The Two-dimensional Fourth Proportional and a Functional Approach to the Geometric Product in a Euclidean Vector Space

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Abstract

This article considers both the scalar product $a \cdot b$ and the exterior product $a \wedge b$ of two vectors $a,\ b$ in a two-dimensional real Euclidean vector space V as functions from V to V. This functional approach to the Clifford Geometric Product $ab = (a \cdot b) \ + \ (a \wedge b)$ arises naturally from the ancient geometric concept of Fourth Proportional.

1 Introduction.

Similar equally oriented plane triangles, coordinate-free products and ratios of vectors in a two-dimensional real Euclidean vector space V, and certain linear operators from V in itself are connected concepts (see Definitions 3, 10 and Theorems 4, 5, 6).

We might think that the algebra of complex numbers $\mathbb{C}$ and its classical correspondence with $\mathbb{R}^2$ would be the proper setting to connect these concepts. However, while a ratio in $\mathbb{C}$ is rotationally invariant, a product is not: if $a = re^{i\alpha}, a' = r'e^{i\alpha'}$, their complex product is $rr'e^{i(\alpha+\alpha')}$, but the product of their rotation by any angle β is $rr'e^{i(\alpha+\alpha'+2\beta)}$.

This article shows that the proper algebra to connect the above mentioned concepts is the Clifford Algebra (also called Geometric Algebra) associated to the real Euclidean vector space V, and denoted by $\mathcal{C}l(V)$.

In particular, the correspondence between similar equally oriented plane triangles, coordinate-free products and ratios of vectors, and certain endomorphisms on V, allows a functional construction of $\mathcal{C}l(V)$.

Throughout this paper V is assumed to be a two-dimensional real Euclidean vector space, whose scalar product will be denoted by $a \cdot b$.

I here restrict my approach to an Euclidean plane because there, in a two-dimensional real Euclidean vector space, one finds the inductive step of advancing to higher dimensions. The cases involving three or higher dimensions require further development, and will be addressed in separate papers. Future papers will also address some applications to Differential Calculus, Integral Calculus, and Functional analysis.

We may ask why we should bother about Clifford multiplication and division of vectors. Here are some possible answers:

- both product and division are invariant under rotations (see Theorem 6), providing a coordinate-free algebra;
- Clifford Algebra can be associated to vector spaces of any dimension.

If you consider the extension from natural numbers to rational numbers an improvement, you can probably perceive how much the mathematical language gains by achieving a coordinate-free notion of product and division between vectors in Euclidean vector spaces of any dimension. The relation between the field of rational numbers $\mathbb{Q}$ and the Clifford Geometric Algebra is more than a simple analogy. In fact, I will show how the concept of the "Fourth Proportional", whose ancient analog was used in the sixteenth century by F. Viète to introduce our actual abstract symbolism for rational numbers, is a crucial instrument for the interpretation of division between vectors.

Since Clifford Algebra is a well established topic (see for instance [13], [11] and [8]), we may also wonder why we should bother about a functional approach to it.

As V is not algebraically closed under the geometric product, we have to construct a minimal set containing V, and closed under the geometric product. In order to avoide a merely formal axiomatic approach, many texts (see for instance [7], [6], [9], [5], [4],[10]) introduce initially the Grassmann algebra $\bigwedge V$ (with its geometric interpretation as the "algebra of linear oriented subspaces of V"), and only afterwards they define the geometric product through $ab = (a \cdot b) + (a \wedge b)$, finally showing that $\bigwedge V$ is closed under such product and so identifying $\mathcal{C}l(V)$ and $\bigwedge V$. Though these recent efforts are commendable in many respects, they have never completely satisfied me because:

- Clifford invented his product before becoming acquainted with Grassmann's work (see [1]);
- after reading [2], I believe that Clifford approach was very different from Grassmann's.

Clifford's article [1] on geometric algebra was published in 1878 and Clifford died on March 3, 1879; I think he simply had no time to clarify his discovery as one would expect from a talented lecturer such as Clifford was (read Bertrand Russell's preface to [2]).

In order to show how tied are the Clifford's approach and the one I propose here, I present some excerpts from his writings.

2 The Dual Nature of Numerical Symbols.

"... formulæ of arithmetic and algebra are capable of a double interpretation." [2, pp. 93].

2.1 Integers as Functions.

The primitive meaning of natural numbers comes from "... the process of counting things which are separate from one another..." [2, pp. 88].

Let us consider dots as our "separate objects". So 1 counts •, 2 counts • •, 3 counts • • • and so on. Such interpretation applies also when summing. For instance,

$$2 + 3 = 5$$

can be viewed as a juxstaposition of separate objects:

$$\bullet\ \bullet\ +\ \bullet\ \bullet\ \bullet\ =\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet$$

However, such interpretation of natural numbers is not equally comfortable with multiplication; for instance, what would it mean

$$\bullet\ \bullet\ \times\ \bullet\ \bullet\ \bullet\ ?$$

Nevertheless, in natural language there is no ambiguity:

$$\bullet\ \bullet \text{ times } \bullet\ \bullet\ \bullet = \text{ two times } \bullet\ \bullet\ \bullet = \text{ twice } \bullet\ \bullet\ \bullet = \begin{matrix}\bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet\end{matrix} = 6$$

This trivial fact reveals that the symbol 2 in the product 2×3 can play a different role from that of symbol 3, as Clifford remarked in his fundamental paper [1, p. 351]: "now there are two sides to the notion of a product. When we say $2 \times 3 = 6$, we may regard the product 6 as a number derived from the numbers 2 and 3 by a process in which they play similar parts; or we may regard it as derived from the number 3 by the operation of doubling. In the former view 2 and 3 are both numbers; in the latter view 3 is a number but 2 is an operation, and the two factors play very distinct parts."

Interpreting the symbol 2 as the function of "doubling", we have

$$2 \times 3 = \text{double}(3) = \text{double}(\bullet\ \bullet\ \bullet) = \begin{matrix}\bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet\end{matrix} = 6$$

Number 3 can be considered as a function as well: that of "tripling",

$$2 \times 3 = \text{double}(3) = \text{double}(\text{triple}(x)) = \text{sextuple}(x) = 6\ ,$$

whatever the object x is.

This elementary situation is an example of how naturally a same mathematical symbol can have different interpretations. New ideas have often been hampered by misunderstanding the double nature of mathematical symbols. You only need think of Fourier's Series: his calculations treated a series of functions as a functional, but Fourier never formalized this approach, and his techniques were initially rejected.

2.2 Numbers and Lengths.

As long as numbers are conceived counting separate objects "the division of one number by another is only possible when the first number happens to be a multiple of the other" [2, p. 98].

Another interpretation of numbers is counting "things which are not separate but all in one piece" [2, pp. 88] as if they measure length; in that case 1 counts for $\vdash\!\!-\!\!\dashv$, 2 counts for $\vdash\!\!-\!\!+\!\!-\!\!\dashv$, 3 counts for $\vdash\!\!-\!\!+\!\!-\!\!+\!\!-\!\!\dashv$ and so on.

Accordingly, any number "may be divided by any number we like ; that is to say, any length (quantities of all kinds being represented by lengths) may be divided into any given number of equal parts" [2, p. 98].

If we interpret numbers as measuring lengths of linear objects, we can still consider the product 2×3 from the functional point of view:

$$2 \times 3 = \text{double}(\vdash\!\!-\!\!+\!\!-\!\!+\!\!-\!\!\dashv) = \vdash\!\!-\!\!+\!\!-\!\!+\!\!-\!\!+\!\!-\!\!+\!\!-\!\!\dashv = 6$$

That is, numbers can also be viewed as functions stretching (or squeezing) lengths. As before:

$$2 \times 3 =^{\text{double}}_{\text{stretch}} (3) =^{\text{double}}_{\text{stretch}} \left(^{\text{triple}}_{\text{stretch}} (x)\right) =^{\text{sextuple}}_{\text{stretch}} (x) = 6 \ ,$$

whatever the object x is.

2.3 Rational Numbers as Functions: the 4th Proportional.

Clifford recalled in [2, p. 98] the functional interpretation of a fraction $\frac{a}{b}$ as well. For instance, the function $\frac{3}{2}(\cdot)$ is a certain operation of stretching which converts length 2 into length 3; by applying the same function $\frac{3}{2}(\cdot)$ on x, it will produce the fourth quantity $y = \dfrac{3}{2}(x)$, that is traditionally called the "4th proportional" because it can also be expressed by the proportion $2 : 3 = x : y$.

Figure 1 shows that "the problem of finding the 4th proportional y is reduced to that of drawing two triangles of the same shape" [2, p. 99], namely $\overset{\triangle}{y0a}$ and $\overset{\triangle}{x0b}$ (where $a = 3$, and $b = 2$).

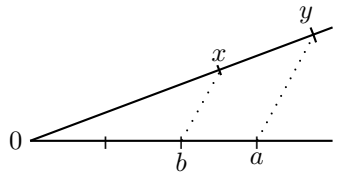

Figure 1: The traditional 4th Proportional y.

The angle between lines $\overline{0x}$ and $\overline{0b}$ is irrelevant for determining the length y (x given), as Figure 2 shows.

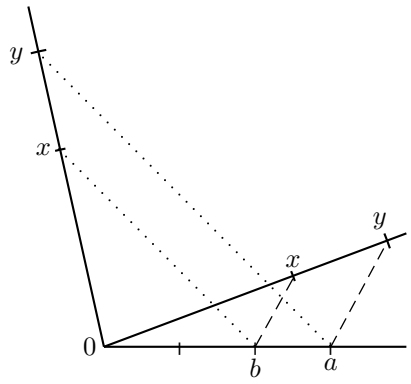

Figure 2: The traditional 4th Proportional y is independent from the angle between line $\overline{0x}$ and line $\overline{0b}$.

2.4 The Ratio of Two Vectors as a Linear Function and the Two-dimensional 4th Proportional.

If we mind the angle between lines $\overline{0x}$ and $\overline{0b}$ in Figure 2, we can consider x, b, a and y as vectors (all lying on the same plane); in this case the vector ratio a/b has a functional interpretation too:

> *the function $(a/b)(\cdot)$ is a certain operation of stretching (or squeezing) and* **turning** *which converts vector b into vector a; by applying the same function $(a/b)(\cdot)$ on vector x, it will produce the fourth quantity $y = (a/b)(x)$, such that the triangle $y\overset{\triangle}{0}x$ is similar to the triangle $a\overset{\triangle}{0}b$, and both triangles have the same orientation* (see Figure 3). (1)

It is reasonable to call the vector $y = (a/b)(x)$ the **"two-dimensional 4th proportional"** vector.

3 From Geometry to Algebra

In this paragraph, after some preliminary considerations, we use the geometric functional interpretation (1) of the vector ratio a/b applied on vector b to algebraically define a/b as a linear operator from V to V. Then (in Paragraph 4) we will write the ratio a/b as an associative product.

Remark 1. *In a vector space we usually write both the product between scalars and the product between scalars and vectors by juxtaposing factors; this cause*

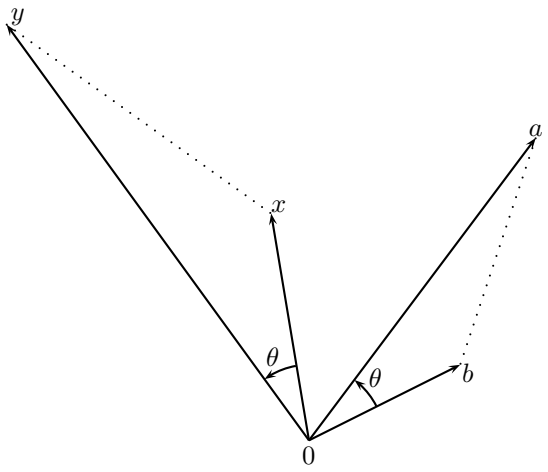

Figure 3: The two-dimensional 4th Proportional vector y.

no harm because in any vector space we have that

$$\alpha(\beta v) = (\alpha\beta)v \quad ,$$

where α, β are scalars and v is a vector. This kind of "associativity" condition allows to write without ambiguity $\alpha\beta v$.

Analogously, we write the geometric product between vectors (or any possible element in $\mathcal{C}l(V)$) by simply juxtaposing factors. Such notation will not cause any harm because we will verify (see Proposition 5 and Proposition 6) that there exist a monomorphism (an injective morphism) $J_{\mathbb{R}}$ from the algebra $\mathbb{R}$ to the real algebra $\mathcal{C}l(V)$, and a linear injection J_V from the real vector space V to the real linear space $\mathcal{C}l(V)$ such that

$$J_{\mathbb{R}}(\alpha)J_V(v) = J_V(\alpha v) \quad .$$

Therefore, instead of writing the Clifford product of two vectors $x, y \in V$ as $J_V(x)J_V(y)$, we simply write xy.

Definition 1. For each vector $x \in V$, $xx = \ x \cdot x$.

A consequence of Definition 1 is the following

Proposition 1. *For each nonzero vector $x \in V$, $1/x = \dfrac{1}{x \cdot x}\ x \in V$.*

Remark 2. *Definition 1 and Proposition 1 imply that the Clifford Algebra associated to a one-dimensional real vector space is isomorphic to $\mathbb{R}$.*

Remark 3. *Algebraically, we prefer to consider a quotient a/b simply as the product of a by the inverse of b. So, we write ab^{-1} instead of a/b.*

Remark 4. *"... it may be said that one of the main differences between the classical and the modern concepts of analysis is that, in classical mathematics, when one writes $f(x)$, f is visualized as "fixed" and x as "variable," whereas nowadays both f and x are considered as "variables" (and sometimes it is x which is fixed, and f which becomes the "varying" object)." [3, pag.1]*

Moreover, here we accept the dual nature of algebraic symbols: an object can be a function as well. For instance, if we consider the object x as fixed, we could interpret the value $f(x)$ as the action (on the left) of the "function" x on the "object" f, being now the function f as an object; so, in this case we should write $(f)x$, instead. That is why in what follows we will simply use as functional notation

$$fx \qquad \textit{instead of} \qquad f(x),$$

reserving brackets only to distinguish different names of functions (or objects, if you like).

3.1 The vector ratio ab^{-1} defined by its action on b: the orthogonal decomposition of a along b.

Following (1), the trivial algebraic product

$$(ab^{-1})b = a \tag{2}$$

can be interpreted functionally as getting vector a from vector b by applying the function (ab^{-1}) on b. In particular, we might consider (ab^{-1}) as the function of stretching and turning b until it becomes a (see Figure 4).

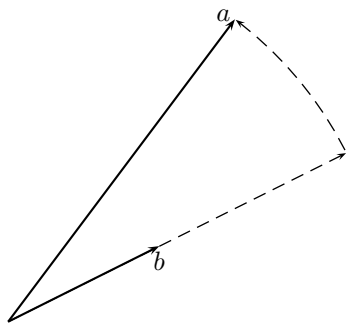

Figure 4: The function (ab^{-1}) as stretching and turning.

In V (a real Euclidean vector space) we do not dispose rotations as a primitive tool, so we describe getting a from b differently: (ab^{-1}) stretches b until it becomes the orthogonal projection v of a along b, and adds the corresponding

component of a, orthogonal to b (as shown in Figure 5). We will then write such decomposition as an associative product.

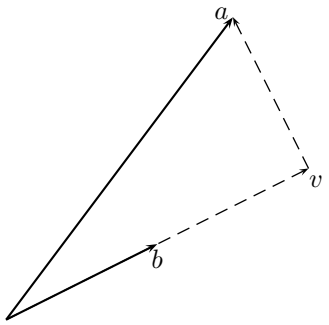

Figure 5: The orthogonal decomposition of a with respect to b.

Through elementary calculations we obtain

$$v = \frac{a \cdot b}{b \cdot b} b \ .$$

By Proposition 1, we can write $v = (a \cdot b^{-1})b = (a \cdot b)b^{-1}$. Expression (2) becomes

$$\begin{array}{rl} (ab^{-1})b & = (a \cdot b^{-1})b \ + \ \big[a - (a \cdot b^{-1})b\big] = \\ & = (a \cdot b^{-1})b \ + \ \big[a(b^{-1} \cdot b) - b^{-1}(a \cdot b)\big] \ . \end{array} \tag{3}$$

Definition 2. Let V be a two-dimensional real Euclidean vector space; for any fixed $u, w \in V$, $(u \wedge w)$ is a function from V to V whose value on $x \in V$ is the vector

$$(u \wedge w)x = u(w \cdot x) \ - \ w(u \cdot x) \ . \tag{4}$$

Remark 5. *The symbol $\wedge$ is generally used for the exterior antisymmetric Grassmann product. Here, instead, we use it to name the function generated by $u, w \in V$ as defined above. This ambivalence of symbols is not new in mathematics. Let simply cite Poincaré's statement in [12, pag.34]:*

"mathematics is the art of giving the same name to different things".

Of course, the function $u \wedge w$ behaves algebraically as the exterior Grassmann product.

Proposition 2. *Let V be a two-dimensional real Euclidean vector space; for any $u, w, x, w_1, w_2, u_1, u_2 \ \in \ V$ and $\alpha \in \mathbb{R}$, we have*

$$
\begin{array}{c}
(u \wedge w)x = -\left[(w \wedge u)x\right] \\
\left[u \wedge (w_1 + w_2)\right] x = \left[(u \wedge w_1)x\right] \ + \ \left[(u \wedge w_2)x\right] \\
\left[(u_1 + u_2) \wedge w\right] x = \left[(u_1 \wedge w)x\right] \ + \ \left[(u_2 \wedge w)x\right] \\
\left[(\alpha u) \wedge w\right] x = \left[u \wedge (\alpha w)\right] x = \alpha \left[(u \wedge w)x\right] \quad .
\end{array}
$$

Proposition 3. *Let V be a two-dimensional real Euclidean vector space; for any $u, w, \in V$,*

$$(u \wedge w) = 0$$

if and only if u and w are linearly dependent.

Remark 6. *Any scalar $\alpha \in \mathbb{R}$ in a vector space V can be considered as a function from V to V; its value on $x \in V$ is simply αx.*

Let us rewrite (3) as a linear operator acting on b:

$$
\begin{array}{rl}
a = (ab^{-1})b & = \left[(a \cdot b^{-1})b\right] \ + \ \left[\left(a \wedge b^{-1}\right) b\right] = \\
& = \left[(a \cdot b^{-1}) \ + \ \left(a \wedge b^{-1}\right)\right] b \quad .
\end{array}
$$

Notice that here $+$ denotes unambiguously both a sum of vectors and a sum of linear operators.

3.2 The geometric product as a linear operator.

At this point we can give the following:

Definition 3. Let V be a two-dimensional real Euclidean vector space; for any fixed $u, w \in V$, the **Clifford geometric product** uw of two vectors is a function from V to V whose value on $x \in V$ is the vector

$$(uw)x = \left[(u \cdot w)x\right] + \left[(u \wedge w)x\right] \ . \tag{5}$$

Remark 7. *In order to consider the vector $(uw)x$ as the triple product of an associative binary product we state how the function $(u \wedge w)$ acts "on its left" on a vector x or, if you prefer, how a vector x acts "on its right" on the operator $(u \wedge w)$, or, simply, what the geometric product $x(u \wedge w)$ is.*

Definition 4. Let V be a two-dimensional real Euclidean vector space; for any $u, w, x \in \ V$

$$x(u \wedge w) = -(u \wedge w)x \quad . \tag{6}$$

Remark 8. *As anticipated in the introduction, V is not closed under the above geometric product. In fact, while u, w, x and uwx are vectors, uw and wx are linear operators on V.*

If we want to consider also $u \wedge w$ as an element of our Clifford Algebra, we must wonder about the product

$$(u_1 \wedge u_2)(w_1 \wedge w_2) \ ,$$

where $u_1, u_2, w_1, w_2 \in V$. By functionally interpreting the product as a composition of functions, and using Definition 2 we show that the composition of linear operators $(u_1 \wedge u_2)(w_1 \wedge w_2)$ is a scalar (see Remark 6). Without loss of generality we can consider $(u_1 \wedge u_2) \neq 0$; by Proposition 3, $\{u_1, u_2\}$ is a basis for V, and we can write the generic vector $x \in V$ as $x = \gamma u_1 + \delta u_2$. Then,

$$\begin{aligned}
& (u_1 \wedge u_2)(w_1 \wedge w_2)x \\
= \ & (u_1 \wedge u_2)\left[(w_1 \wedge w_2)x\right] = (u_1 \wedge u_2)\left[w_1(w_2 \cdot x) - w_2(w_1 \cdot x)\right] = \\
= \ & u_1(u_2 \cdot w_1)(w_2 \cdot x) - u_2(u_1 \cdot w_1)(w_2 \cdot x) - u_1(u_2 \cdot w_2)(w_1 \cdot x) + \\
& + u_2(u_1 \cdot w_2)(w_1 \cdot x) = \\
& \langle \text{ substituing } x = \gamma u_1 + \delta u_2 \text{ and simplyfying } \rangle \\
= \ & \gamma u_1[(u_1 \cdot w_2)(u_2 \cdot w_1) - (u_1 \cdot w_1)(u_2 \cdot w_2)] + \\
& + \delta u_2[(u_1 \cdot w_2)(u_2 \cdot w_1) - (u_1 \cdot w_1)(u_2 \cdot w_2)] = \\
= \ & [(u_1 \cdot w_2)(u_2 \cdot w_1) - (u_1 \cdot w_1)(u_2 \cdot w_2)]x \ \ .
\end{aligned}$$

We obtain the same result if we similarly calculate composition of functions on the left $[x(u_1 \wedge u_2)](w_1 \wedge w_2)$. Since we read the geometric product as the composition of functions, the foregoing result justifies the following Definition.

Definition 5. Let V be a two-dimensional real Euclidean vector space. If $u_1, u_2, w_1, w_2 \in V$, we define the **Clifford geometric product** of $(u_1 \wedge u_2)$ and $(w_1 \wedge w_2)$ as their functional composition,

$$(u_1 \wedge u_2)(w_1 \wedge w_2) = [(u_1 \cdot w_2)(u_2 \cdot w_1) - (u_1 \cdot w_1)(u_2 \cdot w_2)] \in \mathbb{R} \ \ .$$

4 The Clifford Geometric Algebra $\mathcal{C}l(V)$ of a Two-dimesional Real Euclidean Vector Space V.

Definition 6. Let V be a two-dimensional real Euclidean vector space:

$$\mathcal{C}l(V) = \mathbb{R} \oplus V \oplus \bigwedge^2 V \ \ ,$$

where

$$\bigwedge^2 V = \left\{ (u \wedge w) \ : \ u, w \in V \right\}$$

is a set of linear operators from V into itself, and $\oplus$ is the usual symbol for the direct sum of vector spaces.

Proposition 4. *Let V be a two-dimensional real Euclidean vector space, then $\bigwedge^2 V$ is a one-dimensional real vector space.*

Definition 7. Let V be a two-dimensional real Euclidean vector space, $\alpha \oplus u \oplus (u_1 \wedge u_2), \beta \oplus w \oplus (w_1 \wedge w_2) \in \mathcal{C}l(V)$, where $\alpha, \beta \in \mathbb{R}$, $u, u_1, u_2, w, w_1, w_2 \in V$; we define the product in $\mathcal{C}l(V)$ as follows

$$\begin{array}{rl} & [\alpha \oplus u \oplus (u_1 \wedge u_2)]\,[\beta \oplus w \oplus (w_1 \wedge w_2)] = \\ = & \{\alpha\beta + (u \cdot w) + [(u_1 \wedge u_2)(w_1 \wedge w_2)]\} \oplus \\ & \oplus \{\alpha w + \beta u + [u(w_1 \wedge w_2)] + [(u_1 \wedge u_2)w]\} \oplus \\ & \oplus [\alpha(w_1 \wedge w_2) + (u \wedge w) + \beta(u_1 \wedge u_2)] \quad . \end{array} \tag{7}$$

Proposition 5. *Let V be a two-dimensional real Euclidean vector space, the function $J_{\mathbb{R}} : \mathbb{R} \to \mathcal{C}l(V)$ such that $J_{\mathbb{R}}(\alpha) = \alpha \oplus 0 \oplus 0$ is an injective morphism of algebras.*

Proposition 6. *Let V be a two-dimensional real Euclidean vector space, the function $J_V : V \to \mathcal{C}l(V)$ such that $J_V(v) = 0 \oplus v \oplus 0$ is an injective linear operator.*

Remark 9. *The distributivity property of the geometric product follows immediately from distributivity of the scalar product in V and from that of $\wedge$ (see Proposition 2). Thanks to Proposition 5 and Proposition 6, there is no ambiguity in using the unique symbol $+$ when adding in $\mathcal{C}l(V)$ scalars, vectors, or linear operators in $\bigwedge^2 V$, neither using juxtaposition of elements when multiplying any element in $\mathcal{C}l(V)$. This, in fact, is a great comfort of Clifford's formalism.*

Remark 10. *The product (7) is associative. The proof of that involves long and mechanical calculations. However, we do not need of a such general result to verify (1). In this paper we just need to prove the following Theorem.*

Theorem 1. *Let V be a two-dimensional real Euclidean vector space, then the Clifford geometric product between vectors is associative in $\mathcal{C}l(V)$.*

Proof. Let $u, w, x \in V$, then

$$\begin{array}{rl} u(wx) & = u\,[(w \cdot x) + (w \wedge x)] = u(w \cdot x) + [u(w \wedge x)] = \\ & = u(w \cdot x) - [(w \wedge x)u] = u(w \cdot x) - w(x \cdot u) + x(w \cdot u) = \\ & = (uw)x \quad . \end{array}$$

□

5 Verifying the Two-dimensional 4th Proportional.

Apart from the foregoing algebraic properties, geometrically we just know that ab^{-1} acts on vector b giving vector a. Now we verify that the geometric property stated in (1) holds for the generic vector

$$y = ab^{-1}x = (a \cdot b^{-1})x \ + \ \left[\left(a \wedge b^{-1}\right) x\right] = x\frac{a \cdot b}{b \cdot b} \ + \ a\frac{b \cdot x}{b \cdot b} \ - \ b\frac{a \cdot x}{b \cdot b} \ ,$$

where $x \in V$. Plane trigonometry calculations could be used to this aim. However, we prefer to deduce some other algebraic properties of $\mathcal{C}l(V)$, and to prove validity of (1) at the end, just using the following basic trigonometric notions.

Definition 8. Given two vectors $u, w \in V$, we denote by $\overset{\frown}{wu}$ the radian measure of the anticlockwise oriented angle from vector u to vector w.

Remark 11. *Since radians are real numbers (modulo 2π), we have that $\overset{\frown}{uw} = 2\pi - \overset{\frown}{wu} = -\overset{\frown}{wu}$. Besides, for any $a, b, c, d \in V$,*

$$\overset{\frown}{ab} = \overset{\frown}{cd} \iff \begin{cases} & \cos \overset{\frown}{ab} = \cos \overset{\frown}{cd} \\ \text{and} & \\ & \sin \overset{\frown}{ab} = \sin \overset{\frown}{cd} \ . \end{cases} \tag{8}$$

Proposition 7. *Let V be a two-dimensional real Euclidean vector space, then for all $u, w \in V$*

$$\begin{aligned} u \cdot w &= |u| \ |w| \ \cos \overset{\frown}{uw} = |u| \ |w| \ \cos \overset{\frown}{wu} \\ u \wedge w &= |u| \ |w| \ \sin \overset{\frown}{uw} \ (e_2 \wedge e_1) \ , \end{aligned}$$

where $|u|^2 = (u \cdot u)$ and $\{e_1, e_2\}$ is any positively oriented orthonormal basis for V $\left(i.e. \ (e_1 \wedge e_2) \neq 0, (e_1 \cdot e_2) = 0, |e_1| = |e_2| = 1, \text{ and } \overset{\frown}{e_2 e_1} = \frac{\pi}{2}\right)$.

Proof. We can use trigonometric functions to write $u = e_1|u| \cos \overset{\frown}{ue_1} + e_2|u| \sin \overset{\frown}{ue_1}$ and $w = e_1|w| \cos \overset{\frown}{we_1} + e_2|w| \sin \overset{\frown}{we_1}$. Then we have

$$\begin{aligned} u \cdot w &= |u| \ |w| \ \cos(\overset{\frown}{ue_1} - \overset{\frown}{we_1}) = |u| \ |w| \ \cos \overset{\frown}{uw} \\ u \wedge w &= |u| \ |w| \ [\sin(\overset{\frown}{ue_1} - \overset{\frown}{we_1})](e_2 \wedge e_1) = |u| \ |w| \ (\sin \overset{\frown}{uw})(e_2 \wedge e_1) \ . \end{aligned}$$

□

Lemma 1. *Let $\{a, b\}$ be a basis for the real Euclidean vector space V, then for all $x \in V$*

$$xab = bax$$

Proof. $x = \alpha a + \beta b$, so

$$\begin{aligned} xab &= (\alpha a + \beta b)ab = (\alpha|a|^2 + \beta ba)b = \alpha|a|^2 b + \beta bab \\ bax &= ba(\alpha a + \beta b) = b(\alpha|a|^2 + \beta ab) = \alpha|a|^2 b + \beta bab \ . \end{aligned}$$

□

Theorem 2 proves the scalar part of statement (1), while Theorem 3 proves the angular part of statement (1) (see Figure 3).

Theorem 2. *Let $\{a, b\}$ be a basis for the real Euclidean vector space V, if $y = ab^{-1}x$ then the following proportion holds*

$$|y| : |x| = |a| : |b| \ .$$

Proof. By Proposition 1, $\{a, b\}$ is a basis for V if and only if $\{a, b^{-1}\}$ is a basis for V, then $|y|^2 = yy = ab^{-1}xab^{-1}x = ab^{-1}(xab^{-1})x = ab^{-1}b^{-1}axx = \frac{|a|^2}{|b|^2}|x|^2$. □

Theorem 3. *Let V be a two-dimensional real Euclidean vector space, then for all nonzero vectors $a, b, x \in V$ and $y = ab^{-1}x$ we have that*

$$\overset{\curvearrowleft}{yx} = \overset{\curvearrowleft}{ab} \ .$$

Proof. By hypothesis, relation $y = ab^{-1}x$ is equivalent to relation $yx^{-1} = ab^{-1}$. By Definiton 6, we have that $yx^{-1} = ab^{-1}$ if and only if $(y \cdot x^{-1}) = (a \cdot b^{-1})$ and $(y \wedge x^{-1}) = (a \wedge b^{-1})$; then, by Proposition 7, Theorem 2, and equation (8), we have that $\overset{\curvearrowleft}{yx^{-1}} = \overset{\curvearrowleft}{ab^{-1}}$. The thesis follows from the fact that the inverse of any nonzero vector is a positive multiple of itself. □

Remark 12. *Provided $x \neq 0$, relation $y = ab^{-1}x$ is equivalent to relation $yx^{-1} = ab^{-1}$. This suggests a natural two-dimensional extension of the traditional notation, $y : x = a : b$.*

6 Oriented Triangles, Vector Ratios, and Rotations in V.

Definition 9. Given three vectors u_1, u_2, u_3 in a two-dimensional real Euclidean vector space V not all lying on the same line $\big($i.e. $[(u_3 - u_1) \wedge (u_2 - u_1)] \neq 0\big)$, each ordered triple (u_1, u_2, u_3) identifies an oriented triangle that we denote by $[u_1, u_2, u_3]$ (instead of $u_2 \overset{\triangle}{u_1} u_3$), whose unoriented support is the convex hull of the three vectors u_1, u_2, u_3.

Remark 13. *Every nontrivial even permutation $[u_2, u_3, u_1]$, and $[u_3, u_1, u_2]$ of the oriented triangle $[u_1, u_2, u_3]$ corresponds to the same oriented triangle.*

Definition 10. Two oriented triangles $[a, b, c]$ and $[d, e, f]$ have the same orientation (are equally oriented) if $\frac{(c-a)\wedge(b-a)}{(f-d)\wedge(e-d)} > 0$.

Having verified (1), we can state the following correspondence between oriented triangles and vector ratios.

Theorem 4. *Let V be a two-dimensional real Euclidean vector space, then two equally oriented triangles $[u_1, u_2, u_3]$, and $[w_1, w_2, w_3]$ are similar if and only if there exists an even permutation $[w_{1'}, w_{2'}, w_{3'}]$ of $[w_1, w_2, w_3]$ such that*

$$(u_3 - u_1)[(u_2 - u_1)^{-1}] = (w_{3'} - w_{1'})[(w_{2'} - w_{1'})^{-1}] \ .$$

The property stated in (1), gives also the following functional interpretation of isosceles oriented triangles such as $[0, w, u]$ (where $|w| = |u|$).

Theorem 5. *Let V be a two-dimensional real Euclidean vector space, any rotation in V corresponds to a quotient uw^{-1}, for some $u, w \in V$ such that $|u| = |w|$, or to a product uw if $|u| = |w| = 1$ (the identity in V corresponds to multiplying vectors by $uu = 1 \in \mathbb{R}$).*

7 Vector Product and Ratio are Coordinate-free.

Theorem 6. *Let V be a two-dimensional real Euclidean vector space, then both the geometric product and the geometric ratio of vectors in V are rotationally invariant.*

Proof. Let us fix a generic rotation uw in V, where $u, w \in V$ such that $|u| = |w| = 1$. Without loss of generality we can assume $(u \wedge w) \neq 0$.We want to prove that, given any two vectors $a, b \in V$, then

$$(uwa)(uwb) = ab \quad \text{and} \quad (uwa)(uwb)^{-1} = ab^{-1} \ .$$

By Lemma 1 and associativity of the geometric product, we have $uwauwb = uw(auw)b = uwwuab = ab$. Without loss of generality we can assume $(a \wedge b) \neq 0$. By hypothesis, Proposition 1, and Lemma 1, then

$$\begin{aligned}(uwa)(uwb)^{-1} \quad &= uwab^{-1}w^{-1}u^{-1} = uwab^{-1}wu = uw(ab^{-1}w)u = \\ &= uwwb^{-1}au = ub^{-1}au = (ub^{-1}a)u = ab^{-1}uu = ab^{-1} \ .\end{aligned}$$

□

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