is again strongly $\mathcal{E}^1$-Birkhoff in the homological category $\mathsf{Ext}^1\mathcal{A}$. Inductively, this defines a family of Galois structures $(\Gamma_n)_{n\geqslant 0}$:

$$\Gamma_n = (\mathsf{Ext}^n\mathcal{A}, \mathsf{CExt}^n_{\mathcal{B}}\mathcal{A}, \mathcal{E}^n, \mathcal{F}^n, I_n, \subseteq),$$

each of which gives rise to a notion of central extension which determines the next structure [34, Theorem 4.6]. (Here $\mathcal{E}^0 = \mathcal{E}$, $\mathcal{F}^0 = \mathcal{F}$ and $I_0 = I$.) In particular, for every $n \geqslant 1$ we obtain a reflector (the centralisation functor)

$$I_n\colon \mathsf{Ext}^n\mathcal{A} \to \mathsf{CExt}^n_{\mathcal{B}}\mathcal{A},$$

left adjoint to the inclusion $\mathsf{CExt}^n_{\mathcal{B}}\mathcal{A} \subseteq \mathsf{Ext}^n\mathcal{A}$.

For any $n \geqslant 1$, the n-fold extension $\langle F\rangle_{\mathsf{CExt}^n_{\mathcal{B}}\mathcal{A}}$ in the short exact sequence

$$0 \longrightarrow \langle F\rangle_{\mathsf{CExt}^n_{\mathcal{B}}\mathcal{A}} \xrightarrow{\mu^n_F} F \xrightarrow{\eta^n_F} I_nF \longrightarrow 0$$

induced by the centralisation of an n-fold extension F is zero everywhere except in its initial object $\langle F\rangle^n = (\langle F\rangle_{\mathsf{CExt}^n_{\mathcal{B}}\mathcal{A}})_n$. In parallel with the case $n = 0$ considered in Subsection 1.5, this object $\langle F\rangle^n$ of $\mathcal{A}$ acts like an n-dimensional commutator which may be computed as the kernel of the restriction of the kernel pair projection $(\mathrm{pr}_0)_{n-1}\colon \mathrm{R}[F]_{n-1} \to \mathrm{dom}\, F_{n-1}$ to a morphism

$$\langle \mathrm{R}[F]\rangle^{n-1} \to \langle \mathrm{dom}\, F\rangle^{n-1}.$$

in $\mathcal{A}$. Furthermore, an n-fold extension F is central if and only if the induced morphisms

$$\langle \mathrm{pr}_0\rangle^{n-1}, \langle \mathrm{pr}_1\rangle^{n-1}\colon \langle \mathrm{R}[F]\rangle^{n-1} \to \langle \mathrm{dom}\, F\rangle^{n-1}$$

coincide; see [30, 34] for more details. The notation $\langle F\rangle^n$ not mentioning the Birkhoff subcategory $\mathcal{B}$ need not lead to confusion as the only case which we shall use it in is $\mathcal{B} = \mathsf{Ab}\mathcal{A}$; keeping this in mind, we also write $\langle X\rangle^0 = \langle X\rangle$ for the kernel of $\eta_X\colon X \to \mathsf{ab}X$ when X is an object of $\mathcal{A}$.

Example 2.5 (The simplicial objects $\mathbb{K}(Z, A, n)$)**.** Given any integer $n \geqslant 1$, any object Z and any abelian object A in $\mathcal{A}$, the $(n + 1)$-fold extension underlying $\mathbb{K}(Z, A, n)$ is always trivial with respect to abelianisation. This follows by induction from the fact that both its domain and its codomain are trivial n-fold extensions. Note, however, that the $(n + 2)$-fold arrow underlying $\mathbb{K}(Z, A, n)$ is not even an extension!

Example 2.6 (One-fold central extensions)**.** Recall that an extension of groups $f\colon X \to Z$ is central (with respect to Ab) if and only if $[\mathrm{K}[f], X] = 0$. This result was adapted to a semi-abelian context in [13, 41]: when $\mathcal{A}$ is a semi-abelian category and $\mathcal{B} = \mathsf{Ab}\mathcal{A}$ is the Birkhoff subcategory determined by all abelian objects in $\mathcal{A}$, the one-fold central extensions induced by the Galois structure (the "categorically central" ones) are the central extensions in the algebraic sense. These may be characterised through the Smith/Pedicchio commutator of equivalence relations as those $f\colon X \to Z$ such that $[\mathrm{R}[f], \nabla_X] = \Delta_X$, which means that the kernel pair of the extension f is a central equivalence relation (Subsection 1.4). A characterisation closer to the group case appears in [43] where the condition is reformulated in terms of the Huq commutator of normal subobjects so that it becomes $[\mathrm{K}[f], X] = 0$.

Example 2.7 (Double central extensions)**.** One level up, the double central extensions of groups vs. abelian groups were first characterised in [49]: a double extension such as (**J**) above is central if and only if

$$[\mathrm{K}[d], \mathrm{K}[c]] = 0 = [\mathrm{K}[d] \cap \mathrm{K}[c], X].$$

General versions of this characterisation were given in [42] for Mal'tsev varieties, then in [67] for semi-abelian categories and finally in [37] for exact Mal'tsev categories: the double extension (**J**) is central (with respect to abelianisation) if and only if

$$[\mathrm{R}[d], \mathrm{R}[c]] = \Delta_X = [\mathrm{R}[d] \cap \mathrm{R}[c], \nabla_X]. \tag{\textbf{L}}$$

This means that the span (X, d, c) is a special kind of pregroupoid in the slice category $\mathcal{A}/Z$.

The main technical problem here is that later on, we shall have to use the Huq commutator of normal monomorphisms rather than the Smith/Pedicchio commutator of equivalence relations—and the correspondence between the two which exists in level one is no longer there when we go up in degree. In fact, it is well known and easily verified that if the Smith/Pedicchio commutator of two equivalence relations is trivial, then the Huq commutator of their normalisations is also trivial [15]. But, in general, the converse is false; in [3, 11] a counterexample is given in the category of digroups, which is a semi-abelian variety, even a variety of Ω-groups [44]. The equivalence of these commutators is known as the **Smith is Huq condition (SH)** and it is shown in [59] that, for a semi-abelian category, this condition holds if and only if every star-multiplicative graph is an internal groupoid, which is important in the study of internal crossed modules [51]. Moreover, the Smith is Huq condition is also known to hold for pointed strongly protomodular categories [15] (in particular, for any Moore category [65]) and in action accessible categories [19] (in particular, for any category of interest [61, 62]).

The condition (SH) also implies that every action of an object on an abelian object is a module: here, the equality $[\mathrm{R}[f], \mathrm{R}[f]] = \Delta_X$ in Subsection 1.5 follows from $[\mathrm{K}[f], \mathrm{K}[f]] = [A, A] = 0$.

2.8. **Two lemmas on higher centrality.** The centrality of a higher extension implies that certain induced lower-dimensional extensions are also central. The present proof of Lemma 2.10 was offered to us by Everaert and Gran; it is more general and more elegant than our original proof. In the case of abelianisation, it also follows easily from Theorem 3.8.

Lemma 2.9. *In a semi-abelian category with a chosen Birkhoff subcategory, the kernel of an n-fold central extension is an $(n-1)$-fold central extension.*

Proof. This is well known and easily seen from the definition. □

Lemma 2.10. *Let F be an n-fold central extension in a semi-abelian category with a chosen Birkhoff subcategory. Then the one-fold extension $l_F \colon F_n \to \mathsf{L}F$ induced by F is always central.*

Proof. The case $n = 1$ is trivial, so take $n \geqslant 2$. We shall prove that for an n-fold central extension, considered as a square (**J**) of $(n-1)$-fold extensions, the induced comparison $\langle d, c\rangle \colon X \to D \times_Z C$ is an $(n-1)$-fold central extension. Then the claim follows easily by induction. Since $\langle d, c\rangle$ is an extension by definition, we just have to show its centrality.

First we may reduce the situation to trivial extensions. Indeed, taking kernel pairs to the left, we obtain the diagram

$$\begin{tikzcd}
\mathrm{R}[c] \arrow[r, shift left, "c_1"] \arrow[r, shift right, "c_0"'] \arrow[d, "r"'] & X \arrow[r, "c"] \arrow[d, "d"'] & C \arrow[d, "g"] \\
\mathrm{R}[f] \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_0"'] & D \arrow[r, "f"'] & Z.
\end{tikzcd}$$

It is not hard to see that the induced comparison $\langle r, c_0\rangle\colon \mathrm{R}[c] \to \mathrm{R}[f] \times_D X$ is a pullback of the extension $\langle d, c\rangle$. Hence if $\langle r, c_0\rangle$ is central then so is $\langle d, c\rangle$, because pulling back reflects centrality.

Now we reduce to n-fold extensions between $(n-1)$-fold central extensions. Suppose that the square $(\mathbf{J})$, viewed as an arrow from d to g, is a trivial extension. Consider the following cube, which displays the centralisation of d and of g:

$$\begin{tikzcd}
 & & I_{n-1}[d] \arrow[rrr] \arrow[dr, squiggly] \arrow[dddd, dotted] & & & I_{n-1}[g] \arrow[dddd, "I_{n-1}g"] \\
 & & & \cdot \arrow[urr, dotted] \arrow[dddl, dotted] \arrow[dr, phantom, "\lrcorner"] & & \\
X \arrow[uurr, squiggly] \arrow[rrrr] \arrow[dr, squiggly] \arrow[dddd, "d"'] & & & & C \arrow[uur] \arrow[dddd, "g"] & \\
 & \cdot \arrow[uur, squiggly] \arrow[urrr] \arrow[dddl] \arrow[dr, phantom, "\lrcorner"] & & & & \\
 & & D \arrow[rrr, dotted] \arrow[ddll, dotted, equal] & & & Z \arrow[ddl, equal] \\
 & & & & & \\
D \arrow[rrrr] & & & & Z &
\end{tikzcd}$$

Since the front square is a trivial extension, the top square is a pullback. By pullback cancellation, the top square of the prism between the front and back pullbacks is also a pullback, and it follows that the square of wiggly arrows is a pullback too. This completes the reduction, as central extensions are pullback-stable.

Finally, for an n-fold extension $(\mathbf{J})$ between $(n-1)$-fold central extensions d and g the property holds, since $\langle d, c\rangle$ is a subobject of the extension d—a monomorphism of extensions is a square of which the top map is a monomorphism—and central extensions are closed under subobjects. $\square$

2.11. **(Central) extensions over a fixed base object.** Let Z be an object of $\mathcal{A}$ and $n \geqslant 1$. Denote by $\mathsf{Ext}^n_Z\mathcal{A}$ the category of **n-fold extensions of** Z, defined as the fibre over Z (the pre-image of the identity 1_Z) of the functor

$$\mathrm{cod}^n = \underbrace{\mathrm{cod}\circ\cdots\circ\mathrm{cod}}_{n \text{ times}} = (-)_0\colon \mathsf{Ext}^n\mathcal{A} \to \mathcal{A}\colon F \mapsto F_0.$$

Thus the objects are n-fold extensions with “terminal object” Z, and the morphisms are those morphisms in $\mathsf{Ext}^n\mathcal{A}$ which restrict to the identity on Z under the functor cod^n. Similarly $\mathsf{CExt}^n_Z\mathcal{A}$ is the full subcategory of $\mathsf{Ext}^n_Z\mathcal{A}$ determined by the n-fold extensions of Z that are central with respect to $\mathcal{B}$. (The index $\mathcal{B}$ being dropped here is not really problematic as we shall take $\mathcal{B}$ equal to $\mathsf{Ab}\mathcal{A}$ anyway from Subsection 2.13 on.)

Lemma 2.12. *Consider a semi-abelian category $\mathcal{A}$ with a chosen Birkhoff subcategory. Let Z be an object of $\mathcal{A}$ and $n \geqslant 1$. Then $\mathsf{Ext}^n_Z\mathcal{A}$ and $\mathsf{CExt}^n_Z\mathcal{A}$ have binary* *products: the product of two n-fold (central) extensions over Z is an n-fold (central) extension over Z.*

Proof. Given two n-fold extensions F and G over Z, their product $F \times G$ in $\mathsf{Ext}^n_Z\mathcal{A}$ is given pointwise by pullbacks in $\mathcal{A}$:

$$(F \times G)_I = F_I \times_Z G_I$$

for $I \subseteq n$. This n-fold arrow is indeed an extension by Proposition 1.12, as

$$\begin{aligned}(F \times G)_I \to \lim_{J \subsetneq I}(F \times G)_J &= (F_I \times_Z G_I) \to \lim_{J \subsetneq I}(F_J \times_Z G_J)\\ &= (F_I \times_Z G_I) \to (\lim_{J \subsetneq I} F_J \times_Z \lim_{J \subsetneq I} G_J)\\ &= (F_I \to \lim_{J \subsetneq I} F_J) \times_Z (G_I \to \lim_{J \subsetneq I} G_J)\end{aligned}$$

for all $\varnothing \neq I \subseteq n$. Note that in particular,

$$l_{F \times G} = l_F \times_Z l_G \colon (F_n \to \lim_{J \subsetneq n} F_J) \times_Z (G_n \to \lim_{J \subsetneq n} G_J).$$

The n-fold extension $F \times G$ is central as a subobject of the product of F and G in $\mathsf{CExt}^n_{\mathcal{B}}\mathcal{A}$, which is computed pointwise as in $\mathsf{Ext}^n\mathcal{A}$ since $\mathsf{CExt}^n_{\mathcal{B}}\mathcal{A}$ is a reflective subcategory. □

2.13. **The direction of a higher (central) extension.** From now on we assume that $\mathcal{A}$ is a semi-abelian category and $\mathcal{B} = \mathsf{Ab}\mathcal{A}$ is the Birkhoff subcategory determined by the abelian objects of $\mathcal{A}$. We introduce the concept of *direction* for n-fold (central) extensions in $\mathcal{A}$, which is crucial in the definition and in the study of the groups $\mathrm{Centr}^n(Z, A)$. As explained in [67], this notion is based on Bourn's concept of direction for internal groupoids [9].

Definition 2.14. The **direction** of an n-fold extension F is the object $\mathrm{K}^n[F]$, obtained by taking kernels n times. If F is central then the direction of F is an abelian object of $\mathcal{A}$ by Lemma 2.9 and the convention regarding zero-fold central extensions (see Figure 1 for the case $n = 3$). Given any object Z of $\mathcal{A}$, this defines the **direction functor**

$$\mathsf{D}_{(n,Z)} \colon \mathsf{CExt}^n_Z\mathcal{A} \to \mathsf{Ab}\mathcal{A}.$$

The fibre $\mathsf{D}^{-1}_{(n,Z)}A$ of this functor over an abelian object A is the category of ***n*-fold central extensions of Z by A**. Two n-fold central extensions of Z by A which are connected by a zigzag in $\mathsf{D}^{-1}_{(n,Z)}A$ are called **equivalent**. As explained in Subsection 1.6, the equivalence classes (which we shall denote $[F]$ for F in $\mathsf{D}^{-1}_{(n,Z)}A$) form the set

$$\mathrm{Centr}^n(Z, A) = \pi_0(\mathsf{D}^{-1}_{(n,Z)}A)$$

of connected components of the category $\mathsf{D}^{-1}_{(n,Z)}A \subseteq \mathsf{CExt}^n_Z\mathcal{A}$.

The direction may be calculated in one step:

Lemma 2.15. *For any n-fold central extension F in a semi-abelian category we have*

$$\mathsf{D}_{(n,Z)}F = \mathrm{K}[l_F] = \bigcap_{i \in n} \mathrm{K}[f_i].$$

Proof. The chain $\mathrm{K}[l_F] = \mathrm{K}[l_{\mathrm{K}[F]}] = \cdots = \mathrm{K}[l_{\mathrm{K}^{n-1}[F]}] = \mathrm{K}[\mathrm{K}^{n-1}[F]]$ gives us the first equality; the second is immediate from the definition. □

Remark 2.16. For an n-fold extension F, an "element" x of F_n is an n-dimensional hyper-tetrahedron with faces $x_i = f_i(x)$. Such a tetrahedron is in the direction of F precisely when all its faces x_i are zero—see Figure 1 on page 5 for the case $n = 3$.

2.17. **The group structure on** $\mathrm{Centr}^n(Z,A)$**.** We are now ready to show that the set $\mathrm{Centr}^n(Z,A)$ of equivalence classes of n-fold central extensions of Z by A carries a canonical abelian group structure (Corollary 2.20).

Lemma 2.18. *For any object Z of a semi-abelian category $\mathcal{A}$ and any $n \geqslant 1$, the direction functor $\mathsf{D}_{(n,Z)}\colon \mathsf{CExt}^n_Z\mathcal{A} \to \mathsf{Ab}\mathcal{A}$ preserves finite products.*

Proof. The terminal object 1 of $\mathsf{CExt}^n_Z\mathcal{A}$ is the "constant" n-fold central extension of Z formed out of the identities 1_Z; it is clear that the direction of 1 is zero.

Given two n-fold central extensions F and G over Z with respective directions A and B, we have to prove that their product over Z has direction $A \times B$. Lemma 2.12 tells us that the product in question does indeed exist. Lemma 2.15, together with the calculation in the proof of Lemma 2.12, gives us the direction: the kernel of $l_{F\times G} = l_F \times_Z l_G$ is $A \times B$. □

Proposition 2.19. *Let Z be an object of a semi-abelian category $\mathcal{A}$. Mapping any abelian object A of $\mathcal{A}$ to the set $\mathrm{Centr}^n(Z,A)$ of equivalence classes of n-fold central extensions of Z by A gives a finite product-preserving functor*

$$\mathrm{Centr}^n(Z,-)\colon \mathsf{Ab}\mathcal{A} \to \mathsf{Set}.$$

Proof. The functoriality of $\mathrm{Centr}^n(Z,-)$ is a consequence of the same property for $\mathrm{Centr}^1(\mathsf{L}F,-)$; we follow the construction behind [43, Proposition 6.1]. Given an n-fold central extension F of Z by A, we have an induced one-fold central extension (Lemma 2.10) with kernel A (Lemma 2.15).

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, tail, "k_F"] & F_n \arrow[r, two heads, "l_F"] & \mathsf{L}F \arrow[r] & 0 \end{tikzcd}$$

Now let $a\colon A \to B$ be a morphism of abelian objects in $\mathcal{A}$. Then, applying the function $\mathrm{Centr}^1(\mathsf{L}F,a)$ to $[l_F]$, we obtain an element $[l_{F'}]$ of $\mathrm{Centr}^1(\mathsf{L}F,B)$ through the following construction.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, tail, "k_F"] \arrow[d, "\langle 1_A,0\rangle"'] & F_n \arrow[r, two heads, "l_F"] \arrow[d, "\langle 1_{F_n},0\rangle"] & \mathsf{L}F \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A\oplus B \arrow[r, tail, "k_F\times 1_B"] \arrow[d, two heads, "\langle {a \atop 1_B}\rangle"'] & F_n\times B \arrow[r, two heads] \arrow[d, two heads] & \mathsf{L}F \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r, tail] & F'_n \arrow[r, two heads, "l_{F'}"'] & \mathsf{L}F \arrow[r] & 0 \end{tikzcd}$$

(Here $A \oplus B$ is the biproduct of A and B in $\mathsf{Ab}\mathcal{A}$, which may be computed as their product $A \times B$ in $\mathcal{A}$.) We define $\mathrm{Centr}^n(Z,a)[F] = [F']$, where F' is the n-fold extension with initial object F'_n, with initial morphisms $f'_i = \mathrm{pr}_i \circ l_{F'}$ for $i \in n$, and with $F'_I = F_I$ for all $I \subsetneq n$. The centrality of F' is a consequence of F being central: the extension F' is a quotient of $F \times \mathbb{K}(B, n-1)$, which is central as a product of central extensions (cf. Example 2.5). The functoriality properties of $\mathrm{Centr}^n(Z,-)$ are an immediate consequence of the corresponding properties of $\mathrm{Centr}^1(\mathsf{L}F,-)$.

The functor $\mathrm{Centr}^n(Z,-)$ preserves terminal objects: indeed, $\mathrm{Centr}^n(Z,0)$ is a singleton, because the terminal object of $\mathsf{CExt}^n_Z\mathcal{A}$ has direction 0 by Lemma 2.18; if F is an n-fold central extension of Z by 0, there is the unique morphism $F \to 1$ to testify that $[F] = [1]$. As for binary products, we must define an inverse to the map

$$\begin{tikzcd}[column sep=huge] \mathrm{Centr}^n(Z,A\times B) \arrow[r, shift left, "(\mathrm{Centr}^n(Z,\mathrm{pr}_A),\mathrm{Centr}^n(Z,\mathrm{pr}_B))"'] & \mathrm{Centr}^n(Z,A)\times\mathrm{Centr}^n(Z,B). \arrow[l, dotted, shift left] \end{tikzcd}$$

This inverse takes a couple $([F],[F'])$ and sends it to $[F \times_Z F']$: Lemma 2.18 insures that the direction of $F \times_Z F'$ is $A \times B$. □

Corollary 2.20. *When $\mathcal{A}$ is a semi-abelian category, the functor* $\mathrm{Centr}^n(Z,-)$ *factors uniquely over the forgetful functor* $\mathsf{Ab} \to \mathsf{Set}$ *to yield a functor*

$$\mathrm{Centr}^n(Z,-)\colon \mathsf{Ab}\mathcal{A} \to \mathsf{Ab}.$$

In particular, any $\mathrm{Centr}^n(Z,A)$ *carries a canonical abelian group structure.* □

3. The geometry of higher central extensions

We give a geometrical interpretation of the concept of higher central extension, essentially a higher-dimensional version of Bourn and Gran's result [13] that a one-fold extension $f\colon X \to Z$ is central if and only if its kernel A is abelian and its kernel pair is the product $A \times X$. Our Theorem 3.8 says that an n-fold extension F is central if and only if

(i) the direction of F is abelian, and
(ii) any face in any n-fold diamond in F is uniquely determined by an element of the direction of F.

In the following sections this will lead to an equivalence between torsors and central extensions, Theorem 5.9, which in turn will lead to our main result on cohomology, Theorem 6.5.

3.1. **Higher equivalence relations.** Recall that a **double equivalence relation** is an equivalence relation of equivalence relations: given two (internal) equivalence relations R_0 and R_1 on an object X, it is an equivalence relation $R \rightrightarrows R_1$ on the relation $R_0 \rightrightarrows X$ as in the diagram below:

$$\begin{tikzcd} R \arrow[r, shift left, "\mathrm{pr}^1_1"] \arrow[r, shift right, "\mathrm{pr}^1_0"'] \arrow[d, shift right, "\mathrm{pr}^0_0"'] \arrow[d, shift left, "\mathrm{pr}^0_1"] & R_1 \arrow[d, shift right, "r^1_0"'] \arrow[d, shift left, "r^1_1"] \\ R_0 \arrow[r, shift left, "r^0_1"] \arrow[r, shift right, "r^0_0"'] & X. \end{tikzcd}$$

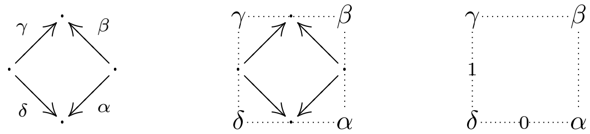
That is, each of the four pairs of parallel morphisms on this diagram represents an equivalence relation, and these relations are compatible in an obvious sense. For instance, $R_1 \square R_0$ denotes the largest double equivalence relation on R_0 and R_1, a two-dimensional version of ∇_X; see [3, 10, 54]. It "consists of" all quadruples $(\alpha,\beta,\gamma,\delta)$ in X^4 in the configuration

$$\begin{tikzcd} \gamma \arrow[r, dotted, dash] \arrow[d, dotted, dash, "1"'] & \beta \arrow[d, dotted, dash] \\ \delta \arrow[r, dotted, dash, "0"] & \alpha, \end{tikzcd}$$

a 2×2 matrix where (δ,α), $(\gamma,\beta) \in R_0$ and (α,β), $(\delta,\gamma) \in R_1$. We shall be especially interested in the particular case where R is induced by a double extension F as in Diagram (**J**), as follows: $R_0 = \mathrm{R}[c]$ is the kernel pair of c, the relation $R_1 = \mathrm{R}[d]$ is the kernel pair of d and $R = \mathrm{R}[d] \square \mathrm{R}[c]$. It is easily seen that then the rows and columns of the induced diagram

$$
\begin{tikzcd}
\mathrm{R}[d]\,\square\,\mathrm{R}[c] \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & \mathrm{R}[d] \arrow[r, "p"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & \mathrm{R}[g] \arrow[d, shift right, "g_0"'] \arrow[d, shift left, "g_1"] \\
\mathrm{R}[c] \arrow[r, shift left, "c_1"] \arrow[r, shift right, "c_0"'] \arrow[d, "r"'] & X \arrow[r, "c"] \arrow[d, "d"'] & C \arrow[d, "g"] \\
\mathrm{R}[f] \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_0"'] & D \arrow[r, "f"'] & Z
\end{tikzcd}
\tag{$\mathbf{M}$}
$$

are exact forks, i.e., consist of (effective) equivalence relations with their coequalisers; it is a denormalised 3×3 diagram as studied in [10]. Since the "elements" of X may now be viewed as arrows with a domain in D and a codomain in C, any "element" of $\mathrm{R}[d]\,\square\,\mathrm{R}[c]$ corresponds to a **(two-fold) diamond** [54] in the double extension F:



($\mathbf{N}$)

Note the geometrical duality here, which at this level is almost invisible since the dual of a square is a square. This will become more manifest in higher degrees. In some sense $\mathrm{R}[d]\,\square\,\mathrm{R}[c]$ is a kind of *denormalised direction* of F (where the kernels are replaced by kernel pairs), also in that $\mathrm{R}[d]\,\square\,\mathrm{R}[c]$ may be considered as $\mathrm{R}^2[F]$—see Diagram ($\mathbf{M}$) and compare with Definition 2.14 for $n = 2$.

Inductively, an **n-fold equivalence relation** may be defined as an equivalence relation of $(n-1)$-fold equivalence relations. Considered as a diagram in the base category $\mathcal{A}$, it has n underlying equivalence relations $R_0, \ldots, R_{n-1}$ on a common object X. An internal n-fold equivalence relation is the same thing as an internal n-fold groupoid in which all pairs of projections are jointly monomorphic. The

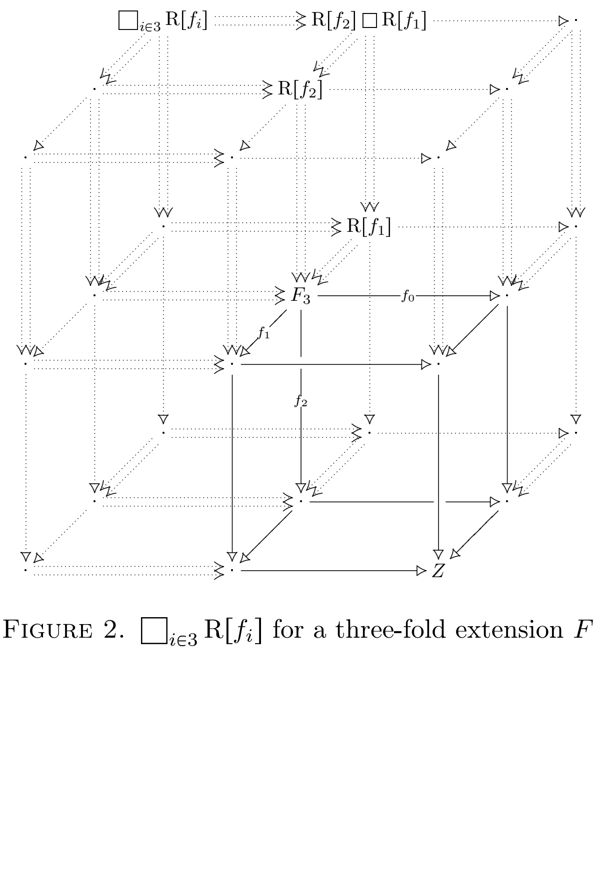


FIGURE 2. $\square_{i\in 3}\mathrm{R}[f_i]$ for a three-fold extension F

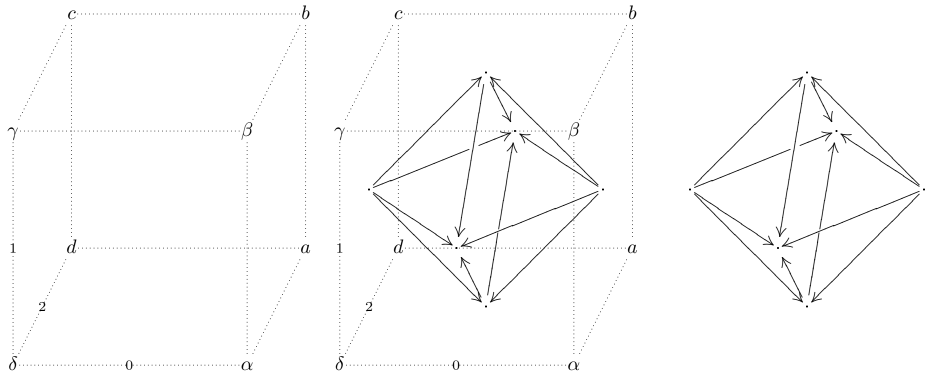


FIGURE 3. Matrix and diamond for a three-fold extension

largest n-fold equivalence relation on n given equivalence relations $R_0, \ldots, R_{n-1}$ on an object X is denoted

$$\mathop{\square}_{i\in n} R_i.$$

It has projections pr_0^i and pr_1^i to R_i, for all $i \in n$, and thus consists of 2^n commutative cubes of projections, one for each choice of projection (either pr_0^i or pr_1^i) in each direction $i \in n$. The elements of $\square_{i\in n} R_i$ are n-dimensional matrices in X, in fact matrices of order

$$\underbrace{2 \times \cdots \times 2}_{n}.$$

In the i-th direction of the matrix (counting from 0 to $n-1$) the elements are related by the equivalence relation R_i. We are again mostly interested in the case where the n-fold equivalence relation is induced by an n-fold extension F: simply take $R_i = \mathrm{R}[f_i]$. The induced object $\square_{i\in n} \mathrm{R}[f_i] = \mathrm{R}^n[F]$ may again be considered as a denormalised direction of F. Its elements are called **(n-fold) diamonds in** F because of their shape in the lower dimensions.

When F is a three-fold extension (see Figure 2) such a diamond is a hollow octahedron (see Figure 3) of which the faces are elements of F_3. We name the faces of the octahedron by the vertices of a cube which is formally a three-dimensional matrix where $\mathrm{R}[f_0]$ is the left-right relation, $\mathrm{R}[f_1]$ is bottom-top and $\mathrm{R}[f_2]$ is front-back. Note how the geometrical duality between the octahedron and the cube is explicit here.

3.2. **Indexing the elements of $\square_{i\in n} \mathrm{R}[f_i]$.** Consider an n-fold extension F. An element of $\square_{i\in n} \mathrm{R}[f_i]$ being an n-dimensional matrix, its entries are indexed by the elements of 2^n, i.e., the subsets of the ordinal n. An entry x_I in a matrix $x \in \square_{i\in n} \mathrm{R}[f_i]$ finds itself in the first entry of the i-th direction when $i \notin I$ and in the second entry of the i-th direction when $i \in I$. Hence the entry $x_I = \mathrm{pr}_I(x)$ is

$$(\mathrm{pr}^0_{\delta_I(0)} \circ \mathrm{pr}^1_{\delta_I(1)} \circ \cdots \circ \mathrm{pr}^n_{\delta_I(n)})(x)$$

where

$$\delta_I(i) = \begin{cases} 0 & \text{if } i \notin I \\ 1 & \text{if } i \in I \end{cases}$$

and pr_0^i and pr_1^i are the first and second projection of $\mathrm{R}[f_i]$. Two entries x_I and x_J are related by $\mathrm{R}[f_i]$ when the only difference between I and J is that one does, and the other does not, contain i. So $(x_I, x_J) \in \mathrm{R}[f_i]$ when $J = I \cup \{i\}$ or $I = J \cup \{i\}$.

For instance, in Figure 3, the face β corresponds to the entry x_2: the set $2 \subseteq 3$ contains 0 and 1 but it doesn't contain 2.

3.3. **The induced n-cubes.** As explained above, given an n-fold extension F, any choice of a set $I \subseteq n$ corresponds to one of the commutative n-cubes in the n-fold equivalence relation $\square_{i\in n} \mathrm{R}[f_i]$. We shall denote it $\square(F, I)$. In fact, it again forms an n-fold extension in $\mathcal{A}$, and its initial morphisms are the

$$\mathrm{pr}^i_{\delta_I(i)} \colon \square_{i\in n} \mathrm{R}[f_i] \to \mathrm{R}[f_i].$$

The extension property follows, for instance, from the fact that all its morphisms are compatibly split (by the reflexivity of all the equivalence relations involved). Since no confusion with the other arrows is possible (cf. the notation introduced in Subsection 1.10), we shall denote such a composed splitting

$$\square(F, I)^J_K \colon \square(F, I)_J \to \square(F, I)_K$$

when $J \subseteq K \in 2^n$.

3.4. **The objects $\boxdot^I_{i\in n} \mathrm{R}[f_i]$.** Given an n-fold extension F and $I \subseteq n$, the elements of the object $\boxdot^I_{i\in n} \mathrm{R}[f_i]$ are diamonds in F with the I-face missing, or equivalently, n-dimensional matrices (of order $2 \times \cdots \times 2$) with the I-entry left out; it is the limit $\mathsf{L}(\square(F, n\backslash I))$ from Subsection 1.10 determined by the n-fold extension $\square(F, n\backslash I)$. Let

$$\pi^I = l_{\square(F,n\backslash I)} \colon \square_{i\in n} \mathrm{R}[f_i] \to \boxdot^I_{i\in n} \mathrm{R}[f_i]$$

denote the canonical projection which forgets the I-face, then clearly the kernel of π^I is isomorphic to the direction of F. (All faces but one in the diamond are zero, and of course this face has its boundary zero.) In fact, this gives us a version of Lemma 1.21, valid for higher extensions: any square

$$\begin{tikzcd}
\square_{i\in n} \mathrm{R}[f_i] \arrow[r, "\pi^I"] \arrow[d, "\mathrm{pr}_I"'] & \boxdot^I_{i\in n} \mathrm{R}[f_i] \arrow[d] \\
F_n \arrow[r, "\langle f_i \rangle_i"'] & \mathsf{L}F
\end{tikzcd} \qquad (\mathbf{O})$$

is a pullback (Lemma 1.2).

For instance, in $\boxdot^2_{i\in 3} \mathrm{R}[f_i]$ we have 3-fold diamonds as in Figure 3 in which the face $\beta = x_2$ is missing.

In degree two the pullback $\mathrm{R}[d] \times_X \mathrm{R}[c]$ (mentioned in the introduction, and computed as in Diagram (**H**)) that contains two-fold diamonds in which the face δ is missing, is nothing but $\mathrm{R}[d] \boxdot^{\varnothing} \mathrm{R}[c]$, and the projection π is $\pi^{\varnothing}$.

3.5. **Analysis of centrality in degree two.** As explained in [67] (and, in full generality, in the proof of Theorem 3.8 below), the double extension F from Diagram (**J**) is central if and only if in the diagram

$$\begin{tikzcd}
 & \langle \mathrm{R}[d] \square \mathrm{R}[c] \rangle \arrow[r, "\langle \pi \rangle", "\cong"'] \arrow[d] & \langle \mathrm{R}[d] \times_X \mathrm{R}[c] \rangle \arrow[d] \\
A \arrow[r] \arrow[d, "\cong"'] & \mathrm{R}[d] \square \mathrm{R}[c] \arrow[r, "\pi"] \arrow[d] \arrow[dr, phantom, "\mathtt{(i)}"] & \mathrm{R}[d] \times_X \mathrm{R}[c] \arrow[d] \\
\mathrm{K}[\mathsf{ab}\pi] \arrow[r] & \mathsf{ab}(\mathrm{R}[d] \square \mathrm{R}[c]) \arrow[r, shift right, "\mathsf{ab}\pi"'] & \mathsf{ab}(\mathrm{R}[d] \times_X \mathrm{R}[c]) \arrow[l, shift right]
\end{tikzcd}$$

the morphism $\langle\pi\rangle$ is an isomorphism. By Lemma 1.2, this occurs when the square `(i)` is a pullback, which is precisely saying that π is a trivial extension. (Indeed π is an extension as it is the comparison to the pullback in a double extension, in fact in a double split epimorphism.)

Note that $\mathsf{ab}\pi$ is a split epimorphism by Lemma 1.8, because ab preserves the pullback $\mathrm{R}[d]\times_X \mathrm{R}[c]$: in fact, it preserves all pullbacks of split epimorphisms along split epimorphisms, or even all pullbacks of split epimorphisms along extensions (Lemma 2.3). This makes $\mathsf{ab}\pi$ a product projection. Further recall that the kernel of π is the direction of F.

Hence if F is central then π is a split epimorphism, in fact a product projection, and thus we see that

$$\mathrm{R}[d]\,\square\,\mathrm{R}[c] \cong A \times (\mathrm{R}[d] \times_X \mathrm{R}[c]) \tag{$\mathbf{P}$}$$

where A is the direction of F, an abelian object. Conversely, whenever A is abelian and π is the projection in the product ($\mathbf{P}$), the extension π is trivial, so that the square `(i)` is a pullback, and F is a double central extension.

We may also view this slightly differently: the condition $[\mathrm{R}[d], \mathrm{R}[c]] = \Delta_X$ in ($\mathbf{L}$) is equivalent to the morphism $\pi\colon \mathrm{R}[d]\,\square\,\mathrm{R}[c] \to \mathrm{R}[d] \times_X \mathrm{R}[c]$ being a split epimorphism [14, Lemma 3.3]. Also $\Delta_X = [\mathrm{R}[d] \cap \mathrm{R}[c], \nabla_X]$ if and only if π is central [41]. Now π is trivial as a split epimorphic central extension.

3.6. **Higher degrees.** This characterisation of centrality goes up to higher dimensions. The basic idea is that by induction, an n-fold extension F is central if and only if the morphisms

$$\langle\pi^I\rangle\colon \langle \square_{i\in n} \mathrm{R}[f_i]\rangle \to \langle \boxdot^{I}_{i\in n} \mathrm{R}[f_i]\rangle$$

are isomorphisms. This eventually implies that

$$\square_{i\in n} \mathrm{R}[f_i] \cong A \times \boxdot^{I}_{i\in n} \mathrm{R}[f_i] \tag{$\mathbf{Q}$}$$

where A is the direction of F: any missing face in an n-fold diamond is completely determined by an element in A.

Lemma 3.7. *When $\mathcal{A}$ is a semi-abelian category, the functor $\mathsf{ab}\colon \mathcal{A} \to \mathsf{Ab}\mathcal{A}$ preserves any limit $\boxdot^{I}_{i\in n} \mathrm{R}[f_i]$ induced by any n-fold extension F.*

Proof. This follows from Lemma 2.3, as such a limit may be computed by repeated pullbacks of regular epimorphisms along split epimorphisms. $\square$

Theorem 3.8. *In a semi-abelian category, let F be an n-fold extension with direction A. Then the following are equivalent:*

(i) *F is central;*
(ii) *the n-fold extension $\langle\square(F, I)\rangle$ is a limit n-cube;*
(iii) *the morphism $\langle\pi^I\rangle\colon \langle\square_{i\in n} \mathrm{R}[f_i]\rangle \to \langle\boxdot^{I}_{i\in n} \mathrm{R}[f_i]\rangle$ is an isomorphism;*
(iv) *A is abelian and $\square_{i\in n} \mathrm{R}[f_i] \cong A \times \boxdot^{I}_{i\in n} \mathrm{R}[f_i]$;*

for any, hence for all, $I \subseteq n$.

Proof. First we show that (i) and (ii) are equivalent. The n-fold extension F, considered as a morphism $\operatorname{dom} F \to \operatorname{cod} F$, is central if and only if either one of the projections $\mathrm{R}[F] \to \operatorname{dom} F$ is trivial, which occurs when the morphisms

$$\langle \mathrm{R}[F]\rangle^{n-1} \rightrightarrows \langle \operatorname{dom} F\rangle^{n-1}$$

are isomorphisms (see Subsection 2.4). By Lemma 1.2 this happens when either one of the commutative squares in

$$\begin{tikzcd}
\langle \mathrm{R}^2[F]\rangle^{n-2} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \langle \mathrm{R}[\operatorname{dom} F]\rangle^{n-2} \arrow[d, shift left] \arrow[d, shift right] \\
\langle \operatorname{dom} \mathrm{R}[F]\rangle^{n-2} \arrow[r, shift left] \arrow[r, shift right] & \langle \operatorname{dom}^2 F\rangle^{n-2}
\end{tikzcd}$$

is a pullback. This, in turn, is equivalent to either one of the commutative cubes in

$$\begin{tikzcd}
& \langle \mathrm{R}^3[F]\rangle^{n-2} \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dl, shift left] \arrow[dl, shift right] \arrow[dd, dotted, shift left] \arrow[dd, dotted, shift right] & & \langle \mathrm{R}^2[\operatorname{dom} F]\rangle^{n-2} \arrow[dl, shift left] \arrow[dl, shift right] \arrow[dd, shift left] \arrow[dd, shift right] \\
\langle \mathrm{R}[\operatorname{dom} \mathrm{R}[F]]\rangle^{n-2} \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd, shift left] \arrow[dd, shift right] & & \langle \mathrm{R}[\operatorname{dom}^2 F]\rangle^{n-2} \arrow[dd, shift left] \arrow[dd, shift right] & \\
& \langle \operatorname{dom} \mathrm{R}^2[F]\rangle^{n-2} \arrow[rr, dotted, shift left] \arrow[rr, dotted, shift right] \arrow[dl, dotted, shift left] \arrow[dl, dotted, shift right] & & \langle \operatorname{dom} \mathrm{R}[\operatorname{dom} F]\rangle^{n-2} \arrow[dl, shift left] \arrow[dl, shift right] \\
\langle \operatorname{dom}^2 \mathrm{R}[F]\rangle^{n-2} \arrow[rr, shift left] \arrow[rr, shift right] & & \langle \operatorname{dom}^3 F\rangle^{n-2} &
\end{tikzcd}$$

being a limit cube. This process continues until we obtain a cube of dimension n.

The equivalence between (ii) and (iii) is clear as $\langle \square(F, I)\rangle$ is just one of the cubes induced by choosing an n-fold arrow (i.e., making a choice of projections) in the n-fold equivalence relation $\langle \square_{i\in n} \mathrm{R}[f_i]\rangle$; so $\langle \pi^I\rangle$ is an isomorphism if and only if this cube is a limit. The functor $\langle -\rangle$ does indeed preserve the limit $\boxdot^I_{i\in n} \mathrm{R}[f_i]$, since so does ab by Lemma 3.7.

Now we prove the equivalence between (iii) and (iv). We actually mean a bit more in (iv): we have a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, tail] & \square_{i\in n} \mathrm{R}[f_i] \arrow[r, "\pi^I"] & \boxdot^I_{i\in n} \mathrm{R}[f_i] \arrow[r] & 0
\end{tikzcd}$$

where π^I is a product projection. Indeed (iii) is equivalent to the square

$$\begin{tikzcd}
\square_{i\in n} \mathrm{R}[f_i] \arrow[r, "\pi^I"] \arrow[d] & \boxdot^I_{i\in n} \mathrm{R}[f_i] \arrow[d] \\
\mathsf{ab}(\square_{i\in n} \mathrm{R}[f_i]) \arrow[r, "\mathsf{ab}\pi^I"'] & \mathsf{ab}(\boxdot^I_{i\in n} \mathrm{R}[f_i])
\end{tikzcd} \qquad (\mathbf{R})$$

being a pullback, which means that π^I is a trivial extension. Since its kernel is the abelian object A, the extension π^I is a product projection if and only if it is a split epimorphism. But this is the case, since the limit $\mathsf{ab}\boxdot^I_{i\in n} \mathrm{R}[f_i]$ may be computed by successive pullbacks of regular epimorphisms along split epimorphisms (Lemma 3.7), which become pullbacks of split epimorphisms along split epimorphisms by induction and repeatedly using Lemma 1.8. $\square$

In what follows we shall use this result to obtain one half of the equivalence between torsors and central extensions.

Remark 3.9. Note that the splitting of π^I constructed in the proof above is natural in F, so that also the product decompositions (iv) are natural in the extension considered.

Remark 3.10. The proof of Theorem 3.8 shows that an n-fold extension F is central precisely when, for any $I \subseteq n$, the induced $(n+1)$-fold extension

$$\square(F, I) \to \mathsf{ab}(\square(F, I))$$

is a limit $(n+1)$-cube. In fact, these $(n+1)$-fold extensions are part of the regular epimorphism of n-fold groupoids

$$\eta_{\square_{i\in n}\mathrm{R}[f_i]}\colon \mathop{\Box}_{i\in n}\mathrm{R}[f_i] \to \mathsf{ab}(\mathop{\Box}_{i\in n}\mathrm{R}[f_i]),$$

which therefore is a discrete fibration if and only if F is central. (The concept of **discrete fibration** between higher-dimensional internal groupoids is the obvious extension of the one-fold groupoid case: any of its induced n-fold arrows must be a pullback. In the situation at hand this gives precisely the condition on the $(n+1)$-cubes $\square(F, I) \to \mathsf{ab}(\square(F, I))$ mentioned above.) In the article [33], the authors study the Galois structure for n-fold groupoids in a semi-abelian category (cat^n-groups in Gp, for instance [56]) induced by the reflection

$$\begin{tikzcd} \mathsf{Gpd}^n\mathcal{A} \arrow[r, shift left=1.5, "\Pi_0^n"] \arrow[r, phantom, "\perp"] & \mathsf{Dis}^n\mathcal{A} \arrow[l, shift left=1.5, "\supset"] \end{tikzcd} \simeq \mathcal{A}$$

to $\mathcal{A}$ via the "connected components" functor to discrete n-fold groupoids. It turns out [33, Proposition 2.9] that the central extensions with respect to this reflection are again the regular epimorphisms of internal n-fold groupoids which are discrete fibrations. Hence an n-fold extension F in $\mathcal{A}$ is central relative to $\mathsf{Ab}\mathcal{A}$ if and only if the induced extension of n-fold groupoids $\eta_{\square_{i\in n}\mathrm{R}[f_i]}$ is central relative to $\mathcal{A}$.

3.11. **Higher Mal'tsev operations.** The isomorphisms (**Q**) determine "multiplications" or "compositions" of $(n-1)$-dimensional hyper-tetrahedra (or n-dimensional hyper-triangles) in an n-fold central extension, in the sense that any aggregation of hyper-tetrahedra in the shape of an n-fold diamond with a face missing "composes" to the missing face. That is to say, the composite morphism

$$p^I\colon \boxdot^I_{i\in n}\mathrm{R}[f_i] \xrightarrow{\langle 0,1\rangle} A \times \boxdot^I_{i\in n}\mathrm{R}[f_i] \xrightarrow{\cong} \square_{i\in n}\mathrm{R}[f_i] \xrightarrow{\mathrm{pr}_I} F_n$$

acts as a higher-dimensional Mal'tsev operation: the symmetries of $\square_{i\in n}\mathrm{R}[f_i]$ force it to satisfy certain higher-dimensional Mal'tsev laws.

For instance, in the two-dimensional case, $\delta = p^{\varnothing}(\alpha, \beta, \gamma)$ is the unique choice of δ which makes the diamond (**N**) "commute" (in which case one may think of δ as a composite $\gamma\beta^{-1}\alpha$), i.e., which is such that the projection $a = \mathrm{pr}_A(\alpha, \beta, \gamma, \delta)$ of the diamond $(\alpha, \beta, \gamma, \delta)$ on the direction A is zero. Furthermore, $p^{\varnothing}(\alpha, \alpha, \gamma) = \gamma$, since once $\alpha = \beta$ we have to take $\delta = \gamma$: there is no other choice possible for δ as the diamond has to commute, and $\delta = \gamma$ is a *valid* choice, one which does make the diamond commute, so it is the *uniquely valid* one.

In higher degrees the algebraic properties of the p^I still have to be further studied, but we can already say the following.

Proposition 3.12. *In a semi-abelian category, let F be an n-fold central extension with direction A. Then in any product diagram*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, shift right=1, "\ker \pi^I"'] & \square_{i\in n}\mathrm{R}[f_i] \arrow[l, shift right=1, dotted, "\mathrm{pr}_A"'] \arrow[r, shift right=1, "\pi^I"'] & \boxdot^I_{i\in n}\mathrm{R}[f_i] \arrow[l, shift right=1, dotted, "\iota^I"'] \arrow[r] & 0 \end{tikzcd}$$

induced by Theorem 3.8, the projection pr_A is an alternating sum

$$\sum_{J\subseteq n}(-1)^{|J|}\eta_{F_n}\circ \mathrm{pr}_J \qquad (\mathbf{S})$$

where $\mathrm{pr}_J\colon \square_{i\in n}\mathrm{R}[f_i] \to F_n$ sends a diamond to its J-face.

Proof. The idea behind the proof may be illustrated as follows in dimension two. (Here we let F be the double extension from Diagram (**J**) to simplify notations.) When the following calculation, in which we denote the equivalence classes in the quotient by respresentative elements, is made in the abelian object $\mathsf{ab}(\mathrm{R}[d]\square\mathrm{R}[c])$,

$$\begin{tikzcd} \gamma \arrow[r, dash, dotted] \arrow[d, dash, dotted, "1"'] & \beta \arrow[d, dash, dotted] \\ \delta \arrow[r, dash, dotted, "0"'] & \alpha \end{tikzcd} - \begin{tikzcd} \gamma \arrow[r, dash, dotted] \arrow[d, dash, dotted, "1"'] & \beta \arrow[d, dash, dotted] \\ \gamma \arrow[r, dash, dotted, "0"'] & \beta \end{tikzcd} + \begin{tikzcd} \beta \arrow[r, dash, dotted] \arrow[d, dash, dotted, "1"'] & \beta \arrow[d, dash, dotted] \\ \beta \arrow[r, dash, dotted, "0"'] & \beta \end{tikzcd} - \begin{tikzcd} \beta \arrow[r, dash, dotted] \arrow[d, dash, dotted, "1"'] & \beta \arrow[d, dash, dotted] \\ \alpha \arrow[r, dash, dotted, "0"'] & \alpha \end{tikzcd} = \begin{tikzcd} 0 \arrow[r, dash, dotted] \arrow[d, dash, dotted, "1"'] & 0 \arrow[d, dash, dotted] \\ \delta-\gamma+\beta-\alpha \arrow[r, dash, dotted, "0"'] & 0 \end{tikzcd}$$

we see that the result belongs to the kernel A of the projection $\mathsf{ab}\pi^{\varnothing}$. Indeed, the pullback $\mathrm{R}[d]\times_X\mathrm{R}[c]$ is preserved by the functor ab, and the projections to $\mathsf{ab}\,\mathrm{R}[d]$ and $\mathsf{ab}\,\mathrm{R}[c]$ send the above sum to zero. This gives us the morphism

$$\eta_{\square}\circ\mathrm{pr}_{\varnothing}-\eta_{\square}\circ\mathrm{pr}_{\{1\}}+\eta_{\square}\circ\mathrm{pr}_2-\eta_{\square}\circ\mathrm{pr}_1\colon \mathrm{R}[d]\square\mathrm{R}[c]\to A,$$

clearly a splitting for $\ker\pi^{\varnothing}$; hence by Lemma 1.3 it is the needed product projection.

For general n, let us again consider the commutative square (**R**)—which is a pullback by centrality of F—and the induced kernels:

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, equal] & \square_{i\in n}\mathrm{R}[f_i] \arrow[r, "\pi^I"] \arrow[d, "\eta_{\square}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \boxdot^I_{i\in n}\mathrm{R}[f_i] \arrow[d, "\eta_{\boxdot}"] \\ A \arrow[r, tail] & \mathsf{ab}(\square_{i\in n}\mathrm{R}[f_i]) \arrow[r, "\mathsf{ab}\pi^I"'] & \mathsf{ab}(\boxdot^I_{i\in n}\mathrm{R}[f_i]) \end{tikzcd}$$

Since $\mathsf{ab}(\boxdot^I_{i\in n}\mathrm{R}[f_i])=\boxdot^I_{i\in n}\mathsf{ab}\,\mathrm{R}[f_i]$ by Lemma 3.7, the abelian object A being the kernel of $\mathsf{ab}\pi^I$ implies that it is the direction of $\mathsf{ab}(\square(F,n\backslash I))$, which means $A=\bigcap_{i\in n}\mathrm{K}[\mathsf{ab}\,\mathrm{pr}^i_{\delta_{n\backslash I}(i)}]$. In order to define a morphism with codomain A, we now only need to define a morphism with codomain $\mathsf{ab}(\square_{i\in n}\mathrm{R}[f_i])$ which becomes zero when composed with the

$$\mathsf{ab}\,\mathrm{pr}^i_{\delta_{n\backslash I}(i)}\colon \mathsf{ab}(\mathop{\boxdot}^{I}_{i\in n}\mathrm{R}[f_i])\to\mathsf{ab}\,\mathrm{R}[f_i].$$

We shall use this procedure to define a splitting for $\ker\pi^I$ as an alternating sum, which will then automatically be the needed product projection by Lemma 1.3.

Recall the notation introduced in Subsection 3.3. Then, for any $J\subseteq n$, write $I\ominus J$ for the symmetric difference $(I\cup J)\backslash(I\cap J)$ of I and J, and put

$$\xi_{(F,I,J)}=\square(F,J)_n^{n\backslash(I\ominus J)}\circ\square(F,J)^n_{n\backslash(I\ominus J)}\colon \mathop{\square}_{i\in n}\mathrm{R}[f_i]\to\mathop{\square}_{i\in n}\mathrm{R}[f_i].$$

Now note that, given any element x of $\square_{i\in n}\mathrm{R}[f_i]$, the I-entry of $\xi_{(F,I,J)}(x)$ is x_J. Furthermore, after projecting in any direction $i\in n$ onto $\mathrm{R}[f_i]$, every morphism $\mathrm{pr}^i_{\delta_{n\backslash I}(i)}\circ\xi_{(F,I,J)}$ occurs twice: indeed

$$\mathrm{pr}^i_{\delta_{n\backslash I}(i)}\circ\xi_{(F,I,J)}=\mathrm{pr}^i_{\delta_{n\backslash I}(i)}\circ\xi_{(F,I,J\cup\{i\})}$$

when $i\notin J$. Hence the induced morphism

$$\sum_{J\subseteq n}(-1)^{|J|}\eta_{\square}\circ\xi_{(F,I,J)}\colon \mathop{\square}_{i\in n}\mathrm{R}[f_i]\to\mathsf{ab}(\mathop{\square}_{i\in n}\mathrm{R}[f_i])$$

satisfies the conditions required to factor over A, and its I-entry is precisely the needed formula (**S**), so that in particular it splits the kernel of π^I. $\square$

Note that the formula for the projection pr_A is independent of the chosen index $I \subseteq n$.

When $n = 1$, Proposition 3.12 reduces to a well-known property of (one-fold) central extensions (cf. [13]): if $f\colon X \to Z$ is central and $x_0, x_1\colon W \to X$ are such that $f \circ x_0 = f \circ x_1$, then they induce a unique morphism $x_1 - x_0\colon W \to A$ to the kernel A of f such that x_0 and $x_1 - x_0$ together determine x_1.

4. Torsors and centrality

We analyse the concept of torsor from the point of view of centrality of higher extensions. We prove that a truncated simplicial resolution of an object Z is a torsor of Z by an abelian object A if and only if the underlying extension is central with direction A (Theorem 5.9; one implication is Proposition 4.12, the other Proposition 5.8).

Let Z be an object and (A, ξ) a Z-module in a semi-abelian category $\mathcal{A}$. Recall from Subsection 1.22 that an n-torsor of Z by (A, ξ) is an augmented simplicial object $\mathbb{T}$ together with a simplicial morphism $\mathbb{t}\colon \mathbb{T} \to \mathbb{K}((A, \xi), n)$ such that

(T1) $\mathbb{t}$ is a fibration which is exact from degree n on;
(T2) $\mathbb{T} \cong \mathsf{Cosk}_{n-1}\mathbb{T}$;
(T3) $\mathbb{T}$ is a resolution.

4.1. **Why extensions?** Condition (T2) in the definition of n-torsor means that (the simplicial object-part $\mathbb{T}$ of) an n-torsor $(\mathbb{T}, \mathbb{t})$ *is* the $(n-1)$-truncated simplicial object $T = \mathsf{tr}_{n-1}\mathbb{T}$ (Subsection 1.16), in the sense that this is the only information $\mathbb{T}$ contains. Its initial object is $T_n = \mathbb{T}(n) = \mathbb{T}_{n-1}$ due to the shift in numbering mentioned in Remark 1.15. Condition (T3) means that the underlying n-fold arrow of T is an extension (Subsection 1.18).

4.2. **Why trivial actions?** We shall prove that for an n-torsor $(\mathbb{T}, \mathbb{t})$ of an object Z by a Z-module (A, ξ) in a semi-abelian category, the action ξ is trivial if and only if the induced one-fold extension

$$\langle \partial_i \rangle_i = l_T\colon T_n = \mathbb{T}_{n-1} \to \triangle(\mathbb{T}, n-1) = \mathsf{L}T$$

is central with respect to abelianisation. In other words, an n-torsor $(\mathbb{T}, \mathbb{t})$ has a trivial action if and only if

$$\Big[\bigcap_{i \in n} \mathrm{R}[\partial_i], \nabla_{T_n}\Big] = \Delta_{T_n} \qquad \text{or, equivalently,} \qquad \Big[\bigcap_{i \in n} \mathrm{K}[\partial_i], T_n\Big] = 0;$$

see Example 2.6. This extends Proposition 3.3 in [67] to higher dimensions. It also explains why only cohomology *with trivial coefficients* can ever classify higher central extensions: this commutator condition is part of the centrality by Lemma 2.10.

Proposition 4.3. *In a semi-abelian category, consider an object Z and a Z-module (A, ξ). For any n-torsor $(\mathbb{T}, \mathbb{t})$ of Z by (A, ξ), the following conditions are equivalent:*

(i) *the action ξ is trivial;*
(ii) *the one-fold extension $\langle \partial_i \rangle_i = l_T\colon T_n \to \triangle(\mathbb{T}, n-1) = \mathsf{L}T$ is central;*
(iii) *$\triangle(\mathbb{T}, n) \cong A \times \wedge^i(\mathbb{T}, n)$ for all $i \in n$.*

In any case, the kernel of $\langle \partial_i \rangle_i$ is A.

Proof. For any $i \in n$, Lemma 1.21 tells us that the square

$$\begin{tikzcd}
\triangle(\mathbb{T}, n) \arrow[r, "\widehat{\partial}_i"] \arrow[d, "\partial_i"'] & \bigwedge^i(\mathbb{T}, n) \arrow[d] \\
T_n \arrow[r, "\langle \partial_i \rangle_i"'] & \triangle(\mathbb{T}, n-1)
\end{tikzcd}$$

is a pullback. Note that all its arrows are extensions: the morphism ∂_i as any split epimorphism; $\langle \partial_i \rangle_i$ since $\mathbb{T}$ is a resolution; and $\widehat{\partial}_i$ either by the Kan property, which all simplicial objects in a semi-abelian category have, or as a pullback of $\langle \partial_i \rangle_i$. We see that the kernel of $\langle \partial_i \rangle_i$ is isomorphic to the kernel of $\widehat{\partial}_i$ (Lemma 1.2), and furthermore $\langle \partial_i \rangle_i$ is central if and only if so is $\widehat{\partial}_i$—indeed, central extensions are preserved and reflected by pullbacks of extensions along extensions. Since $(\mathbb{T}, \mathfrak{t})$ is an n-torsor, also the square

$$\begin{tikzcd}
\triangle(\mathbb{T}, n) \arrow[r, shift right, "\widehat{\partial}_i"'] \arrow[d, "\langle \varsigma, \partial_0^{n+1} \rangle"'] & \bigwedge^i(\mathbb{T}, n) \arrow[l, dotted, shift right] \arrow[d, "\partial_0^n"] \\
(A, \xi) \rtimes Z \arrow[r, shift right, "p"'] & Z \arrow[l, shift right, "s"']
\end{tikzcd}$$

is a pullback, by the exact fibration property. This already proves that the kernel of $\langle \partial_i \rangle_i$ is A (again Lemma 1.2). Note that a split epimorphism with abelian kernel represents a trivial action if and only if it is a product projection, if and only if it is a trivial extension, if and only if it is a central extension. Again using that central extensions are preserved and reflected by pullbacks of extensions along extensions we obtain the claimed result. □

Hence, from now on, we shall only have to consider torsors of Z by a trivial module (A, τ)—we called them *n-torsors of Z by A* in Subsection 1.22—and restrict our cohomology theory accordingly.

Remark 4.4. It is clear from the proof that the product decomposition (iii) is natural in $(\mathbb{T}, \mathfrak{t})$, i.e., any morphism of torsors will be compatible with the induced product decompositions.

Remark 4.5. Note that for any simplicial resolution $\mathbb{X}$, the kernel of any induced extension $\widehat{\partial}_i \colon \triangle(\mathbb{X}, n) \to \bigwedge^i(\mathbb{X}, n)$ is the direction A of the underlying n-fold extension X. If indeed an (n, i)-sub-horn $\widehat{x}_i$ of an n-cycle x in $\mathbb{X}$ is zero, then the i-face x_i which is missing in the horn must have boundary zero, so that x_i belongs to A. More formally, this also follows from Lemma 2.15 combined with Lemma 1.21, since $\langle \partial_i \rangle_i = l_X \colon \mathbb{X}_{n-1} \to \triangle(\mathbb{X}, n-1)$.

4.6. **Multiplying simplices in a torsor.** As explained in [28], given an n-torsor $(\mathbb{T}, \mathfrak{t})$ of Z by A and an integer $i \in n$, the isomorphism

$$\triangle(\mathbb{T}, n) \cong A \times \bigwedge^i(\mathbb{T}, n)$$

induces a multiplication or composition of the simplices in a horn to the "missing face" such that the thus completed n-cycle "commutes", in the sense that its projection on A is zero. So a horn may be considered as a *composable aggregation of simplices*—compare with the higher Mal'tsev structures p^I from Subsection 3.11. Indeed, we may simply use the morphism

$$m^i \colon \bigwedge^i(\mathbb{T}, n) \xrightarrow{\langle 0,1 \rangle} A \times \bigwedge^i(\mathbb{T}, n) \xrightarrow{\cong} \triangle(\mathbb{T}, n) \xrightarrow{\partial_i} T_n.$$

This composition of (n,i)-horns satisfies certain additional properties [28], of which for us the most important one is compatibility with degeneracies. From the axioms of torsor (the requirement that $\mathbb{t}\colon \mathbb{T} \to \mathbb{K}(Z,A,n)$ be a simplicial morphism) it follows that a degenerate n-cycle commutes. Hence any (n,i)-horn in $\mathbb{T}$ which *may be* completed to a degenerate n-cycle *has to be* completed this way, and hence composes to the i-face of this degenerate n-cycle.

For instance, in degree two, the left hand side $(2,1)$-horn

$$\begin{tikzcd}[column sep=small] & \cdot \arrow[dr, "\sigma_0\partial_0\alpha"] & \\ \cdot \arrow[ur, "\alpha"] & & \cdot \end{tikzcd} \qquad\qquad \begin{tikzcd}[column sep=small] & \cdot \arrow[dr, "\sigma_0\partial_0\alpha"] \arrow[d, phantom, "\sigma_1\alpha"] & \\ \cdot \arrow[ur, "\alpha"] \arrow[rr, dotted, "\alpha"'] & {} & \cdot \end{tikzcd}$$

fits into the right hand side degenerate 2-simplex $\sigma_1\alpha$. It follows by uniqueness that $m^1(\sigma_0\partial_0\alpha, \alpha) = \alpha$. Likewise, $m^0(\alpha,\alpha) = \sigma_0\partial_0\alpha$, etc.

4.7. **The exact fibration property.** Most of the fibration property (T1) of a torsor comes for free, since a regular epimorphism of simplicial objects in a regular Mal'tsev category is always a fibration [35, Proposition 4.4]. Given a simplicial morphism $\mathbb{t}\colon \mathbb{T} \to \mathbb{K}(Z,A,n)$ satisfying (T2) and (T3), already the $\mathbb{t}_i = \partial_0^{i+1}$ are regular epimorphisms for all $i \in n$, so it suffices to check the regularity of $\mathbb{t}_n$ and $\mathbb{t}_{n+1}$. Then there is the exactness, but this reduces to one square being a pullback—Diagram (**T**) for any $i \in n$—which in turn corresponds to a direction property.

Proposition 4.8. *Suppose that Z is an object and A is an abelian object in a semi-abelian category. Let $\mathbb{t}\colon \mathbb{T} \to \mathbb{K}(Z,A,n)$ be as in the definition of torsors, satisfying conditions* (T2) *and* (T3)*. Then for every i the square*

$$\begin{tikzcd} \triangle(\mathbb{T},n) \arrow[r, "\widehat{\partial}_i"] \arrow[d, "\langle \varsigma, \partial_0^{n+1}\rangle"'] & \Lambda^i(\mathbb{T},n) \arrow[d, "\partial_0^n"] \\ A \times Z \arrow[r, "\mathrm{pr}_Z"'] & Z \end{tikzcd} \tag{\textbf{T}}$$

is a pullback if and only if the induced morphism $\bigcap_i \mathrm{K}[\partial_i] \to A$ is an isomorphism. When this is the case, the simplicial morphism $\mathbb{t}$ is a fibration, exact from degree n on, so that $(\mathbb{T},\mathbb{t})$ is an n-torsor of Z by A.

Proof. Again, $\widehat{\partial}_i$ is a regular epimorphism by the Kan property. As in the proof of the previous proposition, the kernel of $\widehat{\partial}_i$ is $\mathrm{K}[\langle \partial_i \rangle_i] = \bigcap_i \mathrm{K}[\partial_i]$. Via Lemma 1.2 this already proves the equivalence.

Recall that every regular epimorphism of simplicial objects in a semi-abelian category is a fibration. When the above square (**T**) is a pullback (for any $i \in n$), the morphism ∂_0^n being regular epimorphic implies that also $\langle \varsigma, \partial_0^{n+1}\rangle$ is a regular epimorphism.

One degree up, the corresponding squares are automatically pullbacks: indeed, any comparison $\triangle(\mathbb{T}, n+1) \to \Lambda^i(\mathbb{T}, n+1)$ is an isomorphism by the axiom (T2) which tells us that every n-simplex in $\mathbb{T}$ is an n-cycle, as is any morphism

$$\widehat{\partial}_i\colon A^{n+1} \times Z \to \Lambda^i(\mathbb{K}(Z,A,n), n+1) = A^{n+1} \times Z.$$

In higher degrees there is nothing to be checked because $\mathbb{t}\colon \mathbb{T} \to \mathbb{K}(Z,A,n)$ is completely determined by the coskeleton construction. This implies that $\mathbb{t}$ is a regular epimorphism in all degrees, hence it is a fibration; moreover, this fibration is exact from degree n on. □

Thus we see that an n-torsor $(\mathbb{T}, \mathbb{t})$ of Z by A has an underlying n-fold extension of Z of which the direction is A. Furthermore, the squares $(\mathbf{T})$ are pullbacks, which means that $\triangle(\mathbb{T}, n) \cong A \times \wedge^i(\mathbb{T}, n)$. Note that the projection on $\wedge^i(\mathbb{T}, n)$ is $\widehat{\partial}_i$ and the projection on A is ς.

In what follows we shall prove that this condition is equivalent to the centrality of the underlying n-fold extension. Given an n-fold central extension T of Z by A, we construct a simplicial morphism $\mathbb{t}\colon \mathbb{T} = \mathsf{cosk}_{n-1}T \to \mathbb{K}(Z, A, n)$ such that the squares $(\mathbf{T})$ are all pullbacks. As explained above, this is enough for $(\mathbb{T}, \mathbb{t})$ to be an n-torsor. Furthermore, Remark 4.13 tells us that such a simplicial morphism $\mathbb{t}$ is uniquely determined, so that its existence is a *property* of T, not additional structure—as it should be, because centrality is also a property.

The other implication (which says that the underlying n-fold extension of an n-torsor is always central) will be treated in the following section.

4.9. **Embedding cycles into diamonds.** Up to symmetry of the diamond, there is a unique way a cycle may be embedded into a diamond using degeneracies to fill up missing faces. In degree two there is the morphism

$$s_2(\mathbb{X})\colon \triangle(\mathbb{X}, 2) \to \mathrm{R}[\partial_1]\square\mathrm{R}[\partial_0]\colon \langle x_0, x_1, x_2\rangle \mapsto \begin{tikzcd} \sigma_0\partial_1 x_0 \arrow[r, dotted, no head] & x_2 \\ x_0 \arrow[u, dotted, no head, "1"] \arrow[r, dotted, no head, "0"'] & x_1 \arrow[u, dotted, no head] \end{tikzcd}$$

which sends the left hand side (empty) triangle

$$\begin{tikzcd}[column sep=small] & \cdot \arrow[dr, "x_0"] & \\ \cdot \arrow[ur, "x_2"] \arrow[rr, "x_1"'] & & \cdot \end{tikzcd} \qquad\qquad \begin{tikzcd}[column sep=small] & \cdot & \\ \cdot \arrow[ur, "\sigma_0\partial_1 x_0"] \arrow[dr, "x_0"'] & & \cdot \arrow[ul, "x_2"'] \arrow[dl, "x_1"] \\ & \cdot & \end{tikzcd}$$

to the right hand side diamond. In degree three we have

$$s_3(\mathbb{X})\colon \triangle(\mathbb{X}, 3) \to \mathop{\square}_{i\in 3}\mathrm{R}[\partial_i]\colon \langle x_0, x_1, x_2, x_3\rangle \mapsto \begin{tikzcd}[column sep=small, row sep=small] & \sigma_0\partial_2 x_0 \arrow[rr, dotted, no head] \arrow[dd, dotted, no head] & & x_3 \arrow[dd, dotted, no head] \\ \sigma_0\partial_1 x_0 \arrow[ur, dotted, no head] \arrow[rr, dotted, no head] & & x_2 \arrow[ur, dotted, no head] & \\ & \sigma_1\partial_2 x_0 \arrow[rr, dotted, no head] & & \sigma_1\partial_2 x_1 \\ x_0 \arrow[uu, dotted, no head, "1"] \arrow[ur, dotted, no head, "2"] \arrow[rr, dotted, no head, "0"'] & & x_1 \arrow[uu, dotted, no head] \arrow[ur, dotted, no head] & \end{tikzcd}$$

and in general we have an inductive formula, as follows.

Notation 4.10 (Décalage)**.** Let $^{-}\mathbb{X}$ denote the **décalage** of $\mathbb{X}$, the augmented simplicial object constructed out of $\mathbb{X}$ by forgetting the lowest degree $\mathbb{X}_{-1}$ and the last face operators $\partial_n\colon \mathbb{X}_n \to \mathbb{X}_{n-1}$, so that $^{-}\mathbb{X}_n = \mathbb{X}_{n+1}$. We obtain a morphism of simplicial objects $\mathbb{d}\colon {}^{-}\mathbb{X} \to \mathbb{X}$ by $\mathbb{d}_n = \partial_{n+1}\colon {}^{-}\mathbb{X}_n = \mathbb{X}_{n+1} \to \mathbb{X}_n$.

Proposition 4.11. *For any simplicial object $\mathbb{X}$ in a semi-abelian category and any $n \geqslant 2$ there is a canonical natural inclusion*

$$s_n(\mathbb{X})\colon \triangle(\mathbb{X}, n) \to \mathop{\square}_{i\in n}\mathrm{R}[\partial_i].$$

Proof. Suppose $s_n(\mathbb{X})$ is defined for every $\mathbb{X}$ and natural in $\mathbb{X}$; we then construct a morphism $s_{n+1}(\mathbb{X})$, natural in $\mathbb{X}$. Given an $(n+1)$-cycle

$$x = \langle x_0, \dots, x_n, x_{n+1}\rangle \in \triangle(\mathbb{X}, n+1),$$

note that both $\widehat{x}_{n+1} = \langle x_0, \dots, x_n\rangle$ and

$$\widehat{y}_{n+1} = \langle \sigma_{n-1}\partial_0 x_{n+1}, \dots, \sigma_{n-1}\partial_{n-1} x_{n+1}, x_{n+1}\rangle,$$

where

$$y = \sigma_n x_{n+1} = \langle \sigma_{n-1}\partial_0 x_{n+1}, \dots, \sigma_{n-1}\partial_{n-1} x_{n+1}, x_{n+1}, x_{n+1}\rangle,$$

are in $\triangle(^-\mathbb{X}, n)$. The induction hypothesis gives us a pair of diamonds, and we define

$$s_{n+1}(\mathbb{X})(x) = \langle s_n(^-\mathbb{X})(\widehat{x}_{n+1}), s_n(^-\mathbb{X})(\widehat{y}_{n+1})\rangle \in \mathop{\square}_{i\in n} \mathrm{R}[^-\partial_i] \times \mathop{\square}_{i\in n} \mathrm{R}[^-\partial_i].$$

Now we only have to show that this pair does belong to $\square_{i\in n+1} \mathrm{R}[\partial_i]$, which means that $\partial_n(s_n(^-\mathbb{X})(\widehat{x}_{n+1})) = \partial_n(s_n(^-\mathbb{X})(\widehat{y}_{n+1}))$. This equality follows from the naturality of s_n, which makes the square

$$\begin{tikzcd}
\triangle(^-\mathbb{X}, n) \arrow[r, "\triangle(\mathbb{d}, n)"] \arrow[d, "s_n(^-\mathbb{X})"'] & \triangle(\mathbb{X}, n) \arrow[d, "s_n(\mathbb{X})"] \\
\square_{i\in n} \mathrm{R}[^-\partial_i] \arrow[r, "\mathbb{d}"'] & \square_{i\in n} \mathrm{R}[\partial_i]
\end{tikzcd}$$

commute, and the fact that $\triangle(\mathbb{d}, n)(\widehat{x}_{n+1})$ is equal to $\triangle(\mathbb{d}, n)(\widehat{y}_{n+1})$. Indeed, we have $\partial_n x_n = \partial_n x_{n+1}$ and

$$\partial_n x_i = \partial_i x_{n+1} = \partial_n \sigma_{n-1} \partial_i x_{n+1}$$

for every $i \in n$, so that the latter equality holds. This completes the construction of $s_{n+1}(\mathbb{X})$, which is evidently natural in $\mathbb{X}$. $\square$

The morphism $s_n(\mathbb{X})$ constructed above takes an element $x = \langle x_0, \dots, x_n\rangle$ of $\triangle(\mathbb{X}, n)$ and maps it to the diamond $s_n(\mathbb{X})(x)$ which has x_i on its i-entry and degeneracies elsewhere (see Subsection 3.2). Clearly, $s_n(\mathbb{X})$ restricts to morphisms

$$\dot{s}^i_n(\mathbb{X})\colon \Lambda^i(\mathbb{X}, n) \to \mathop{\boxdot}_{j\in n}^{i} \mathrm{R}[\partial_j],$$

natural in $\mathbb{X}$.

When we say that an $(n-1)$-truncated simplicial resolution is **central**, we mean that such is the underlying n-fold extension.

Proposition 4.12. *If, in a semi-abelian category, an $(n-1)$-truncated simplicial resolution central, then it is an n-torsor.*

Proof. Let $\mathbb{T}$ be a simplicial resolution and let A be the direction of $T = \mathsf{tr}_{n-1}\mathbb{T}$, considered as a trivial Z-module. We have to define a morphism of augmented simplicial objects $\mathbb{t}\colon \mathbb{T} \to \mathbb{K}(Z, A, n)$:

$$\begin{tikzcd}
\cdots & \triangle(\mathbb{T}, n+1) \arrow[r, shift left=2] \arrow[r, shift left=1] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[d, "\langle\langle \varsigma\circ\partial_i\rangle_i, \partial_0^{n+2}\rangle"'] & \triangle(\mathbb{T}, n) \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[d, "\langle \varsigma, \partial_0^{n+1}\rangle"] & \mathbb{T}_{n-1} \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[d, "\partial_0^n"] & \mathbb{T}_{n-2} \arrow[d, "\partial_0^{n-1}"] & \cdots & \mathbb{T}_0 \arrow[r, "\partial_0"] \arrow[d, "\partial_0"] & \mathbb{T}_{-1} \arrow[d, equal] \\
\cdots & A^{n+1} \times Z \arrow[r, shift left=2, "\partial_{n+1}\times 1_Z"] \arrow[r, shift left=1, "\mathrm{pr}_n \times 1_Z"'] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\mathrm{pr}_0 \times 1_Z"'] & A \times Z \arrow[r, shift left=2, "\mathrm{pr}_Z"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\mathrm{pr}_Z"'] & Z \arrow[r, shift left=2, equal] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, equal] & Z & \cdots & Z \arrow[r, equal] & Z
\end{tikzcd}$$

Such a simplicial morphism is completely determined by the choice of a suitable morphism $\varsigma\colon \triangle(\mathbb{T}, n) \to A$.

Consider, for $i \in n+1$, the commutative square of solid arrows

$$\begin{tikzcd}
0 \arrow[r, dotted] & A \arrow[r, dotted, shift right] \arrow[d, dotted, equal] & \triangle(\mathbb{T}, n) \arrow[l, dotted, shift right, "\varsigma"'] \arrow[r, shift right, "\widehat{\partial}_i"'] \arrow[d, "s_n(\mathbb{T})"'] & \Lambda^i(\mathbb{T}, n) \arrow[l, dotted, shift right] \arrow[r, dotted] \arrow[d, "\dot{s}^i_n(\mathbb{T})"] & 0 \\
0 \arrow[r, dotted] & A \arrow[r, dotted, shift right] & \square_{j\in n} \mathrm{R}[\partial_j] \arrow[l, dotted, shift right, "\mathrm{pr}_A"'] \arrow[r, shift right, "\pi^i"'] & \boxdot^i_{j\in n} \mathrm{R}[\partial_j] \arrow[l, dotted, shift right] \arrow[r, dotted] & 0
\end{tikzcd}$$

which embeds cycles into diamonds. By assumption, the kernel of π^i is A; moreover, by Theorem 3.8,

$$\mathop{\square}_{j\in n} \mathrm{R}[\partial_j] \cong A \times \mathop{\boxdot}_{j\in n}^{i} \mathrm{R}[\partial_j]$$

with π^i the projection on $\boxdot^i_{j\in n} \mathrm{R}[\partial_j]$. The square above is a pullback as a consequence of Lemma 1.2, since $\widehat{\partial}_i$ is a regular epimorphism by the extension property of T, and since the kernel of $\widehat{\partial}_i$ is A: cf. Remark 4.5. This implies that

$$\triangle(\mathbb{T}, n) \cong A \times \wedge^i(\mathbb{T}, n)$$

with $\widehat{\partial}_i$ the projection on $\wedge^i(\mathbb{T}, n)$. We may now complete the square with the dotted arrows.

We choose ς to be $\mathrm{pr}_A \circ s_n(\mathbb{T})\colon \triangle(\mathbb{T}, n) \to A$, the projection of $\triangle(\mathbb{T}, n)$ on A. We must prove that this does indeed give us a genuine morphism $\mathbb{t}\colon \mathbb{T} \to \mathbb{K}(Z, A, n)$; then the exact fibration property holds by Proposition 4.8, so that $(\mathbb{T}, \mathbb{t})$ is an n-torsor.

For this, we only need to check that all the squares in the diagram

$$\begin{tikzcd}
\triangle(\mathbb{T}, n+1) \arrow[r, shift left=2, "\partial_{n+1}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[d, "\langle\langle \varsigma\circ\partial_i\rangle_i, \partial_0^{n+2}\rangle"'] & \triangle(\mathbb{T}, n) \arrow[d, "\langle \varsigma, \partial_0^{n+1}\rangle"] \\
A^{n+1} \times Z \arrow[r, shift left=3, "\partial_{n+1}\times 1_Z"] \arrow[r, shift left=1, "\mathrm{pr}_n \times 1_Z"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "\mathrm{pr}_0\times 1_Z"'] & A \times Z
\end{tikzcd}$$

commute. This condition reduces to the commutativity of just one square, the one "on top":

$$\varsigma\circ\partial_{n+1} = (-1)^n \sum_{i=0}^{n} (-1)^i \varsigma\circ\partial_i. \qquad (\mathbf{U})$$

In fact the morphism $\langle\langle \varsigma\circ\partial_i\rangle_i, \partial_0^{n+2}\rangle$ is already the unique one that makes all the other squares commute. But this equality follows from Proposition 3.12, which tells us that the morphism ς itself may be considered as an alternating sum,

$$\varsigma = \sum_{J\subseteq n} (-1)^{|J|} \eta_{\mathbb{T}_{n-1}} \circ \mathrm{pr}_J \circ s_n(\mathbb{T}).$$

The equality $(\mathbf{U})$ may now be obtained via a direct calculation in the abelian object A. $\square$

Remark 4.13. A morphism of central truncated simplicial objects which keeps the terminal object and the direction fixed is automatically a morphism of torsors, as also the projections to the directions are compatible with the given morphism of truncated simplicial objects. That is to say, given n-torsors $(\mathbb{X}, \mathbb{x})$ and $(\mathbb{Y}, \mathbb{y})$ of Z by A of which the underlying n-fold arrows are central extensions and a simplicial morphism $\mathbb{f}\colon \mathbb{X} \to \mathbb{Y}$, always $\mathbb{y}\circ\mathbb{f} = \mathbb{x}$. To see this we only need to consider the diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "\varsigma_{\mathbb{X}}"] \arrow[d, equal] & \triangle(\mathbb{X}, n) \arrow[l, shift left] \arrow[r] \arrow[d, "\triangle(\mathbb{f}, n)"] & \wedge^i(\mathbb{X}, n) \arrow[d, "\wedge^i(\mathbb{f}, n)"] \\
A \arrow[r, shift left, "\varsigma_{\mathbb{Y}}"] & \triangle(\mathbb{Y}, n) \arrow[l, shift left] \arrow[r] & \wedge^i(\mathbb{Y}, n)
\end{tikzcd}$$

and note that $\varsigma_{\mathbb{Y}}\circ\triangle(\mathbb{f}, n) = \varsigma_{\mathbb{X}}$ by naturality of the product decompositions *induced by centrality*—see Remark 3.9 or the above proof.

5. The commutator condition

In general it is not clear how an isomorphism on the simplicial level may be extended to an isomorphism on the level of higher-dimensional diamonds. Therefore, to prove that every n-torsor is an n-fold central extension, we shall add an assumption on the base category: we ask that higher central extensions may be characterised in terms of binary Huq commutators. This happens in many cases, but thus far we have no precise characterisation of the categories which satisfy this condition.

It is proved in Section 9.1 of [34] that an n-fold extension of groups F is central with respect to Ab if and only if $\big[\bigcap_{i\in I}\mathrm{K}[f_i], \bigcap_{i\in n\setminus I}\mathrm{K}[f_i]\big] = 0$ for all $I \subseteq n$. The theory which we develop depends crucially on a similar characterisation of higher central extensions, valid in a sufficiently general context.

Definition 5.1. We say that an n-fold extension F in a semi-abelian category $\mathcal{A}$ is **algebraically central** when

$$\Big[\bigcap_{i\in I}\mathrm{K}[f_i], \bigcap_{i\in n\setminus I}\mathrm{K}[f_i]\Big] = 0$$

for all $I \subseteq n$. The category $\mathcal{A}$ satisfies the **commutator condition on n-fold central extensions** when the algebraically central n-fold extensions in $\mathcal{A}$ coincide with the **categorically central** ones, i.e., those which are central with respect to $\mathsf{Ab}\mathcal{A}$ in the Galois-theoretic sense used throughout the rest of the paper. We say that $\mathcal{A}$ satisfies the **commutator condition (CC)** when it satisfies the commutator condition on n-fold central extensions for all n.

5.2. **The case $n = 2$.** As explained in Example 2.7, in a semi-abelian category, the commutator condition on double central extensions is weaker than the Smith is Huq condition [59]. So far, the question whether or not this implies the commutator condition (in all degrees) is unanswered, as is the question whether or not (SH) and "(CC) for $n = 2$" are equivalent.

5.3. **Some examples.** It is shown in [34] that, next to the category of groups, also the categories Lie algebras and non-unitary rings have (CC). The examples of Leibniz and Lie n-algebras were treated in [25].

A general context where many examples may be found is given by those semi-abelian categories for which the abelianisation functor is protoadditive.

Example 5.4 (Protoadditive abelianisation)**.** Recall from [33] that a functor between semi-abelian categories is **protoadditive** when it preserves split short exact sequences

$$0 \longrightarrow K \overset{k}{\rightarrowtriangle} X \underset{f}{\overset{s}{\leftrightarrows}} Y \longrightarrow 0$$

(the cokernel f is split by some morphism s). It is explained in [32] that, when $\mathcal{A}$ is semi-abelian and the abelianisation functor $\mathsf{ab}\colon \mathcal{A} \to \mathsf{Ab}\mathcal{A}$ is protoadditive, the Huq commutator $[K, L]$ of two normal subobjects K, L of an object X is $\langle K \cap L\rangle = [K \cap L, K \cap L]$. This gives us

$$\Big[\bigcap_{i\in I}\mathrm{K}[f_i], \bigcap_{i\in n\setminus I}\mathrm{K}[f_i]\Big] = \Big[\bigcap_{i\in n}\mathrm{K}[f_i], \bigcap_{i\in n}\mathrm{K}[f_i]\Big] = \langle \mathsf{D}_{(n,Z)}F\rangle$$

for any n-fold extension F of Z and any $I \subseteq n$. Furthermore, by another result in [32], an n-fold extension is categorically central if and only if its direction is abelian; hence the commutator condition (CC) holds.

A non-trivial instance of this situation, mentioned in [32], is the variety of non-unitary rings that satisfy the law $abab = ab$. We now explain another special case, one which is less interesting from a cohomological point of view, but which does give a class of extreme examples.

Example 5.5 (Arithmetical categories)**.** Recall from [64] that an exact Mal'tsev category is **arithmetical** when every internal groupoid is an equivalence relation. We restrict ourselves to semi-abelian arithmetical categories, examples of which are the dual of the category of pointed sets, more generally, the dual of the category of pointed objects in any topos, and also the categories of von Neumann regular rings, Boolean rings and Heyting semi-lattices [3]. Since in such a category all abelian objects are trivial, the abelianisation functor is protoadditive, so that the commutator condition (CC) holds. Here an n-fold extension is categorically central if and only if its direction is zero, which means that the extension is a limit n-cube (or an isomorphism, when $n = 1$). Hence the interpretation of cohomology in terms of higher central extensions (Theorem 6.5) just means that any two n-fold central extensions of an object Z, i.e., limit n-cubes over Z, are connected, as $\mathrm{Centr}^n(Z, 0) \cong \mathrm{H}^{n+1}(Z, 0)$ is trivial—which is, however, not difficult to prove directly.

At the other end of the spectrum we find the context of abelian categories where (CC) also holds, and where the cohomology theory reduces to Yoneda's interpretation of $\mathrm{Ext}^n(Z, A)$.

Example 5.6 (Abelian categories)**.** In an abelian category all Huq commutators are zero while all extensions are categorically central, which already gives us (CC). On top of that the abelianisation functor is an identity, so that it is trivially protoadditive. Via the classical Dold–Kan theorem [26], an $(n-1)$-truncated simplicial resolution, considered as an extension of Z by A, corresponds to an exact sequence

$$0 \longrightarrow A \rightarrowtail C_{n-1} \longrightarrow C_{n-2} \longrightarrow \cdots \longrightarrow C_0 \twoheadrightarrow Z \longrightarrow 0.$$

Thus we regain Yoneda's interpretation of the Ext groups [70, 57]

$$\mathrm{Ext}^n(Z, A) \cong \mathrm{Centr}^n(Z, A) \cong \mathrm{H}^{n+1}(Z, A)$$

as a consequence of the results in Section 6. (A proof of the analogous result for torsors is given in [39]). The dimension shift is there because our numbering of the cohomology objects agrees with the classical non-abelian examples (groups, Lie algebras, etc.) rather than with the abelian case.

5.7. **From torsors to central extensions.** We are now ready to prove the equivalence between torsors and central extensions we need for our cohomological interpretation of higher centrality.

Proposition 5.8. *In a semi-abelian category, the underlying n-fold extension of an n-torsor is algebraically central.*

Proof. Let $(\mathbb{T}, \mathfrak{t})$ be an n-torsor of Z by A with underlying n-fold extension T. Then already the commutator $[T_n, A]$ is zero: by Proposition 4.3, since A is a trivial Z-module, and by Example 2.6.

Now suppose that $\varnothing \neq I \subsetneq n$ and consider $x\colon X = \bigcap_{j\in I} \mathrm{K}[\partial_j] \to T_n$ and $y\colon Y = \bigcap_{j\in n\setminus I} \mathrm{K}[\partial_j] \to T_n$. We are to show that x and y Huq-commute, so that T is algebraically central.

Without any loss of generality we may assume that $\partial_0 y = 0$. (If not, reverse the roles of x and y.) Let i be the smallest element of I. Then $\partial_i x = 0$ and $\partial_j y = 0$ for all $j < i$, and the boundaries

$$\partial\sigma_{i-1}x = \langle \sigma_{i-2}\partial_0 x, \dots, \sigma_{i-2}\partial_{i-2}x, x, x, 0, \sigma_{i-1}\partial_{i+1}x, \dots, \sigma_{i-1}\partial_{n-1}x \rangle$$

and

$$\partial\sigma_i y = \langle 0,\ldots,0,0,y,y,\sigma_i\partial_{i+1}y,\ldots,\sigma_i\partial_{n-1}y\rangle$$

of $\sigma_{i-1}x$ and $\sigma_i y$ determine (n,i)-horns

$$\overline{x} = (\widehat{\partial\sigma_{i-1}x})_i = \langle\sigma_{i-2}\partial_0 x,\ldots,\sigma_{i-2}\partial_{i-2}x,x,0,\sigma_{i-1}\partial_{i+1}x,\ldots,\sigma_{i-1}\partial_{n-1}x\rangle$$

and

$$\overline{y} = (\widehat{\partial\sigma_i y})_i = \langle 0,\ldots,0,0,y,\sigma_i\partial_{i+1}y,\ldots,\sigma_i\partial_{n-1}y\rangle$$

in $\mathbb{T}$ which Huq-commute with each other. That is to say, there is a morphism $\widehat{\varphi}_i$ such that the diagram

$$\begin{tikzcd}
X \arrow[dr, "\langle 1_X,0\rangle"'] \arrow[drr, bend left, "\overline{x}"] & & \\
& X\times Y \arrow[r, dashed, "\widehat{\varphi}_i"] & \bigwedge^i(\mathbb{T},n) \\
Y \arrow[ur, "\langle 0,1_Y\rangle"] \arrow[urr, bend right, "\overline{y}"'] & &
\end{tikzcd}$$

is commutative, namely, the morphism determined by the family

$$\varphi_j = \begin{cases} \overline{x}_j\circ \mathrm{pr}_X & \text{if } j<i \text{ (so that } j\notin I) \\ \overline{x}_j\circ \mathrm{pr}_X & \text{if } j>i \text{ and } j-1\notin I \\ \overline{y}_j\circ \mathrm{pr}_Y & \text{if } j>i \text{ and } j-1\in I; \end{cases}$$

note that indeed $\partial_k\circ\varphi_j = \partial_{j-1}\circ\varphi_k$ for all $k<j$ such that $i\notin\{j,k\}$. Furthermore, being induced by degeneracies, $\overline{x}$ and $\overline{y}$ compose to the face left out—see Subsection 4.6—so that the diagram

$$\begin{tikzcd}
X \arrow[dr, "\langle 1_X,0\rangle"'] \arrow[drr, bend left, "\overline{x}"] \arrow[drrr, bend left, "x"] & & & \\
& X\times Y \arrow[r, "\widehat{\varphi}_i"] & \bigwedge^i(\mathbb{T},n) \arrow[r, "m^i"] & T_n \\
Y \arrow[ur, "\langle 0,1_Y\rangle"] \arrow[urr, bend right, "\overline{y}"'] \arrow[urrr, bend right, "y"'] & & &
\end{tikzcd}$$

is commutative, and x and y Huq-commute. □

Thus we proved that, in a semi-abelian category, any categorically central truncated simplicial resolution gives a torsor (Proposition 4.12) and any torsor gives an algebraically central truncated simplicial resolution (Proposition 5.8). To complete the circle, what we need is precisely the commutator condition (CC).

Theorem 5.9. *In a semi-abelian category which satisfies the commutator condition* (CC)*, an augmented simplicial object is an n-torsor if and only if its underlying n-fold arrow is an n-fold central extension.* □

6. Cohomology classifies higher central extensions

In this last section we prove our main result, Theorem 6.5, which states that, for any object Z, any abelian object A, and any $n\geqslant 1$, we have a natural group isomorphism

$$\mathrm{H}^{n+1}(Z,A)\cong \mathrm{Centr}^n(Z,A).$$

To do so, we only need to use the results of the previous sections and establish a natural bijection between the underlying sets.

We already know that, for truncated simplicial resolutions, being a torsor is equivalent to being central. Now we have to explain how any (central) extension may be approximated by a truncated augmented simplicial object so that the two types of objects may be compared. In fact, any equivalence class of central extensions of Z by A contains a truncated simplicial object.

6.1. **Simplicial approximation of higher-dimensional arrows.** Using a classical Kan extension argument, every n-dimensional arrow may be universally approximated by an $(n-1)$-truncated simplicial object. Indeed, the functor from Subsection 1.16

$$\mathsf{arr}_n = \mathsf{Fun}(-, \mathsf{a}_n)\colon \mathsf{S}^n\mathcal{A} \to \mathsf{Arr}^n\mathcal{A}$$

has a right adjoint

$$\mathsf{s}_n = \mathsf{Ran}_{\mathsf{a}_n}(-)\colon \mathsf{Arr}^n\mathcal{A} \to \mathsf{S}^n\mathcal{A}$$

which takes an n-fold arrow $F\colon (2^n)^{\mathrm{op}} \to \mathcal{A}$ and maps it to the right Kan extension

$$\mathsf{Ran}_{\mathsf{a}_n}F\colon (\Delta_n^+)^{\mathrm{op}} \to \mathcal{A}$$

of F along the functor $\mathsf{a}_n\colon 2^n \to \Delta_n^+$.

Proposition 6.2. *Let $\mathcal{A}$ be a regular category. Then for all $n \geqslant 1$, the functors arr_n and s_n preserve n-fold extensions.*

Proof. Since an $(n-1)$-truncated simplicial object is by definition an extension if and only if so is its underlying n-fold arrow, the functor arr_n preserves and reflects extensions. The case of s_n for $n \geqslant 2$ is more complicated: given an n-fold arrow F, the Kan extension $G = \mathsf{s}_nF = \mathsf{Ran}_{\mathsf{a}_n}F$ is computed pointwise as a limit (see e.g. [58]). For instance, a double extensions such as ($\mathbf{J}$) has

$$\mathrm{R}[d] \times_X \mathrm{R}[c] \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X \longrightarrow Z$$

for its simplicial approximation. It is easily seen that also in general, G has $F_n \to F_0$ as its augmentation. In higher degrees, the objects of G are all subobjects of powers of F_n. The $(n-1)$-truncated simplicial object G is an extension when such is F: the objects of G contain certain configurations built up out of n-simplices in F, and if in such a configuration certain n-simplices are missing, the resulting holes may be filled by using the extension property of F. □

We now return to the semi-abelian context and prove that then these adjunctions also preserve centrality. These two results together extend Proposition 5.1 in [67] to higher degrees and beyond the case of central extensions.

Proposition 6.3. *Let $\mathcal{A}$ be a semi-abelian category. For all $n \geqslant 1$, the functors arr_n and s_n preserve centrality. Furthermore, both functors preserve the direction of a central extension.*

Proof. Let F be an n-fold central extension. Then the direction $A = \bigcap_{i\in n} \mathrm{K}[f_i]$ of F (Lemma 2.15) induces a short exact sequence

$$0 \longrightarrow A \rightarrowtail F_n \xrightarrow{l_F} \mathsf{L}F \longrightarrow 0$$

where $\mathsf{L}F$ is the limit described in Subsection 1.10 and l_F the comparison morphism. Let $G = \mathsf{arr}_n\mathsf{s}_nF$ be the n-fold arrow underlying the simplicial approximation of F, and $u\colon G \to F$ the counit of the adjunction at F; consider the induced limit $\mathsf{L}G$

and the comparison morphism l_G. By construction of G out of F we have that the square

$$\begin{tikzcd} G_n \arrow[r, "u_n"] \arrow[d, "l_G"'] \arrow[dr, phantom, "\lrcorner", very near start] & F_n \arrow[d, "l_F"] \\ \mathsf{L}G \arrow[r, "\mathsf{L}u"'] & \mathsf{L}F \end{tikzcd} \qquad (\mathbf{V})$$

is a pullback. Via Lemma 2.10, this already implies that l_G is a central extension when F is central; moreover, we have $A = \bigcap_{i\in n} \mathrm{K}[g_i]$ by Lemma 1.2, so that the functor s_n preserves directions. By Theorem 3.8, now we only need to prove that any square

$$\begin{tikzcd} \square_{i\in n} \mathrm{R}[g_i] \arrow[r] \arrow[d, "\pi^I_G"'] & \square_{i\in n} \mathrm{R}[f_i] \arrow[d, "\pi^I_F"] \\ \boxdot^I_{i\in n} \mathrm{R}[g_i] \arrow[r] & \boxdot^I_{i\in n} \mathrm{R}[f_i] \end{tikzcd}$$

is a pullback. This is the case, since any "missing face in a diamond in G" is completely determined by its boundary and a filling in F: this is precisely what the pullback property of the square (**V**) gives us when combined with the pullback (**O**). $\square$

6.4. **The equivalence between central extensions and torsors.** By Proposition 6.3 the functors arr_n and s_n not only preserve central extensions, but also the directions of those central extensions. Hence for any object Z and any abelian object A, these functors (co)restrict to an adjunction

$$\begin{tikzcd} \mathsf{d}^{-1}_{(n,Z)}A \arrow[r, shift left, "\mathsf{arr}_n"] \arrow[r, phantom, "\perp"] & \mathsf{D}^{-1}_{(n,Z)}A \arrow[l, shift left, "\mathsf{s}_n"] \end{tikzcd}$$

where

$$\mathsf{d}_{(n,Z)} \colon \mathsf{SCExt}^n_Z\mathcal{A} \to \mathsf{Ab}\mathcal{A}$$

is the restriction of $\mathsf{D}_{(n,Z)}$ to those $(n-1)$-truncated simplicial objects which, as n-fold arrows, are central extensions. Taking connected components gives a bijection of sets (cf. Remark 5.2 in [67]), which by Theorem 5.9 takes the shape

$$\pi_0 \operatorname{Tors}^n(Z, A) \cong \pi_0(\mathsf{d}^{-1}_{(n,Z)}A) \cong \pi_0(\mathsf{D}^{-1}_{(n,Z)}A)$$

when also the commutator condition (CC) holds. Thus we see that the underlying sets of the abelian groups

$$\mathrm{H}^{n+1}(Z, A) = \operatorname{Tors}^n[Z, A] = \pi_0 \operatorname{Tors}^n(Z, A)$$

and $\operatorname{Centr}^n(Z, A) = \pi_0(\mathsf{D}^{-1}_{(n,Z)}A)$ are isomorphic. Since both

$$\mathrm{H}^{n+1}(Z, -) \quad \text{and} \quad \operatorname{Centr}^n(Z, -) \colon \mathsf{Ab}\mathcal{A} \to \mathsf{Set}$$

are product-preserving functors (Proposition 2.19), they lift to isomorphic functors $\mathsf{Ab}\mathcal{A} \to \mathsf{Ab}$, which gives us

Theorem 6.5. *Let Z be an object and A an abelian object in a semi-abelian category satisfying the commutator condition* (CC). *Then for every $n \geqslant 1$ we have an isomorphism $\mathrm{H}^{n+1}(Z, A) \cong \operatorname{Centr}^n(Z, A)$.* $\square$

Thus—see the end of the introduction—we obtain the claimed duality between homology and cohomology.

Theorem 6.6. *Consider $n \geqslant 1$ and let Z be an object in a semi-abelian category $\mathcal{A}$ which satisfies the commutator condition* (CC). *Then*

$$\mathrm{H}_{n+1}(Z, \mathsf{Ab}\mathcal{A}) = \lim \mathsf{D}_{(n,Z)} \qquad \textit{and} \qquad \mathrm{H}^{n+1}(Z, A) = \pi_0(\mathsf{D}^{-1}_{(n,Z)} A),$$

where A is any abelian object in $\mathcal{A}$. When, in particular, $\mathcal{A}$ is monadic over Set*, the homology and the cohomology are comonadic Barr–Beck (co)homology with respect to the canonical comonad on $\mathcal{A}$.* □

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