THE TERNARY COMMUTATOR OBSTRUCTION FOR INTERNAL CROSSED MODULES

MANFRED HARTL AND TIM VAN DER LINDEN

Abstract. We study the notion of internal crossed module in terms of cross-effects of the identity functor. These cross-effects give rise to a concept of commutator which allows a description of internal categories, (pre)crossed modules, Beck modules, and abelian extensions in finitely cocomplete homological categories in a way which is very close to the case of groups. We single out the obstruction which prevents a Peiffer graph being a groupoid—which in a semi-abelian context is known to vanish precisely when the *Smith is Huq* condition holds, so is invisible in the category of groups—as a certain ternary commutator. Such a ternary commutator appears in the Hopf formula for the third homology with coefficients in the abelianisation functor and in the interpretation of the second cohomology of an object with coefficients in a module. It is generally not decomposable into nested binary commutators: this happens, for instance, in the category of loops, where *Smith is Huq* is shown not to hold.

Introduction

Internal crossed modules in a semi-abelian category [42] may be axiomatised in several equivalent ways. In all approaches the starting point is a desired correspondence between crossed modules and internal categories, which determines the basic properties that such an axiomatisation should satisfy.

In the article [41] Janelidze presents axioms for internal crossed modules in terms of the internal actions he introduced in his article [20] with Bourn. His analysis is elegant and efficient and captures all appropriate examples. It also explains that the extension of the case of groups to semi-abelian categories is not entirely without difficulties. The most straightforward description of the concept of crossed module merely gives so-called *star-multiplicative graphs*—in which the composition of morphisms is only defined locally around the origin—and not the internal groupoids one would expect, in which any composable pair of morphisms can actually be composed. Of course this defect can be mended, as it is indeed done in [41]. Unfortunately, the resulting characterisation of internal crossed modules becomes slightly less natural.

The question thus arose when the two concepts (star-multiplicative graphs and internal groupoids) are equivalent. It turns out [53] that the gap between the two is precisely as big as the gap between the Huq commutator of normal subobjects

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The second author works as *chargé de recherches* for Fonds de la Recherche Scientifique–FNRS. His research was supported by Centro de Matemática da Universidade de Coimbra and by Fundação para a Ciência e a Tecnologia (grant number SFRH/BPD/38797/2007). He wishes to thank the Max-Planck-Institut für Mathematik for its kind hospitality during his stay at Bonn.

and the Smith/Pedicchio commutator of internal equivalence relations. That is to say, in a semi-abelian category they are equivalent if and only if the *Smith is Huq* condition holds. This explains why the difference between the two concepts is invisible in the category of groups, in fact in any of the categories where internal crossed modules were ever considered: all of those are strongly protomodular or at least action accessible, so that the Smith/Pedicchio commutator and the Huq commutator are equivalent. One solution would be to restrict ourselves to such a somewhat less general context, but then we would choose to ignore the problem rather than to face it.

Alternatively, another axiomatisation might be found, one which maybe stays closer to the case of groups, but which shows clearly what has to be added to make the theory work in general. In the present article we try to do precisely this. We present a new approach to internal crossed modules, one which is based on a different notion of internal action, and which from the start takes the *Smith is Huq* problem explicitly into account. The resulting characterisation is an obvious extension of the groups case, but at the same time the ingredient missing there is brought into focus: it appears as a ternary commutator. A byproduct of this alternative analysis is that the context is enlarged to a non-exact setting, as we may work in finitely cocomplete homological categories instead of semi-abelian ones.

Cross-effects of functors. The main technical innovation which allows us to consider higher-order commutators and the corresponding actions is the general theory of *cross-effects*. The concept of cross-effect of a functor between abelian categories was introduced by Eilenberg and Mac Lane in the article [29], where it was used in the study of polynomial functors. This definition does, however, not generalise to non-additive contexts. The approach due to Baues and Pirashvili [5], worked out in the case of groups, does extend easily to more general situations. It is such an extension we shall consider in the present article. Here we will not use cross-effects to define polynomial functors—this is, for instance, done in the articles [35] and [37]. In the present article we shall rather construct higher-order commutators, and express the higher-order coherence conditions internal actions may satisfy in terms of higher cross-effects. In fact, ternary cross-effects will suffice for our present purposes.

Commutators and actions. The concept of internal action introduced by Hartl and Loiseau [36] blends naturally with the theory of cross-effects. It is based on the idea that the second cross-effect

$$(K|L) = \mathrm{Ker}\left(\left\langle \begin{smallmatrix} \langle 1_K, 0\rangle \\ \langle 0, 1_L\rangle \end{smallmatrix} \right\rangle : K + L \to K \times L\right)$$

of the identity functor $1_{\mathcal{A}}$ of a finitely cocomplete homological category $\mathcal{A}$ evaluated in the objects K, $L \in \mathrm{Ob}(\mathcal{A})$ behaves as a kind of "formal commutator" of K and L. (This was also discovered independently by Mantovani and Metere, see [50].) If now $k\colon K \to X$ and $l\colon L \to X$ are subobjects of an object X, their **(Higgins) commutator** $[K, L] \leqslant X$ is the image of the induced morphism

$$(K|L) \xrightarrow{\iota_{K,L}} K + L \xrightarrow{\left\langle \begin{smallmatrix} k \\ l \end{smallmatrix} \right\rangle} X.$$

Using higher cross-effects it is easy to extend this definition to higher-order commutators: for instance, given a third subobject $m\colon M \to X$ of X, the ternary commutator $[K, L, M] \leqslant X$ is the image of the composite

$$(K|L|M) \xrightarrow{\iota_{K,L,M}} K + L + M \xrightarrow{\left\langle \begin{smallmatrix} k \\ l \\ m \end{smallmatrix} \right\rangle} X,$$

where $\iota_{K,L,M}$ is the kernel of

$$K+L+M \xrightarrow{\left\langle \left\langle \begin{smallmatrix} i_K \\ i_L \\ 0 \end{smallmatrix} \right\rangle, \left\langle \begin{smallmatrix} i_K \\ 0 \\ i_M \end{smallmatrix} \right\rangle, \left\langle \begin{smallmatrix} 0 \\ i_L \\ i_M \end{smallmatrix} \right\rangle \right\rangle} (K+L)\times(K+M)\times(L+M).$$

The basic properties of the (binary) Higgins commutator are explored in the articles [36] and [50]. In the former it is also explained how this commutator gives rise to a concept of internal action. We shall recall some of this in sections 2 and 3; for now it suffices to mention that an action of an object G on an object A is a morphism $\psi\colon (A|G) \to A$ satisfying a certain condition, and that such an action contains just enough information for reconstructing the semi-direct product $A \rtimes_{\psi} G$.

The ternary commutator obstruction. The *Smith is Huq* condition for homological categories may be expressed in terms of cross-effects as the vanishing of a ternary commutator. Thus a condition which is about *locally defined internal categorical structures admitting a global extension* is characterised as a computational obstruction.

Indeed, we prove that for equivalence relations R and S on X with respective normalisations K, $L \lhd X$, the relations R and S commute in the sense of Smith and Pedicchio if and only if $[K,L]$ and $[K,L,X]$ are trivial. As $[K,L]=0$ precisely when K and L commute in the Huq-sense, the object $[K,L,X]$ is the **ternary commutator obstruction** which should vanish for the Smith/Pedicchio commutator and the Huq commutator to be equivalent in the given situation.

In the category of groups, this condition on ternary commutators is invisible, as all ternary commutators are expressible in terms of binary ones. This explains why crossed modules of groups, which correspond to internal categories in the category Gp, may be characterised using just a binary commutator as is done in the final section of Mac Lane [48]. In general, though, ternary commutators cannot be written in terms of repeated binary ones.

This new viewpoint on the *Smith is Huq* condition gives new examples of categories which satisfy it. A *nilpotent category of class* 2 is a semi-abelian category whose identity functor is *quadratic*, i.e., it has a trivial triple cross-effect [35]. Hence, almost by definition, any such category satisfies (SH). In particular, the *Smith is Huq* condition holds for modules over a square ring, and specifically for algebras over a nilpotent operad of class two [3].

On the other hand, the category of loops (quasigroups with an identity) does not satisfy (SH): we give an example of a loop X with an abelian subloop A and elements $a \in A$, $x \in X$ such that the associator element $[\![a,a,x]\!]$ is non-trivial. (In fact, one of the first examples of a non-associative structure ever considered will do, see Example 4.9). This proves that the triple commutator $[A,A,X]$ need not vanish even when the binary commutator $[A,A]$ does. As a consequence, Loop is not action accessible or strongly protomodular—though it is well known to be semi-abelian [8].

Definition of internal crossed modules. Consider a quadruple (G,A,μ,∂) in which G and A are objects, $\mu\colon (A|G) \to A$ is an action of G on A, and $\partial\colon A \to G$ is a morphism. This quadruple is a *crossed module* when the following three squares commute.

$$\begin{tikzcd} (A|G) \arrow[r, "\mu"] \arrow[d, "(\partial|1_G)"'] & A \arrow[d, "\partial"] \\ (G|G) \arrow[r, "c^{G,G}"'] & G \end{tikzcd} \qquad \begin{tikzcd} (A|A) \arrow[r, "c^{A,A}"] \arrow[d, "(1_A|\partial)"'] & A \arrow[d, equal] \\ (A|G) \arrow[r, "\mu"'] & A \end{tikzcd} \qquad \begin{tikzcd} (A|A|G) \arrow[r, "\mu_{2,1}"] \arrow[d, "(1_A|\partial|1_G)"'] & A \arrow[d, equal] \\ (A|G|G) \arrow[r, "\mu_{1,2}"'] & A \end{tikzcd}$$

The first square expresses the *precrossed module condition* which says that the morphism ∂ is G-equivariant with respect to μ and the conjugation action $c^{G,G}$ of G on itself. Quadruples which satisfy this first condition correspond to internal reflexive graphs. Commutativity of the middle square is the *Peiffer condition*: the conjugation action $c^{A,A}$ of A on itself coincides with the pullback $\partial^*(\mu)$ of μ along ∂. Quadruples which satisfy the first two conditions correspond to so-called *Peiffer graphs*, which admit some kind of composition locally around the origin, and which are equivalent to star-multiplicative graphs ([49], see also [53]). The square on the right commutes when the local composition of the Peiffer graph extends to a globally defined internal groupoid structure.

Internal categories in a homological category. Our analysis of internal crossed modules depends on a new characterisation of internal categories in terms of internal actions, valid in any finitely cocomplete homological category. Let us just mention here that an internal reflexive graph

$$\begin{tikzcd} R \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & G \arrow[l, "e" description] \end{tikzcd} \qquad\qquad d\circ e = c\circ e = 1_G$$

is an internal category when either one of the following equivalent conditions holds (Theorem 5.2):

- $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = 0 = [\mathrm{Ker}(d), \mathrm{Ker}(c), R]$;
- $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = 0 = [\mathrm{Ker}(d), \mathrm{Ker}(c), \mathrm{Im}(e)]$;
- the conjugation action $c^{A,R} \colon (A|R) \to A$ of R on $A = \mathrm{Ker}(d)$ factors through the morphism $(1_A|c) \colon (A|R) \to (A|G)$;
- $c^{A,R} = (e\circ c)^*(c^{A,R})$.

Abelian extensions, Beck modules. The concepts of *abelian extension* and *Beck module* allow a similar analysis in terms of internal actions and ternary commutators. Both are certain internal Mal'tsev operations in a slice category, known to exist if and only if a Smith/Pedicchio commutator vanishes. We shall see that a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, tail, "a"] & X \arrow[r, two heads, "p"] & G \arrow[r] & 0 \end{tikzcd} \qquad (\mathbf{A})$$

is an abelian extension if and only if A is abelian and $[A, A, X]$ is zero. Likewise, an internal G-action $\psi \colon (A|G) \to A$ on an abelian object A is a G-module structure on A if and only if a certain induced morphism $\psi_{2,1} \colon (A|A|G) \to A$ is trivial. Alternatively, one could ask that the induced diagram

$$\begin{tikzcd} (A \times A|G) \arrow[r, "\langle(\pi_1|1_G),(\pi_2|1_G)\rangle"] \arrow[d, "(+|1_G)"'] & (A|G) \times (A|G) \arrow[r, "\psi\times\psi"] & A \times A \arrow[d, "+"] \\ (A|G) \arrow[rr, "\psi"'] & & A \end{tikzcd}$$

commutes, to mention two of several equivalent conditions.

Applications in homology and cohomology. We give two concrete applications of these results in semi-abelian (co)homology. First we characterise double central extensions [31, 40, 60] in terms of binary and ternary commutators, and obtain a Hopf formula for the third homology of an object Z with coefficients in the abelianisation functor:

$$\mathrm{H}_3(Z, \mathsf{ab}) = \frac{K \wedge L \wedge [X, X]}{[K, L, X] \vee [K, L] \vee [K \wedge L, X]},$$

where K, $L \lhd X$ are the kernels induced by a double presentation of Z. This formula is valid in any semi-abelian category with enough projectives, whether the *Smith is Huq* condition holds or not.

Then we focus on cohomology in semi-abelian categories, and explain how to connect the main result of [34] with the torsor theories from [21, 27]. The central idea here is that for any abelian extension such as ($\mathbf{A}$), the conjugation action of X on A factors through p to yield an action of G on A. This action, called the **direction** of ($\mathbf{A}$), is always a module, and thus we obtain the **direction functor**

$$\mathsf{d}_G\colon \mathsf{AbExt}_G(\mathcal{A}) \to \mathsf{Mod}_G(\mathcal{A}).$$

We give an interpretation of the second cohomology group $\mathrm{H}^2(G,(A,\psi))$ of G with coefficients in a module (A,ψ) as the group of connected components of the fibre $\mathsf{d}_G^{-1}(A,\psi)$ of the direction functor d_G over the module (A,ψ). When, in particular, the action ψ is trivial, we recover the interpretation worked out in [34] of the second cohomology group of G with coefficients in an abelian object A as equivalence classes of central extensions of G by A. On the other hand, the fibre $\mathsf{d}_G^{-1}(A,\psi)$ consists of *torsors of G by* (A,ψ), as in [21] and [27].

Structure of the text. In Section 1 we recall the basic categorical notions we need further on and introduce some conventions and notations. Section 2 is devoted to the definition and first properties of cross-effects and the induced (higher-order) commutators. We recall some properties from [36, 50] and then prove right exactness results for cross-effect functors: preservation of coequalisers of reflexive graphs (Theorem 2.26 and Corollary 2.27) and cokernels inducing certain exact sequences (Proposition 2.31, Corollary 2.32 and Proposition 2.33). These are used in Section 3 where we explain how to deal with internal actions and semi-direct products starting from cross-effects. In Section 4 we give a characterisation of the *Smith is Huq* condition (Theorem 4.6) in terms of ternary commutators, and a formula for the Smith/Pedicchio commutator of equivalence relations in terms of a binary and a ternary commutator of normal subobjects (Theorem 4.14). We also find a characterisation of double central extensions (Proposition 4.16) and a Hopf formula for the third homology of an object (Theorem 4.17). This leads to Section 5 where we give new characterisations of internal categories (Theorem 5.2), which gives an elementary description of the concept of internal crossed module. We use this description to prove some of its most classical properties: Theorem 5.7 and Proposition 5.12. In Section 6 we turn to abelian extensions; we obtain some equivalent characterisations (Theorem 6.4) and an explicit description of the reflection of extensions to abelian extensions (Corollary 6.5). Next, in Section 7, we consider Beck modules; we give several characterisations (Theorems 7.3, 7.9 and 7.13) and study the relation with internal crossed modules (Proposition 7.18). In the final Section 8 these results are used in the study of semi-abelian cohomology (Theorem 8.7).

1. Preliminaries

As a rule we shall work in a finitely cocomplete homological category $\mathcal{A}$ unless where explicitly mentioned. Some proofs need a semi-abelian environment; we always explain where and why.

We start with an overview of those basic categorical notions and results needed throughout the text.

1.1. **Pointed categories.** A **pointed** category has a **zero object**, an initial object that is also terminal. In a pointed category with finite sums, we denote the coproduct inclusion $X_k \to X_1 + \cdots + X_n$ by i_{X_k} or by i_k, and its canonical retraction

$$\left\langle \begin{matrix} 0 \\ \vdots \\ 1_{X_k} \\ \vdots \\ 0 \end{matrix} \right\rangle : X_1 + \cdots + X_n \to X_k$$

by r_{X_k} or by r_k. We further write $\widehat{r}_k \colon X_1 + \cdots + X_n \to X_1 + \cdots + \widehat{X_k} + \cdots + X_n$ for the morphism whose restriction to X_j is i_{X_j} when $j \neq k$ and is the zero morphism when $j = k$.

Dually, when working in a pointed category with finite products, we denote the product projection $X_1 \times \cdots \times X_n \to X_k$ by π_{X_k} or π_k and its canonical section

$$\langle 0, \ldots, 1_{X_k}, \ldots, 0 \rangle \colon X_k \to X_1 \times \cdots \times X_n$$

by σ_{X_k} or σ_k.

1.2. **Regular and exact categories.** Recall that a **regular epimorphism** is the coequaliser of some pair of morphisms. A **regular** category is finitely complete and endowed with a pullback-stable (regular epi, mono)-factorisation system. Given a morphism $f \colon X \to Y$, we write $\mathrm{im}(f) \colon \mathrm{Im}(f) \to Y$ for the mono-part in this **image factorisation** of f. If $M \leqslant X$ is a subobject of X, we write $f(M)$ for the **direct image** of M along f: the image of $f \circ m$, where $m \colon M \to X$ is a monomorphism that represents the subobject.

Regular categories provide a natural context for working with relations. We denote the kernel relation (= kernel pair) of a morphism f, the pullback of f along itself, by $(\mathrm{R}[f], f_1, f_2)$. A regular category is said to be **Barr exact** when every equivalence relation is **effective**, which means that it is the kernel pair of some morphism [1].

1.3. **Homological and semi-abelian categories.** A pointed category with pullbacks is **protomodular** [11] when the Split Short Five Lemma holds. When, moreover, the pointed category is regular, then protomodularity is equivalent to the (Regular) Short Five Lemma: given a commutative diagram (**B**) with regular epimorphisms p, p' and their kernels, if a and g are isomorphisms then also x is an isomorphism. We usually denote the kernel of a morphism f by $(\mathrm{Ker}(f), \ker(f))$, and say that a morphism is **proper** when its image is a kernel. A proper monomorphism is said to be **normal**, and when $M \leqslant X$ is a normal subobject we write $M \lhd X$.

A pointed, regular and protomodular category is called **homological** [7]. This is a context where many of the basic diagram lemmas of homological algebra hold. In particular, here the notion of **(short) exact sequence** has its full meaning: a regular (= normal) epimorphism with its kernel such as (**A**) above. A short exact sequence (**A**) is **split** when there exists a **section** (or **splitting**) $s \colon G \to X$ of p, i.e., a morphism s in $\mathcal{A}$ such that $p \circ s = 1_G$.

Note that a split epimorphism $p \colon X \to G$ may have many splittings. When just one splitting s is chosen, the couple (p, s) is called a **point (over** G**)**. The **category of points** $\mathsf{Pt}(\mathcal{A})$ has points in $\mathcal{A}$ (considered as diagrams $p \circ s = 1_G$) as objects and natural transformations between points as morphisms. The points over a given object G form the full subcategory $\mathsf{Pt}_G(\mathcal{A}) = (1_G)/(\mathcal{A}/G)$ of $\mathsf{Pt}(\mathcal{A})$.

Unless where explicitly mentioned in the text, we shall always work in a finitely cocomplete homological category which we write $\mathcal{A}$.

A **Mal'tsev** category [24] is finitely complete and such that every reflexive relation is necessarily an equivalence relation. It is well known that any finitely complete protomodular category is Mal'tsev [12]. Furthermore, the Mal'tsev property is preserved by slicing. This is a context in which many of the basic constructions in commutator theory make sense. In a Mal'tsev category, internal categories are automatically internal groupoids.

Semi-abelian categories are homological and exact with binary sums [42]. In a semi-abelian category, the direct image of a kernel along a regular epimorphism is still a kernel. In this context, the existence of binary sums entails finite cocompleteness, and any comparison morphism $\langle r_X, r_Y\rangle\colon X+Y \to X\times Y$ is a regular epimorphism.

1.4. **Extensions and double extensions.** An **extension** in a homological category is a regular epimorphism. The extensions in $\mathcal{A}$ form a full subcategory $\mathsf{Ext}(\mathcal{A})$ of the category of arrows $\mathsf{Arr}(\mathcal{A}) = \mathsf{Fun}(2^{\mathrm{op}}, \mathcal{A})$: morphisms are commutative squares between extensions. Since regular epimorphisms are normal, an extension $p\colon X \to G$ may equally well be considered as a short exact sequence (**A**). Then we call p an **extension of G by A.**

A **double extension** in $\mathcal{A}$ is a commutative square

$$
\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "d"'] & C \arrow[d, "g"] \\
D \arrow[r, "f"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rrd, bend left, "c"] \arrow[rd, dotted] \arrow[rdd, bend right, "d"'] & & \\
& D\times_Z C \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
& D \arrow[r, "f"'] & Z
\end{tikzcd}
$$

such that all arrows in the induced right hand side diagram are extensions [31]. In a semi-abelian category this happens when the square is a pushout of regular epimorphisms. Double extensions form a full subcategory $\mathsf{Ext}^2(\mathcal{A})$ of the category $\mathsf{Arr}^2(\mathcal{A}) = \mathsf{Arr}(\mathsf{Arr}(\mathcal{A})) = \mathsf{Fun}((2^2)^{\mathrm{op}}, \mathcal{A})$ of **double arrows** in $\mathcal{A}$. By Lemma 1.5 below, double extensions correspond to short exact sequences *in the category* $\mathsf{Ext}(\mathcal{A})$. In other words, the double extensions in $\mathcal{A}$ are the normal epimorphisms in $\mathsf{Ext}(\mathcal{A})$, and in $\mathsf{Ext}(\mathcal{A})$ kernels of normal epimorphisms are computed degree-wise.

Higher extensions were introduced in [31] following [40] in order to capture the concept of *higher centrality* which is useful in the study of semi-abelian (co)homology: see, for instance, Subsection 4.15 below and the articles [30, 31, 60]. We shall also need double extensions for the proof of Theorem 2.26.

Lemma 1.5. [14, 18, 20, 31] *Consider, in a homological category, a commutative diagram with exact rows*

$$
\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, tail] \arrow[d, "a"'] & X' \arrow[r, "p'"] \arrow[d, "x"'] & G' \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & A \arrow[r, tail] & X \arrow[r, "p"'] & G \arrow[r] & 0.
\end{tikzcd}
\tag{B}
$$

(i) *The right hand square $p\circ x = g\circ p'$ is a pullback if and only if a is an isomorphism.*
(ii) *Suppose that x and g are regular epimorphisms. Then the square $p\circ x = g\circ p'$ is a double extension if and only if a is a regular epimorphism.* $\square$

1.6. **Abelian objects and internal abelian groups.** In a Mal'tsev category $\mathcal{A}$, an object A is **abelian** when it carries a (necessarily unique) internal Mal'tsev operation: a morphism $g\colon A\times A\times A \to A$ such that $g(x,x,z) = z$ and $g(x,z,z) = x$. As soon as $\mathcal{A}$ is moreover pointed, such an internal Mal'tsev operation is the same thing as an internal abelian group structure. However, in general, the two concepts are different: see, for instance, sections 6 and 7 below where we consider them in a slice category. To avoid confusion, we denote the full subcategory of $\mathcal{A}$ determined by the abelian objects $\mathsf{Mal}(\mathcal{A})$, and we write $\mathsf{Ab}(\mathcal{A})$ for the category of internal abelian groups in $\mathcal{A}$. Again, when $\mathcal{A}$ is pointed, $\mathsf{Ab}(\mathcal{A})$ coincides with the full subcategory $\mathsf{Mal}(\mathcal{A})$ of $\mathcal{A}$.

For instance, an abelian object in the category of groups is an abelian group, and an abelian associative algebra over a field is a vector space (equipped with a trivial multiplication).

1.7. **The Huq commutator.** A coterminal pair

$$K \xrightarrow{k} X \xleftarrow{l} L$$

of morphisms in a homological category **Huq-commutes** [19, 39] when there is a (necessarily unique) morphism φ such that the diagram

$$\begin{tikzcd} & K \arrow[ld, "\langle 1_K{,}0\rangle"'] \arrow[rd, "k"] & \\ K \times L \arrow[rr, dotted, "\varphi"] & & X \\ & L \arrow[lu, "\langle 0{,}1_L\rangle"] \arrow[ru, "l"'] & \end{tikzcd}$$

is commutative. We shall mostly be interested in the case where k and l are normal monomorphisms (i.e., kernels). The **Huq commutator**

$$[k, l]^{\mathrm{Huq}} \colon [K, L]^{\mathrm{Huq}} \to X$$

of k and l is the smallest normal subobject of X that should be divided out to make k and l commute—so that they do commute if and only if $[K, L]^{\mathrm{Huq}} = 0$. It may be obtained through the colimit Q of the outer square above, as the kernel of the (normal epi)morphism $X \to Q$. In a homological category, an object X is abelian when $[X, X]^{\mathrm{Huq}} = 0$.

2. Cross-effects and commutators

We explain how the cross-effects of the identity functor of a homological category give rise to (higher-order) commutators. We start with some basic definitions and properties, give some examples and recall how the binary commutator is a categorical version of the Higgins commutator [36, 38, 50]. Then we focus on right exactness results for cross-effect functors, mostly those valid in semi-abelian categories: preservation of coequalisers of reflexive graphs (Theorem 2.26 and Corollary 2.27) and cokernels inducing certain exact sequences (Proposition 2.31 and Corollary 2.32).

2.1. **Cross-effects of functors.** We recall how the definition of cross-effects given in [5] in the case of groups extends to a general categorical framework [36, 37].

Definition 2.2. [36] Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor from a pointed category with finite sums $\mathcal{C}$ to a pointed finitely complete category $\mathcal{D}$. The **n-th cross-effect of F** is the functor

$$\mathrm{cr}_n(F)\colon \mathcal{C}^n \to \mathcal{D},$$

a multi-functor $\mathcal{C} \to \mathcal{D}$, inductively defined by

$$\mathrm{cr}_1(F)(X) = \mathrm{Ker}\big(F(0)\colon F(X) \to F(0)\big)$$

and, for $n > 1$,

$$\mathrm{cr}_n(F)(X_1, \ldots, X_n) = \mathrm{Ker}(\hat{r}),$$

where
$$\hat{r}\colon F(X_1+\cdots+X_n)\to\prod_{k=1}^{n}F(X_1+\cdots+\widehat{X_k}+\cdots+X_n)$$
is such that $\pi_k\circ\hat{r}=F(\hat{r}_k)$. We usually write
$$F(X_1|\cdots|X_n)=\mathrm{cr}_n(F)(X_1,\dots,X_n)$$
and
$$\ker(\hat{r})=\iota_{X_1,\dots,X_n}=\iota^F_{X_1,\dots,X_n}\colon F(X_1|\cdots|X_n)\to F(X_1+\cdots+X_n).$$
The functor $\mathrm{cr}_n(F)$ acts on morphisms in the obvious way that makes $\iota_{X_1,\dots,X_n}$ natural. When F is the identity functor $1_{\mathcal{A}}$ of $\mathcal{A}$ we write
$$(X_1|\cdots|X_n)=1_{\mathcal{A}}(X_1|\cdots|X_n).$$

Example 2.3. Let us make explicit what happens in the lowest-dimensional cases, which are essential in the present article. When $n=2$ we obtain a short exact sequence
$$0\longrightarrow (X|Y)\xrightarrow{\iota_{X,Y}} X+Y\xrightarrow{\langle r_X,r_Y\rangle} X\times Y\longrightarrow 0$$
for any X, Y in $\mathcal{A}$. Note that the object $(X|Y)$ is denoted $X\diamond Y$ in the article [50]. When $n=3$ and X, Y, Z are objects of $\mathcal{A}$, we consider the morphism
$$X+Y+Z\xrightarrow{\left\langle\left\langle\begin{smallmatrix}i_X\\ i_Y\\ 0\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}i_X\\ 0\\ i_Z\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}0\\ i_Y\\ i_Z\end{smallmatrix}\right\rangle\right\rangle}(X+Y)\times(X+Z)\times(Y+Z),$$
which need no longer be a regular epimorphism; the cross-effect $(X|Y|Z)$ of $1_{\mathcal{A}}$ is its kernel.

Example 2.4. In the case of groups
$$(X|Y)=\langle xyx^{-1}y^{-1}\mid x\in X, y\in Y\rangle,$$
a kind of "formal commutator" of X and Y as explained in [50] and [36]. This fact will prove crucial in what follows.

Given groups X, Y and Z with respective chosen elements x, y and z, the word
$$xyx^{-1}y^{-1}zyxy^{-1}x^{-1}z^{-1}$$
is an example of an element of $(X|Y|Z)$.

Example 2.5. In a pointed variety of algebras $\mathcal{V}$, an element of a sum $X+Y+Z$ is of the shape
$$t(x_1,\dots,x_k,y_1,\dots,y_l,z_1,\dots,z_m)$$
where t is a term of arity $k+l+m$ in the theory of $\mathcal{V}$ and x_1, …, $x_k\in X$, y_1, …, $y_l\in Y$ and z_1, …, $z_m\in Z$. It belongs to the cross-effect $(X|Y|Z)$ if and only if
$$\begin{cases}t(x_1,\dots,x_k,y_1,\dots,y_l,0,\dots,0)=0 & \text{in } X+Y,\\ t(x_1,\dots,x_k,0,\dots,0,z_1,\dots,z_m)=0 & \text{in } X+Z,\\ t(0,\dots,0,y_1,\dots,y_l,z_1,\dots,z_m)=0 & \text{in } Y+Z.\end{cases}$$
Here 0 denotes the unique constant of the theory of $\mathcal{V}$.

Proposition 2.6. *The multi-functors* $\mathrm{cr}_n(F)$ *are*

(i) ***reduced:*** $F(X_1|\cdots|X_n)=0$ *if* $X_k=0$ *for some* $k\in\{1,\dots,n\}$*;*

(ii) ***symmetric:*** *for any permutation* $\sigma\in\Sigma_n$ *there is an isomorphism*
$$\sigma_F\colon F(X_1|\cdots|X_n)\to F(X_{\sigma^{-1}(1)}|\cdots|X_{\sigma^{-1}(n)}),$$
natural in X_1, …, X_n, *and moreover* $\sigma_F\circ\tau_F=(\sigma\circ\tau)_F$.

Proof. Assertion (ii) is obvious. For (i), just observe that if $X_k = 0$, then the morphism $\hat{r}_k$ is an isomorphism, while by definition $\hat{r}_k \circ \iota_{X_1,\dots,X_n} = 0$. □

2.7. **Joins and the sum decomposition.** For subobjects

$$L \rightarrowtail^{l} X \leftarrowtail^{m} M$$

of an object X in a homological category $\mathcal{A}$ we write

$$L \vee M = \mathrm{Im}\bigl(\bigl\langle\begin{smallmatrix} l \\ m \end{smallmatrix}\bigr\rangle\colon L + M \to X\bigr)$$

and $L \vee M = L \rtimes M$ when $L \wedge M = 0$ and L is normal in $L \vee M$. Note that we have $L \vee M = L \rtimes M$ if and only if there is a split short exact sequence

$$0 \longrightarrow L \xrightarrow{l} L \vee M \underset{m}{\rightleftarrows} M \longrightarrow 0,$$

which justifies the semi-direct product notation (see Section 3). As for the sum, morphisms defined on $L \rtimes M$ are completely determined by the effect on L and M, and written in a column.

Proposition 2.8. [36] *Suppose that $\mathcal{A}$ is finitely cocomplete homological, $\mathcal{C}$ is pointed with binary sums and $F\colon \mathcal{C} \to \mathcal{A}$ preserves zero. Then we have a decomposition*

$$F(X + Y) = (F(X|Y) \rtimes F(X)) \rtimes F(Y)$$

for any X, Y in $\mathcal{C}$. □

2.9. **Higher-order commutators.** We need the following categorical notion of commutator (of arbitrary length) which was introduced in [36] and [50] and is more thoroughly studied in [35].

Definition 2.10. Let X be an object of a finitely cocomplete homological category. The **n-fold commutator morphism** of X is the composite morphism

$$c_n^X\colon (X|\cdots|X) \xrightarrow{\iota_{X,\dots,X}} X + \cdots + X \xrightarrow{\left\langle\begin{smallmatrix} 1_X \\ \vdots \\ 1_X \end{smallmatrix}\right\rangle} X.$$

When $x_i\colon X_i \to X$ for $1 \leqslant i \leqslant n$ are subobjects of X, their **commutator** is the subobject

$$\begin{aligned}[X_1, \dots, X_n] &= \mathrm{Im}\bigl((X_1|\cdots|X_n) \xrightarrow{(x_1|\cdots|x_n)} (X|\cdots|X) \xrightarrow{c_n^X} X\bigr) \\ &= \mathrm{Im}\bigl((X_1|\cdots|X_n) \xrightarrow{\iota_{X_1,\dots,X_n}} X_1 + \cdots + X_n \xrightarrow{\left\langle\begin{smallmatrix} x_1 \\ \vdots \\ x_n \end{smallmatrix}\right\rangle} X\bigr)\end{aligned}$$

of X.

Example 2.11. In [35] it is shown that in the category of groups $[X_1, \dots, X_n]$ is indeed essentially generated by all n-fold commutators of elements of $X_1, \dots, X_n$. In particular, the n-fold commutator $[X, \dots, X] = \mathrm{Im}(c_n^X)$ coincides with the n-th term of the lower central series of X, i.e., with the (normal) subgroup generated by the commutators of weight n in X.

Remark 2.12. The binary commutator $[K, L]$ is also studied in [50], where it is called the **Higgins commutator**. It is an conceptual generalisation of the commutator constructed in a varietal context in [38].

In contrast with the Huq commutator, the Higgins commutator $[K, L]$ need not be normal in X, even when both K and L are normal subobjects of X. In fact, the Huq commutator $[K, L]^{\mathrm{Huq}}$ of K, $L \lhd X$ is the normal closure of $[K, L]$, so that $[[K, L], X] \vee [K, L] = [K, L]^{\mathrm{Huq}}$ by Proposition 2.15 below.

Remark 2.13. Note that an object X is abelian if and only if its commutator morphism c_2^X is trivial: $[X,X]=0$ precisely when $[X,X]^{\mathrm{Huq}}=0$.

Remark 2.14. The higher-order commutators are generally not built up out of iterated binary commutators (Example 4.9). Moreover, in general, the lower central series mentioned in Example 2.11 does not coincide with the concept considered in [39].

Proposition 2.15. [36, 50] *If K, $L \leqslant X$ in a semi-abelian category then the normal closure of K in the join $K \vee L$ is $[K,L] \vee K$.* □

Proposition 2.16. [36] *In a semi-abelian category, consider K, $L \leqslant X$. The subobject K is normal in $K \vee L$ if and only if $[K,L] \leqslant K$. In particular,*

(i) *$K \lhd X$ if and only if $[K,X] \leqslant K$;*
(ii) *a morphism $f\colon X \to Y$ in $\mathcal{A}$ is proper if and only if the composite morphism*

$$(X|Y) \xrightarrow{(f|1_Y)} (Y|Y) \xrightarrow{c_2^Y} Y$$

factors through $\mathrm{Im}(f)$. □

Remark 2.17. As further explained in Example 5.11, the exactness of $\mathcal{A}$ is fundamental here; in fact, it is also shown in [36] that the characterisation (i) of normal subobjects is valid in a finitely cocomplete homological category if and only if this category is semi-abelian.

The following basic properties commutators have will be useful throughout the text.

Proposition 2.18. [35] *Let X_1, …, X_n be subobjects of an object X and let $f\colon X \to Y$ be a morphism.*

(o) *Commutators are reduced: if $X_i = 0$ for some i then $[X_1,\dots,X_n]=0$.*
(i) *Commutators are symmetric: for any permutation $\sigma \in \Sigma_n$,*

$$[X_1,\dots,X_n] \cong [X_{\sigma^{-1}(1)},\dots,X_{\sigma^{-1}(n)}].$$

(ii) *Commutators are preserved by direct images:*

$$f[X_1,\dots,X_i,\dots,X_n] = [f(X_1),\dots,f(X_i),\dots,f(X_n)].$$

(iii) *Commutators are monotone: if $M \leqslant X_i$ then*

$$[X_1,\dots,X_{i-1},M,X_{i+1},\dots,X_n] \leqslant [X_1,\dots,X_{i-1},X_i,X_{i+1},\dots,X_n].$$

(iv) *Removing brackets enlarges the object:*

$$[[X_1,\dots,X_i],X_{i+1},\dots,X_n] \leqslant [X_1,\dots,X_i,X_{i+1},\dots,X_n].$$

(v) *Removing duplicates enlarges the object:*

$$[X_1,\dots,X_i,X_{i+1},X_{i+2},\dots,X_n] \leqslant [X_1,\dots,X_i,X_{i+2}\dots,X_n]$$

when $X_i = X_{i+1}$.
(vi) *Commutators satisfy a distribution rule with respect to joins:*

$$[X_1,\dots,X_n,A_1 \vee \dots \vee A_m] = \bigvee_{\substack{1\leqslant k\leqslant m\\ 1\leqslant i_1<\dots<i_k\leqslant m}} [X_1,\dots,X_n,A_{i_1},\dots,A_{i_m}].$$

(vii) *When $\mathcal{A}$ is semi-abelian, if $X_1 \vee \dots \vee X_n = X$ then $[X_1,\dots,X_n]$ is normal in X.* □

2.19. **Inductive nature of cross-effects.** We give an alternative inductive description of the higher-order cross-effects.

Lemma 2.20. *For objects $X_1, \ldots, X_n$ in $\mathcal{C}$ and $1 \leqslant k \leqslant n$ there exist factorisations ι_k, ι'_k, ι''_k as indicated in the following commutative diagram, where $\iota = \iota^F_{X_1,\ldots,X_{k-1},X_k+X_{k+1},X_{k+2},\ldots,X_n}$.*

$$\begin{tikzcd}
F(X_1|\cdots|X_{k-1}|-|X_{k+2}|\cdots|X_n)(X_k|X_{k+1}) \arrow[r, dotted, shift right, "\iota_k"'] \arrow[d, tail, "\iota_{X_k,X_{k+1}}"'] & F(X_1|\cdots|X_n) \arrow[l, dotted, shift right, "\iota''_k"'] \arrow[d, tail, "\iota^F_{X_1,\ldots,X_n}"] \arrow[dl, dotted, "\iota'_k"] \\
F(X_1|\cdots|X_{k-1}|X_k+X_{k+1}|X_{k+2}|\cdots|X_n) \arrow[r, tail, "\iota"'] & F(X_1+\cdots+X_n)
\end{tikzcd}$$

Thus ι_k and ι''_k are mutually inverse isomorphisms, which yields an exact sequence

$$\begin{tikzcd}
0 \arrow[d] \\
F(X_1|\cdots|X_n) \arrow[d, tail, "\iota'_k"] \\
F(X_1|\cdots|X_{k-1}|X_k+X_{k+1}|X_{k+2}|\cdots|X_n) \arrow[d, "\langle r'_{X_k}, r'_{X_{k+1}}\rangle"] \\
F(X_1|\cdots|X_{k-1}|X_k|X_{k+2}|\cdots|X_n)\times F(X_1|\cdots|X_{k-1}|X_{k+1}|X_{k+2}|\cdots|X_n) \arrow[d] \\
0
\end{tikzcd}$$

where $r'_{X_j} = (1_{X_1}|\cdots|1_{X_{j-1}}|r_{X_j}|1_{X_{j+1}}|\cdots|1_{X_n})$ for $j \in \{k, k+1\}$.

Proof. The factorisations ι_k, ι'_k, ι''_k are obtained successively by checking that the post-compositions with the morphisms $\hat{r}_j$ are trivial. □

Notation 2.21. If $F\colon \mathcal{C}^m \to \mathcal{D}$ is a multi-functor then for $1 \leqslant k \leqslant m$ we define a multi-functor $\partial_k F\colon \mathcal{C}^{m+1} \to \mathcal{D}$ by

$$\partial_k F(X_1, \ldots, X_{m+1}) = F(X_1, \ldots, X_{k-1}, -, X_{k+2}, \ldots, X_{m+1})(X_k|X_{k+1}).$$

Lemma 2.22. *For any sequence of integers $k_1, \ldots, k_{n-1}$ such that $1 \leqslant k_j \leqslant j$ there is a natural isomorphism $\mathrm{cr}_n(F) \cong \partial_{k_{n-1}} \cdots \partial_{k_1} F$.* □

This follows immediately from Lemma 2.20 and provides an inductive description of cross-effects which allows for inductive proofs. The subsequent result provides an example of this principle.

Proposition 2.23. *Suppose in addition that $\mathcal{D}$ is homological and that F preserves regular epimorphisms. Then for all objects $X_1, \ldots, X_n$ in $\mathcal{C}$ the functor $F(X_1|\cdots|X_{k-1}|-|X_k|\cdots|X_n)\colon \mathcal{C} \to \mathcal{D}$ also preserves regular epimorphisms.*

Proof. This was proved for $n = 1$ in [36] and then follows for all n by an induction based on Lemma 2.22. □

We now refine this preservation of regular epimorphisms to a more precise right exactness property: preservation of coequalisers of reflexive graphs.

2.24. **Right exactness of cross-effects.** We prove that the cross-effects of a functor which preserves coequalisers of reflexive graphs still preserve coequalisers of reflexive graphs. Our proof shall be based on the following **basic principle** concerning those coequalisers, valid in semi-abelian categories. (But not in merely homological ones!) Let

$$\begin{tikzcd} R \arrow[rr, shift left=2, "d"] \arrow[rr, shift right=2, "c"'] \arrow[dr, "r=\langle d,c\rangle"'] & & G \arrow[ll, "e" description] \arrow[r, "q"] & Q \\ & \mathrm{R}[q] \arrow[ur, shift left, "q_1"] \arrow[ur, shift right, "q_2"'] & & \end{tikzcd} \tag{C}$$

be a reflexive graph with its coequaliser, the induced kernel pair $(\mathrm{R}[q], q_1, q_2)$ and the comparison morphism r. Certainly both q and r are regular epimorphisms. But in fact, the converse also holds: any regular epimorphism q which coequalises d and c is their coequaliser if and only if r is a regular epimorphism.

Lemma 2.25. *Suppose that $\mathcal{A}$ is homological, $\mathcal{C}$ is pointed with binary sums and $F\colon \mathcal{C} \to \mathcal{A}$ is reduced. Then for any X, $Y \in \mathrm{Ob}(\mathcal{C})$ the morphism*

$$\langle F(r_X), F(r_Y)\rangle\colon F(X+Y) \to F(X) \times F(Y)$$

is a regular epimorphism. Hence also the comparison natural transformation

$$F(X+(-)) \Rightarrow F(X) \times F(-)$$

is regular epic.

Proof. Since the functor F is reduced, the triangle

$$\begin{tikzcd} F(X)+F(Y) \arrow[rr, "\left\langle\begin{smallmatrix} F(i_X) \\ F(i_Y)\end{smallmatrix}\right\rangle"] \arrow[dr, "\hat{r}"'] & & F(X+Y) \arrow[dl, "\langle F(r_X),F(r_Y)\rangle"] \\ & F(X)\times F(Y) & \end{tikzcd}$$

commutes. The result follows, as $\hat{r}$ is a regular epimorphism. $\square$

Theorem 2.26. *Suppose that $\mathcal{A}$ is semi-abelian and $\mathcal{C}$ is pointed with binary sums. Let $F\colon \mathcal{C} \to \mathcal{A}$ a functor which is reduced and preserves coequalisers of reflexive graphs. For any object X of $\mathcal{C}$, the induced functor $F(X|-)\colon \mathcal{C} \to \mathcal{A}$ also preserves coequalisers of reflexive graphs. Hence, by induction, so do all resulting functors*

$$F(X_1|\cdots|X_{k-1}|-|X_k|\cdots|X_n).$$

Proof. Consider in $\mathcal{C}$ a reflexive graph with its coequaliser (**C**) and the induced diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d, dotted] & 0 \arrow[d] & 0 \arrow[d] \\
F(X|R) \arrow[r, "r'''"] \arrow[d, tail] & \mathrm{R}[F(1_X|q)] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d] & F(X|G) \arrow[l] \arrow[r, "F(1_X|q)"] \arrow[d, tail] & F(X|Q) \arrow[d, tail] \\
F(X+R) \arrow[r, "r''"] \arrow[d] \arrow[dr, phantom, "\texttt{(i)}"] & \mathrm{R}[F(1_X+q)] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d] & F(X+G) \arrow[l] \arrow[r, "F(1_X+q)"] \arrow[d] \arrow[dr, phantom, "\texttt{(ii)}"] & F(X+Q) \arrow[d] \\
F(X)\times F(R) \arrow[r, "r'"'] \arrow[d] & \mathrm{R}[F(1_X)\times F(q)] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, dotted] & F(X)\times F(G) \arrow[l] \arrow[r, "F(1_X)\times F(q)"'] \arrow[d] & F(X)\times F(Q) \arrow[d] \\
0 & 0 & 0 & 0
\end{tikzcd}$$

in $\mathcal{A}$ that shows how the functor $F(X|-)$ works on this reflexive graph. By the "basic principle" it suffices to prove that both $F(1_X|q)$ and r''' are regular epimorphisms.

In the bottom row, the morphisms r' and $F(1_X) \times F(q)$ are regular epic by the assumption that F preserves coequalisers of reflexive graphs and the fact that also the product functor $F(X)\times(-)$ does. Indeed, products preserve regular epimorphisms and kernel pairs. In particular, $F(1_X)\times F(q)$ is the coequaliser of $F(1_X)\times F(d)$ and $F(1_X) \times F(c)$.

In the middle row, the morphisms r'' and $F(1_X+q)$ are regular epic because the sum functor $X + (-)$ and the functor F preserve coequalisers of reflexive graphs. In particular, $F(1_X + q)$ is the coequaliser of $F(1_X + d)$ and $F(1_X + c)$.

The four lower vertical arrows in the diagram are regular epimorphisms by Lemma 2.25 and by the fact that r' is a regular epimorphism.

In the category $\mathsf{Ext}(\mathcal{A})$, coequalisers are computed degree-wise (see, for instance, [31]). Hence the commutative square `(ii)` may be considered as a regular epimorphism in $\mathsf{Ext}(\mathcal{A})$, so that it represents a double extension in $\mathcal{A}$.

Also the square `(i)` is a double extension in $\mathcal{A}$. To see this, consider the following diagram with exact rows, in which $r' = 1_{F(X)} \times \overline{r}$.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(r'') \arrow[r, tail] \arrow[d, dotted] & F(X+R) \arrow[r, "r''"] \arrow[d] \arrow[dr, phantom, "\texttt{(i)}"] & \mathrm{R}[F(1_X+q)] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(r') \arrow[r, tail] \arrow[d, equal] & F(X)\times F(R) \arrow[r, "r'"] \arrow[d, "\pi_{F(R)}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & F(X)\times \mathrm{R}[F(q)] \arrow[r] \arrow[d, "\pi_{\mathrm{R}[F(q)]}"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(r') \arrow[r, tail] & F(R) \arrow[r, "\overline{r}"'] & \mathrm{R}[F(q)] \arrow[r] & 0
\end{tikzcd}$$

The right and middle composed vertical arrows in it are compatibly split epimorphisms, so that also the left hand side dotted arrows is a split, hence a regular, epimorphism. Lemma 1.5 now implies that the square `(i)` is a double extension.

Since kernels commute with kernel pairs, Lemma 1.5 implies that $F(1_X|q)$ and r''' are regular epic, and the result follows by the "basic principle". □

Corollary 2.27. *For any object X in a semi-abelian category $\mathcal{A}$, the induced functor $(X|-)\colon \mathcal{A} \to \mathcal{A}$ preserves coequalisers of reflexive graphs.* □

To understand the behaviour of the functor $F(X|-)$ with respect to cokernels we introduce the following folding operations between cross-effects of different order which play a fundamental role in all what follows.

Definition 2.28. Consider $X_1, \ldots, X_n \in \mathrm{Ob}(\mathcal{C})$ and let $r_1, \ldots, r_n$ be nonzero natural numbers. Let

$$\nabla^{r_i}_{X_i} = \left\langle \begin{matrix} 1_{X_i} \\ \vdots \\ 1_{X_i} \end{matrix} \right\rangle : r_i \cdot X_i \to X_i$$

denote the folding morphism, and write $X_i^k = X_i$ for $1 \leqslant k \leqslant r_i$. Then a **folding operation** $S^{X_1,\ldots,X_n}_{r_1,\ldots,r_n}$ is defined by requiring the following square to commute.

$$\begin{tikzcd}
\mathrm{cr}_{r_1+\cdots+r_n}(F)(X_1^1,\ldots,X_1^{r_1},X_2^1,\ldots,X_2^{r_2},\ldots,X_n^1,\ldots,X_n^{r_n}) \arrow[r, dotted, "S^{X_1,\ldots,X_n}_{r_1,\ldots,r_n}"] \arrow[d, tail, "\iota_{X_1^1,\ldots,X_n^{r_n}}"'] & \mathrm{cr}_n(F)(X_1,\ldots,X_n) \arrow[d, tail, "\iota_{X_1,\ldots,X_n}"] \\
F(r_2 \cdot X_1 + \cdots + r_n \cdot X_n) \arrow[r, "F(\nabla^{r_1}_{X_1}+\cdots+\nabla^{r_n}_{X_n})"'] & F(X_1+\cdots+X_n)
\end{tikzcd}$$

Remark 2.29. Note that the folding operations are natural in their arguments. It is also easily checked by the very definition of cross-effects that the morphism

$$F(\nabla^{r_1}_{X_1} + \cdots + \nabla^{r_n}_{X_n}) \circ \iota_{X_1^1,\ldots,X_n^{r_n}}$$

does indeed factor through $\iota_{X_1,\ldots,X_n}$.

Notation 2.30. Taking $n = 1$ and writing $m = r_1$ we obtain a natural transformation

$$S^F_m \colon \mathrm{cr}_m(F) \circ \Delta^m \Rightarrow F,$$

with $\Delta^m \colon \mathcal{C} \to \mathcal{C}^m$ the m-fold diagonal functor, defined by

$$(S^F_m)_X \colon\ \mathrm{cr}_m(F)(X,\ldots,X) \xrightarrow{S^X_m} \mathrm{cr}_1(F)(X) \xrightarrow{\iota^F_X} F(X)$$

for $X \in \mathrm{Ob}(\mathcal{C})$.

Proposition 2.31. *Suppose that $\mathcal{C}$ is pointed with binary coproducts, $\mathcal{A}$ is semi-abelian and $F \colon \mathcal{C} \to \mathcal{A}$ is reduced. Then F preserves coequalisers of reflexive graphs if and only if any cokernel*

$$A \xrightarrow{\partial} G \xrightarrow{q} Q \longrightarrow 0$$

in $\mathcal{C}$ gives rise to an exact sequence

$$F(A|G) \rtimes F(A) \xrightarrow{\left\langle \begin{smallmatrix} (S^F_2)_G \circ F(\partial|1_G) \\ F(\partial) \end{smallmatrix} \right\rangle} F(G) \xrightarrow{F(q)} F(Q) \longrightarrow 0 \qquad \textbf{(D)}$$

in $\mathcal{A}$.

Proof. Suppose that F preserves coequalisers of reflexive graphs. For a cokernel as above, the diagram

$$\begin{tikzcd}
A + G \arrow[r, shift left=2, "\left\langle \begin{smallmatrix} \partial \\ 1_G \end{smallmatrix} \right\rangle"] \arrow[r, shift right=2, "\left\langle \begin{smallmatrix} 0 \\ 1_G \end{smallmatrix} \right\rangle"'] & G \arrow[l, "i_G" description] \arrow[r, "q"] & Q
\end{tikzcd}$$

is a reflexive graph with its coequaliser, hence so is its image

$$\begin{tikzcd}
F(A + G) \arrow[r, shift left=2, "F\left(\left\langle \begin{smallmatrix} \partial \\ 1_G \end{smallmatrix} \right\rangle\right)"] \arrow[r, shift right=2, "F\left(\left\langle \begin{smallmatrix} 0 \\ 1_G \end{smallmatrix} \right\rangle\right)"'] & F(G) \arrow[l, "F(i_G)" description] \arrow[r, "F(q)"] & F(Q)
\end{tikzcd} \qquad \textbf{(E)}$$

through F. Since the kernel of $F\left(\left\langle\begin{smallmatrix}0\\1_G\end{smallmatrix}\right\rangle\right)$ is $F(A|G)\rtimes F(A)$ by Proposition 2.8, the sequence

$$F(A|G)\rtimes F(A)\xrightarrow{F\left(\left\langle\begin{smallmatrix}\partial\\1_G\end{smallmatrix}\right\rangle\right)\circ j} F(G)\xrightarrow{F(q)} F(Q)\longrightarrow 0$$

is a cokernel. Here $j\colon F(A|G)\rtimes F(A)\to F(A+G)$ is the canonical inclusion, a normal monomorphism. Hence already the morphism $F\left(\left\langle\begin{smallmatrix}\partial\\1_G\end{smallmatrix}\right\rangle\right)\circ j$, as any normalisation of a reflexive graph, is proper: it is a composite of a split epimorphism with a kernel. Furthermore, this morphism decomposes on the semi-direct product as claimed: first of all, $F\left(\left\langle\begin{smallmatrix}\partial\\1_G\end{smallmatrix}\right\rangle\right)\circ F(i_A)=F(\partial)$; secondly,

$$\begin{aligned}F\left(\left\langle\begin{smallmatrix}\partial\\1_G\end{smallmatrix}\right\rangle\right)\circ\iota_{A,G}&=F\left(\nabla_G^2\right)\circ F(\partial+1_G)\circ\iota_{A,G}\\&=F\left(\nabla_G^2\right)\circ\iota_{G,G}\circ F(\partial|1_G)\\&=(S_2^F)_G\circ F(\partial|1_G).\end{aligned}\tag{F}$$

Conversely, let

$$\begin{tikzcd} R \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & G \arrow[l, "e" description] \arrow[r, "q"] & Q \end{tikzcd}$$

be a reflexive graph with its coequaliser. Then its normalisation

$$\mathrm{Ker}(d)\xrightarrow{c\circ\mathrm{ker}(d)} G\xrightarrow{q} Q\longrightarrow 0$$

is a cokernel, hence for $\partial = c\circ\mathrm{ker}(d)\colon A=\mathrm{Ker}(d)\to G$ we obtain the exact sequence (**D**). Then Proposition 2.8 gives the reflexive graph with its coequaliser (**E**). Since R is a regular quotient of $A+G$ this proves the statement. $\square$

Corollary 2.32. *Suppose that $\mathcal{C}$ is pointed with binary coproducts, $\mathcal{A}$ is semi-abelian and $F\colon\mathcal{C}\to\mathcal{A}$ is reduced and preserves coequalisers of reflexive graphs. Consider an object X and a cokernel*

$$A\xrightarrow{\partial} G\xrightarrow{q} Q\longrightarrow 0$$

in $\mathcal{C}$. Then we obtain an exact sequence

$$F(X|A|G)\rtimes F(X|A)\xrightarrow{\left\langle\begin{smallmatrix}S_{1,2}^{X,G}\circ F(1_X|\partial|1_G)\\ F(1_X|\partial)\end{smallmatrix}\right\rangle} F(X|G)\xrightarrow{F(1_X|q)} F(X|Q)\longrightarrow 0$$

in $\mathcal{A}$.

Proof. By Theorem 2.26 the functor $F(X|-)$ preserves coequalisers of reflexive graphs, hence by Proposition 2.31 the sequence

$$F(X|-)(A|G)\rtimes F(X|A)\xrightarrow{\left\langle\begin{smallmatrix}F\left(1_X\left|\left\langle\begin{smallmatrix}\partial\\1_G\end{smallmatrix}\right\rangle\right.\right)\circ\iota_{A,G}\\ F(1_X|\partial)\end{smallmatrix}\right\rangle} F(X|G)\xrightarrow{F(1_X|q)} F(X|Q)\longrightarrow 0$$

is exact. Now we only need to prove that

$$F\left(1_X\left|\left\langle\begin{smallmatrix}\partial\\1_G\end{smallmatrix}\right\rangle\right.\right)\circ\iota_{A,G}=S_{1,2}^{X,G}\circ F(1_X|\partial|1_G);$$

but this equality is easily obtained when post-composing with the monomorphism $\iota_{X,G}$. $\square$

When the morphism q in the statement of Proposition 2.31 happens to be a split epimorphism in a homological category, the proof may be simplified and the result extended to the case where $\mathcal{A}$ is not necessarily Barr exact.

Proposition 2.33. *Suppose that $\mathcal{C}$ is pointed protomodular with binary coproducts, $\mathcal{A}$ is finitely cocomplete homological and $F\colon \mathcal{C} \to \mathcal{A}$ is reduced and preserves regular epimorphisms. Then any split right-exact sequence*

$$A \xrightarrow{\partial} G \underset{s}{\overset{q}{\rightleftarrows}} Q \longrightarrow 0$$

gives rise to split exact sequences

$$F(A|Q) \rtimes F(A) \xrightarrow{\left\langle \begin{smallmatrix} (S_2^F)_G \circ F(\partial|s) \\ F(\partial) \end{smallmatrix} \right\rangle} F(G) \underset{F(s)}{\overset{F(q)}{\rightleftarrows}} F(Q) \longrightarrow 0 \qquad (\mathbf{G})$$

and, given an object X in $\mathcal{C}$,

$$F(X|A|Q) \rtimes F(X|A) \xrightarrow{\left\langle \begin{smallmatrix} S_{1,2}^{X,G} \circ F(1_X|\partial|s) \\ F(1_X|\partial) \end{smallmatrix} \right\rangle} F(X|G) \underset{F(1_X|s)}{\overset{F(1_X|q)}{\rightleftarrows}} F(X|Q) \longrightarrow 0 \qquad (\mathbf{H})$$

in $\mathcal{A}$.

Proof. Consider the following diagram of solid arrows.

$$\begin{tikzcd}
F(A|Q) \rtimes F(A) \arrow[r, "\left\langle \begin{smallmatrix} \iota^F_{A,Q} \\ F(i_A) \end{smallmatrix} \right\rangle"] \arrow[d, dotted] & F(A+Q) \arrow[r, "F(r_Q)"] \arrow[d, "F\left(\left\langle \begin{smallmatrix} \partial \\ s \end{smallmatrix} \right\rangle\right)"'] & F(Q) \arrow[r] \arrow[d, equal] & 0 \\
\mathrm{Ker}(F(q)) \arrow[r] & F(G) \arrow[r, "F(q)"'] & F(Q) \arrow[r] & 0
\end{tikzcd}$$

Its top row is exact by Proposition 2.8. Moreover, $F\left(\left\langle \begin{smallmatrix} \partial \\ s \end{smallmatrix} \right\rangle\right)$ is a regular epimorphism by the protomodularity of $\mathcal{C}$ and the hypothesis on F. Hence by the uniqueness of image factorisations, $\ker(F(q))$ is equal to

$$\mathrm{im}\left(F\left(\left\langle \begin{smallmatrix} \partial \\ s \end{smallmatrix} \right\rangle\right) \circ \left\langle \begin{smallmatrix} \iota^F_{A,Q} \\ F(i_A) \end{smallmatrix} \right\rangle\right) = \mathrm{im}\left(\left\langle \begin{smallmatrix} (S_2^F)_G \circ F(\partial|s) \\ F(\partial) \end{smallmatrix} \right\rangle\right),$$

taking (**F**) into account. Hence the sequence (**G**) is exact. Now the exactness of (**H**) may be deduced as in the proof of Proposition 2.32, noting that the functor $F(X|-)$ preserves regular epimorphisms (Proposition 2.23). $\square$

3. Internal actions and semi-direct products

There are several ways in which the concept of action can be introduced categorically: starting from monoidal structures [9, 10]; as algebras over a certain monad, so that an equivalence between actions and points is obtained [20, 41]; or via cross-effects, as explained in [36]. The interpretation of actions as algebras due to Bourn and Janelidze is conceptually very beautiful and rests on a deep categorical result: when $\mathcal{A}$ is semi-abelian, the kernel functor $\mathsf{Pt}_G(\mathcal{A}) \to \mathcal{A}$ is monadic, so that the resulting category of algebras is equivalent to $\mathsf{Pt}_G(\mathcal{A})$. Thus the construction of semi-direct products is part of the definition of action from the start; being algebras of a monad, actions form a category of which the properties are well-studied.

In comparison, the definition of actions via cross-effects is rather ad hoc. But, even when actions-as-algebras are formally equivalent to actions-via-cross-effects, in some situations the latter notion is easier to work with. And there is one further, and more important, advantage: these actions are defined using binary cross-effects, but there are also higher cross-effects, which may be used to express properties of actions that are not so easily captured by other means. In the present article we shall be studying one instance of this phenomenon.

3.1. **Basic definition.** Let A, G be objects of $\mathcal{A}$ and $\psi\colon (A|G) \to A$ a morphism. Consider the coequaliser

$$(A|G) \underset{\iota}{\overset{i_A \circ \psi}{\rightrightarrows}} A + G \xrightarrow{q} Q.$$

We say that the pair (A, ψ) is an **action (of G on A)** or a **G-action** when the morphism $k_\psi = q \circ i_A$ is a monomorphism. (Compare with the analysis of actions worked out in [54].) When this happens, we write $A \rtimes_\psi G = Q$ and call Q the **semi-direct product of A and G along ψ**. It fits into the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "k_\psi"] & A \rtimes_\psi G \arrow[r, shift left, "p_\psi"] & G \arrow[l, shift left, "s_\psi"] \arrow[r] & 0 \end{tikzcd} \qquad \textbf{(I)}$$

where p_ψ is induced by $r_G\colon A + G \to G$ and $s_\psi = q \circ i_G$.

It is further proved in [36] that assigning to an action (A, ψ) the point

$$\begin{tikzcd} A \rtimes_\psi G \arrow[r, shift left, "p_\psi"] & G \arrow[l, shift left, "s_\psi"] \end{tikzcd}$$

defines one half of an equivalence between the category $\mathsf{Act}_G(\mathcal{A})$ of G-actions in $\mathcal{A}$ and the category $\mathsf{Pt}_G(\mathcal{A})$ of points in $\mathcal{A}$ over G. The other half takes a point

$$\begin{tikzcd} X \arrow[r, shift left, "p"] & G \arrow[l, shift left, "s"] \end{tikzcd}$$

and sends it to the induced dotted arrow ψ in the diagram with short exact rows

$$\begin{tikzcd} 0 \arrow[r] & (A|G) \arrow[r, "\iota_{A,G}"] \arrow[d, dotted, "\psi"'] & A + G \arrow[r, "\langle r_A, r_G \rangle"] \arrow[d, "\left\langle \begin{smallmatrix} a \\ s \end{smallmatrix} \right\rangle"'] & A \times G \arrow[r] \arrow[d, "\pi_G"] & 0 \\ 0 \arrow[r] & A \arrow[r, "a"'] & X \arrow[r, shift left, "p"] & G \arrow[l, shift left, "s"] \arrow[r] & 0. \end{tikzcd}$$

Example 3.2. In the category of groups any action $\psi\colon (A|G) \to A$ is already a G-group, i.e., the function

$$(g, a) \mapsto g \cdot a = \psi(gag^{-1}a^{-1})a$$

does not only satisfy the rules $1 \cdot a = a$ and $(gg') \cdot a = g \cdot (g' \cdot a)$, but also $g \cdot (aa') = (g \cdot a)(g \cdot a')$. This agrees with the fact that in Gp, semi-direct products correspond with G-groups rather than with general actions.

In practice, it is often desirable to construct suitable actions; a rich source of actions is given by normal monomorphisms as they carry a conjugation action of the object they are contained in [36].

Example 3.3. For any object X, the **conjugation action**

$$c^{X,X} = c_2^X = \nabla_X^2 \circ \iota_{X,X}\colon (X|X) \to X$$

of X on itself corresponds to the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\langle 1_X, 0 \rangle"] & X \times X \arrow[r, shift left, "\pi_2"] & X \arrow[l, shift left, "\langle 1_X, 1_X \rangle"] \arrow[r] & 0. \end{tikzcd}$$

Proposition 3.4. *Let $n\colon N \to X$ be a normal monomorphism in $\mathcal{A}$. Then there is a unique action $c^{N,X}\colon (N|X) \to N$ of X on N such that*

$$\begin{tikzcd} (N|X) \arrow[r, dotted, "c^{N,X}"] \arrow[d, "(n|1_X)"'] & N \arrow[d, tail, "n"] \\ (X|X) \arrow[r, "c^{X,X}"'] & X \end{tikzcd}$$

commutes, the ***conjugation action of*** X ***on*** N*. It is natural in the sense that any commutative square as on the left*

$$\begin{tikzcd} N \arrow[r] \arrow[d, tail] & N' \arrow[d, tail] \\ X \arrow[r] & X' \end{tikzcd} \qquad \begin{tikzcd} (N|X) \arrow[r] \arrow[d, "c^{N,X}"'] & (N'|X') \arrow[d, "c^{N',X'}"] \\ N \arrow[r] & N' \end{tikzcd}$$

gives a commutative square as on the right. □

Proposition 3.5 (Co-universal property of the semi-direct product)**.** *Consider in $\mathcal{A}$ an action $\psi\colon (A|G) \to A$ and morphisms*

$$A \xrightarrow{f} Z \xleftarrow{g} G.$$

Then there exists a (necessarily unique) morphism $\left\langle \begin{smallmatrix} f \\ g \end{smallmatrix} \right\rangle\colon A \rtimes_\psi G \to Z$ such that $\left\langle \begin{smallmatrix} f \\ g \end{smallmatrix} \right\rangle \circ k_\psi = f$ and $\left\langle \begin{smallmatrix} f \\ g \end{smallmatrix} \right\rangle \circ s_\psi = g$ if and only if the square

$$\begin{tikzcd} (A|G) \arrow[r, "\psi"] \arrow[d, "(f|g)"'] & A \arrow[d, "f"] \\ (Z|Z) \arrow[r, "c_2^Z"'] & Z \end{tikzcd}$$

commutes. □

Example 3.6. The **trivial action** of an object G on an object A is the zero morphism $0\colon (A|G) \to A$. Then the semi-direct product $A \rtimes_0 G$ is $A \times G$ and p_0 is the product projection $\pi_G\colon A \times G \to G$. Hence two coterminal morphisms f and g as in Proposition 3.5 Huq-commute if and only if $c_2^Z \circ (f|g)$ is trivial. This of course also follows immediately from the fact that $A \times G$ is the cokernel of $\iota_{A,G}\colon (A|G) \to A + G$ and the equality $c_2^Z \circ (f|g) = \left\langle \begin{smallmatrix} f \\ g \end{smallmatrix} \right\rangle \circ \iota_{A,G}$.

Example 3.7 (Centrality)**.** The conjugation action $c^{N,X}$ of an object X on a normal subobject $N \lhd X$ is trivial if and only if N is **central** in X, which means that $n\colon N \to X$ and $1_X\colon X \to X$ Huq-commute; indeed $n \circ c^{N,X} = c^{X,X} \circ (n|1_X)$. (Compare with Theorem 3.2.4 in [25].)

Starting from conjugation actions we may again construct various new actions by the following device (see Lemma 6.1 below for a partial converse).

Proposition 3.8. [36] *Let $\psi\colon (A|G) \to A$ be an action, let $m\colon M \to A$ be a monomorphism and $h\colon H \to G$ a morphism. Suppose that M is H-stable under ψ, i.e., the morphism $\psi \circ (m|h)\colon (M|H) \to A$ factors through a (necessarily unique) morphism $\varphi\colon (M|H) \to M$ such that the square*

$$\begin{tikzcd} (M|H) \arrow[r, dotted, "\varphi"] \arrow[d, "(m|h)"'] & M \arrow[d, tail, "m"] \\ (A|G) \arrow[r, "\psi"'] & A \end{tikzcd}$$

commutes. Then φ is an action of H on M. □

Definition 3.9. When, in particular, $M = A$ in the above proposition, we write $\varphi = h^*(\psi)$

$$\begin{tikzcd} (A|H) \arrow[r, "h^*(\psi)"] \arrow[d, "(1_A|h)"'] & A \arrow[d, equal] \\ (A|G) \arrow[r, "\psi"'] & A \end{tikzcd} \tag{J}$$

and call φ the **pullback** of ψ along h.

Remark 3.10. This choice of terminology may be justified as follows. Through the equivalence of actions and points, the square (**J**) matches the morphism of split short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, tail, "k_\varphi"] \arrow[d, equal] & A \rtimes_\varphi H \arrow[r, shift left, "p_\varphi"] \arrow[d, "1_A \rtimes h"'] & H \arrow[l, shift left, "s_\varphi"] \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r, tail, "k_\psi"'] & A \rtimes_\psi G \arrow[r, shift left, "p_\psi"] & G \arrow[l, shift left, "s_\psi"] \arrow[r] & 0; \end{tikzcd}$$

Lemma 1.5 tells us that the right hand side square of the diagram is a pullback. (In fact, one easily sees that it is also a pushout.)

Example 3.11. When $N \lhd X$ as in Proposition 3.4,

$$n^*(c^{N,X}) = c^{N,N} = c_2^N.$$

Indeed, $n\circ c^{N,X}\circ(1_N|n) = c^{X,X}\circ(n|1_X)\circ(1_N|n) = c_2^X\circ(n|n)$, which equals $n\circ c_2^N$ by naturality of conjugation actions.

Example 3.12. For any action $\psi\colon (A|G) \to A$,

$$\psi = c^{A,A\rtimes_\psi G}\circ(1_A|s_\psi) = s_\psi^*(c^{A,A\rtimes_\psi G}):$$

the action ψ coincides with the restriction to G of the conjugation action of the semi-direct product $A \rtimes_\psi G$ on A.

3.13. **A first encounter with a ternary cross-effect.** Any action induces certain higher-order operations which we shall need in what follows.

Notation 3.14. Let A, G be objects in and $\psi\colon (A|G) \to A$ a morphism. Consider $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$. Define $\psi_{k,n-k}$ to be the composite morphism

$$\psi_{k,n-k}\colon\ (A|\cdots|A|G|\cdots|G) \xrightarrow{S^{A,G}_{k,n-k}} (A|G) \xrightarrow{\ \psi\ } A.$$

In particular, taking ψ to be the conjugation action $c^{N,X}$ of an object X on some normal subobject $N \lhd X$, we get morphisms

$$c^{N,X}_{k,n-k}\colon (N|\cdots|N|X|\cdots|X) \to N.$$

Note that $c^{N,X}_{1,1} = c^{N,X}$. Also the other conjugation actions $c^{N,X}_{k,n-k}$ are interrelated, the generic relation being the following one:

Lemma 3.15. *For any normal monomorphism $n\colon N \to X$ the equality*

$$c^{N,X}_{2,1} = c^{N,X}_{1,2}\circ(1_N|n|1_X)\colon (N|N|X) \to N$$

holds. In particular, $c^{X,X}_{2,1} = c^{X,X}_{1,2} = c^X_3$.

Proof. Post-compose with n and use the commutative diagrams obtained by injecting the various cross-effects into the corresponding sums. □

This coherence condition in terms of ternary cross-effects will appear again in the analysis of crossed modules: see Theorem 5.7 and Example 5.10 and 5.11 below. We shall also investigate some closely related structures, such as Beck modules and extensions with abelian kernel. Those structures all satisfy variations of this condition, variations which may be expressed in terms of higher-order commutators (sections 6 and 7).

4. The *Smith is Huq* condition

We explain how the *Smith is Huq* condition for finitely cocomplete homological categories may be expressed in terms of cross-effects as the vanishing of a ternary commutator. Thus a condition which is about *locally defined internal categorical structures admitting a global extension* is characterised as a computational obstruction. This is *the* key point of the present article—all results in the ensuing sections are based on it.

Theorem 4.4 characterises when two given equivalence relations R, S on a common object X commute in the Smith/Pedicchio sense: if K and L denote their respective denormalisations,

$$[K, L] = [K, L, X] = 0$$

is a necessary and sufficient condition. This immediately gives a characterisation of the *Smith is Huq* condition (Theorem 4.6), and a formula for the Smith/Pedicchio commutator in terms of cross-effects (Theorem 4.14). We also find a characterisation of double central extensions (Proposition 4.16), which allows us to obtain a Hopf formula for the third homology of an object in any semi-abelian category with enough projectives (Theorem 4.17).

4.1. **The Smith/Pedicchio commutator.** Consider a pair of equivalence relations (R, S) on a common object X

$$\begin{tikzcd} R \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & X \arrow[l, "\Delta_R" description] \arrow[r, "\Delta_S" description] & S, \arrow[l, shift right=2, "s_2"'] \arrow[l, shift left=2, "s_1"] \end{tikzcd}$$

and consider the induced pullback of r_1 and s_2.

$$\begin{tikzcd} R \times_X S \arrow[r, "\pi_S"] \arrow[d, "\pi_R"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "s_2"] \\ R \arrow[r, "r_1"'] & X \end{tikzcd} \qquad (\mathbf{K})$$

The equivalence relations R and S **Smith/Pedicchio-commute** [62, 57, 19] when there is a (necessarily unique) morphism θ (a **connector** between R and S) such that the diagram

$$\begin{tikzcd} & R \arrow[dl, "\langle 1_R, \Delta_S \circ r_1 \rangle"'] \arrow[dr, "r_2"] & \\ R \times_X S \arrow[rr, dotted, "\theta"] & & X \\ & S \arrow[ul, "\langle \Delta_R \circ s_2, 1_S \rangle"] \arrow[ur, "s_1"'] & \end{tikzcd}$$

is commutative. The connector θ is a partially defined Mal'tsev operation on X, as the diagram commutes precisely when $\theta(x, x, z) = z$ for $(x, z) \in S$ and $\theta(x, z, z) = x$ for $(x, z) \in R$. It is also the same thing as a *pregroupoid structure* [46, 44] on the span $(d = \mathrm{coeq}(r_1, r_2), c = \mathrm{coeq}(s_1, s_2))$.

The **Smith/Pedicchio commutator** $[R, S]^{\mathrm{S}}$ **of** R **and** S is the smallest equivalence relation on X that should be divided out to make R and S commute, so that they do commute if and only if $[R, S]^{\mathrm{S}} = \Delta_X$. It may be obtained through the colimit Q of the outer square above, as the kernel of the (normal epi)morphism $X \to Q$.

4.2. **The *Smith is Huq* condition.** The **normalisation** K of an equivalence relation (R, r_1, r_2) on X is the monomorphism

$$r_2 \circ \ker(r_1) \colon K = \mathrm{Ker}(r_1) \to X.$$

We say that a monomorphism is an **ideal** when it is the normalisation of some (necessarily unique) equivalence relation [13]. In a homological category, ideals are direct images of kernels along regular epimorphisms—see [50] for an in-depth analysis. We shall give a precise characterisation of ideals in terms of internal actions in Example 5.11. For now, it suffices to note that the normalisation of an *effective* equivalence relation is always a kernel; conversely, any normal subobject $N \lhd X$ (in the strong sense, i.e., it may be represented by a kernel) admits a **denormalisation** R_N, the kernel pair of its cokernel. This process determines an order isomorphism between the normal subobjects of X and the effective equivalence relations on X, which in the semi-abelian case coincides with the correspondence between ideals and equivalence relations.

It is well known that Smith/Pedicchio-commuting equivalence relations have Huq-commuting normalisations [19]. However, the converse need not hold; in [7, 16] a counterexample is given in the category of digroups, which is a semi-abelian variety, even a variety of Ω-groups [38]. (See also Example 4.9.) Thus arises a property homological categories may or may not have:

Definition 4.3. A homological category satisfies the **Smith is Huq condition (SH)** when any two effective equivalence relations on a given object commute as soon as their normalisations do.

It turns out that the condition (SH) is fundamental in the study of internal categorical structures: it is shown in [53] that, for a semi-abelian category, this condition holds if and only if every star-multiplicative graph is an internal groupoid. As explained in [41] and in Section 5 of the present article, this is important in the definition of internal crossed modules.

The *Smith is Huq* condition is known to hold for pointed strongly protomodular exact categories [19] (in particular, for any Moore category [58]) and for action accessible categories [22, 25] (in particular, for any category of interest [55, 56]). Well-known examples are the categories of groups, Lie algebras, associative algebras, non-unitary rings, and (pre)crossed modules of groups.

Theorem 4.4. *For effective equivalence relations R and S on X with respective normalisations K, $L \lhd X$, the following are equivalent:*

(i) *R and S Smith/Pedicchio-commute;*
(ii) *$[K, L] = 0 = [K, L, X]$.* □

Hence a homological category satisfies (SH) when for any pair of effective equivalence relations of which the normalisations commute, *the ternary commutator obstruction vanishes.* The proof is an obvious application of the following fundamental lemma. The basic admissibility condition which appears in it was first discovered by Martins–Ferreira, see e.g. [51]. (Incidentally, we believe Lemma 4.5 answers part of the question asked in the concluding section of that paper; see also [52].) We shall consider diagrams of shape

$$
\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[dr, "\alpha"'] & B \arrow[l, shift left, "r"] \arrow[d, "\beta"] \arrow[r, shift right, "s"'] & C \arrow[l, shift right, "g"'] \arrow[dl, "\gamma"] \\
& D &
\end{tikzcd}
\tag{$\mathbf{L}$}
$$

with $f\circ r = 1_B = g\circ s$ and $\alpha\circ r = \beta = \gamma\circ s$. By taking the pullback of f with g, any diagram such as ($\mathbf{L}$) may be extended to a diagram

$$
\begin{tikzcd}
A \times_B C \arrow[r, shift left, "\pi_C"] \arrow[d, shift right, "\pi_A"'] & C \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \arrow[ddr, bend left, "\gamma"] & \\
A \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "f"] \arrow[drr, bend right, "\alpha"'] & B \arrow[l, shift left, "r"] \arrow[u, shift right, "s"'] \arrow[dr, "\beta"] & \\
& & D
\end{tikzcd}
$$

in which the square is a double split epimorphism (i.e., also the obvious squares involving splittings commute). The triple (α, β, γ) is said to be **admissible with respect to** (f, r, g, s) if there is a (necessarily unique) morphism $\vartheta\colon A \times_B C \to D$ such that $\vartheta\circ e_1 = \alpha$ and $\vartheta\circ e_2 = \gamma$.

Lemma 4.5. *Given any diagram* ($\mathbf{L}$)*, let* $\overline{k}\colon \overline{K} \to D$ *be the image of* $\alpha\circ\ker(f)$*,* $\overline{l}\colon \overline{L} \to D$ *the image of* $\gamma\circ\ker(g)$ *and* $\overline{\beta}\colon \overline{B} \to D$ *the image of* β*. Then the triple* (α, β, γ) *is admissible with respect to* (f, r, g, s) *if and only if*

$$[\overline{K}, \overline{L}] = 0 = [\overline{K}, \overline{L}, \overline{B}].$$

Proof. We decompose A, C and $A\times_B C$ into semi-direct products and then analyse in terms of the induced actions what it means for ϑ to exist. By the equivalence between actions and points there are unique actions φ, ψ that give rise to the morphisms of split short exact sequences

$$
\begin{tikzcd}
0 \arrow[r] & K \arrow[r, tail, "\ker(f)"] \arrow[d, equal] & A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "r"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r, tail, "k_\varphi"'] & K \rtimes_\varphi B \arrow[u, "\rho", "\cong"'] \arrow[r, shift left, "p_\varphi"] & B \arrow[l, shift left, "s_\varphi"] \arrow[r] & 0
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
0 \arrow[r] & L \arrow[r, tail, "\ker(g)"] \arrow[d, equal] & C \arrow[r, shift left, "g"] & B \arrow[l, shift left, "s"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r, tail, "k_\psi"'] & L \rtimes_\psi B \arrow[u, "\sigma", "\cong"'] \arrow[r, shift left, "p_\psi"] & B \arrow[l, shift left, "s_\psi"] \arrow[r] & 0.
\end{tikzcd}
$$

By Remark 3.10 we obtain the following commutative diagram with exact rows, in which $\zeta = (g\circ\sigma)^*(\varphi) = \varphi\circ(1_K|p_\psi)$.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, tail, "k_\zeta"] \arrow[d, equal] & K \rtimes_\zeta (L \rtimes_\psi B) \arrow[r, shift left, "p_\zeta"] \arrow[d, "\kappa", "\cong"'] & L \rtimes_\psi B \arrow[l, shift left, "s_\zeta"] \arrow[r] \arrow[d, "\sigma", "\cong"'] & 0 \\
0 \arrow[r] & K \arrow[r, tail, "\langle \ker(f), 0\rangle"] \arrow[d, equal] & A \times_B C \arrow[r, shift left, "\pi_C"] \arrow[d, "\pi_A"] & C \arrow[l, shift left, "e_2=\langle r\circ g, 1_C\rangle"] \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & K \arrow[r, tail, "\ker(f)"'] & A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "r"] \arrow[r] & 0
\end{tikzcd}$$

Now write $k = \alpha\circ\ker(f)\colon K \to D$ and $l = \gamma\circ\ker(g)\colon L \to D$. If the desired morphism ϑ exists then

$$\begin{aligned}
\vartheta\circ\kappa &= \vartheta\circ\left\langle\begin{smallmatrix}\langle\ker(f),0\rangle\\ e_2\circ\sigma\end{smallmatrix}\right\rangle = \vartheta\circ\left\langle\begin{smallmatrix}\langle 1_A, s\circ f\rangle\circ\ker(f)\\ e_2\circ\sigma\end{smallmatrix}\right\rangle = \left\langle\begin{smallmatrix}\vartheta\circ e_1\circ\ker(f)\\ \vartheta\circ e_2\circ\sigma\end{smallmatrix}\right\rangle \\
&= \left\langle\begin{smallmatrix}\alpha\circ\ker(f)\\ \gamma\circ\sigma\end{smallmatrix}\right\rangle = \left\langle\begin{smallmatrix}\alpha\circ\ker(f)\\ \left\langle\begin{smallmatrix}\gamma\circ\ker(g)\\ \beta\end{smallmatrix}\right\rangle\end{smallmatrix}\right\rangle = \left\langle\begin{smallmatrix}k\\ \left\langle\begin{smallmatrix}l\\ \beta\end{smallmatrix}\right\rangle\end{smallmatrix}\right\rangle.
\end{aligned}$$

Conversely, if the morphism

$$\vartheta' = \left\langle\begin{smallmatrix}k\\ \left\langle\begin{smallmatrix}l\\ \beta\end{smallmatrix}\right\rangle\end{smallmatrix}\right\rangle$$

exists then $\vartheta = \vartheta'\circ\kappa^{-1}$ satisfies the relevant constraints: it is clear from the above calculation that $\vartheta'\circ\kappa^{-1}\circ e_2 = \gamma$ and that $\vartheta'\circ\kappa^{-1}\circ e_1\circ\ker(f) = \alpha\circ\ker(f)$, but we also have

$$\begin{aligned}
\vartheta'\circ\kappa^{-1}\circ e_1\circ r &= \vartheta'\circ\kappa^{-1}\circ\langle 1_C, s\circ f\rangle\circ r = \vartheta'\circ\kappa^{-1}\circ\langle r, s\rangle = \vartheta'\circ\kappa^{-1}\circ\langle r\circ g, 1_C\rangle\circ s \\
&= \vartheta'\circ\kappa^{-1}\circ e_2\circ s = \gamma\circ s = \beta = \alpha\circ r.
\end{aligned}$$

Thus $\vartheta'\circ\kappa^{-1}\circ e_1 = \alpha$. It follows that the desired morphism ϑ exists if and only if ϑ' exists, which according to Proposition 3.5 is the case if and only if the diagram

$$\begin{tikzcd}
(K|L \rtimes_\psi B) \arrow[r, "\zeta"] \arrow[d, "\left(k\middle|\left\langle\begin{smallmatrix}l\\ \beta\end{smallmatrix}\right\rangle\right)"'] & K \arrow[d, "k"] \\
(D|D) \arrow[r, "c^{D,D}"'] & D
\end{tikzcd} \tag{$\mathbf{M}$}$$

commutes. To find conditions for this to happen we use sequence ($\mathbf{H}$) on the identity functor of $\mathcal{A}$ in order to decompose the object $(K|L \rtimes_\psi B)$ in three parts, via the regular epimorphism

$$\left\langle\begin{smallmatrix}S_{1,2}^{K,L\rtimes_\psi B}\circ(1_K|k_\psi|s_\psi)\\ (1_K|k_\psi)\\ (1_K|s_\psi)\end{smallmatrix}\right\rangle\colon (K|L|B) + (K|L) + (K|B) \to (K|L \rtimes_\psi B).$$

First note that by Example 3.12 and naturality of the conjugation action

$$\begin{aligned}
k\circ\zeta\circ(1_K|s_\psi) &= k\circ\varphi\circ(1_K|p_\psi)\circ(1_K|s_\psi) = k\circ\varphi = k\circ c^{K,K\rtimes_\varphi B}\circ(1_K|s_\varphi) \\
&= c^{D,D}\circ(k|\left\langle\begin{smallmatrix}k\\ \beta\end{smallmatrix}\right\rangle)\circ(1_K|s_\psi) = c^{D,D}\circ(k|\beta) \\
&= c^{D,D}\circ(k|\left\langle\begin{smallmatrix}l\\ \beta\end{smallmatrix}\right\rangle)\circ(1_K|s_\psi),
\end{aligned}$$

so that Diagram ($\mathbf{M}$) always commutes on $(K|B)$.

Next, $k\circ\zeta\circ(1_K|k_\psi)) = k\circ\varphi\circ(1_K|p_\psi)\circ(1_K|k_\psi) = k\circ\varphi\circ(1_K|0) = 0$ by reducedness of the cross-effect. Hence, for the equality

$$k\circ\zeta\circ(1_K|k_\psi)) = c^{D,D}\circ(k|\left\langle\begin{smallmatrix}l\\ \beta\end{smallmatrix}\right\rangle)\circ(1_K|k_\psi))$$

to hold, the morphism $c^{D,D}\circ(k|l) = c_2^D\circ(\overline{k}|\overline{l})\circ(k'|l')$ has to be trivial. (Here we write $k = \overline{k}\circ k'$, and similarly for l and β). Noting that $(k'|l')$ is a regular epimorphism by Proposition 2.23, we see that $c^{D,D}\circ(k|l) = 0$ precisely when $[\overline{K}, \overline{L}] = \mathrm{Im}(c_2^D\circ(\overline{k}|\overline{l}))$ is trivial.

Finally,

$$\begin{aligned} k\circ\zeta\circ S_{1,2}^{K,L\rtimes_\psi B}\circ(1_K|k_\psi|s_\psi) &= k\circ\varphi\circ(1_K|p_\psi)\circ S_{1,2}^{K,L\rtimes_\psi B}\circ(1_K|k_\psi|s_\psi)\\ &= k\circ\varphi\circ S_{1,2}^{K,B}\circ(1_K|p_\psi|p_\psi)\circ(1_K|k_\psi|s_\psi)\\ &= k\circ\varphi\circ S_{1,2}^{K,B}\circ(1_K|0|1_B) \end{aligned}$$

is zero, while

$$\begin{aligned} c^{D,D}\circ(k|\langle\begin{smallmatrix} l\\ \beta\end{smallmatrix}\rangle)\circ S_{1,2}^{K,L\rtimes_\psi B}\circ(1_K|k_\psi|s_\psi) &= c^{D,D}\circ S_{1,2}^{D,D}\circ(k|\langle\begin{smallmatrix} l\\ \beta\end{smallmatrix}\rangle|\langle\begin{smallmatrix} l\\ \beta\end{smallmatrix}\rangle)\circ(1_K|k_\psi|s_\psi)\\ &= c_3^D\circ(k|l|\beta)\\ &= c_3^D\circ(\overline{k}|\overline{l}|\overline{\beta})\circ(k'|l'|\beta'). \end{aligned}$$

As $(k'|l'|\beta')$ is a regular epimorphism by Proposition 2.23, this tells us that Diagram (**M**) commutes on $(K|L|B)$ if and only if $[\overline{K}, \overline{L}, \overline{B}] = \mathrm{Im}(c_3^D\circ(\overline{k}|\overline{l}|\overline{\beta})) = 0$, which concludes the proof. $\square$

Theorem 4.6. *The following are equivalent:*

(i) *the* Smith is Huq *condition holds;*
(ii) *any two effective equivalence relations on a given object commute as soon as their normalisations do;*
(iii) *any two equivalence relations on a given object commute as soon as their normalisations do;*
(iv) *for any K, L ideals of X,*

$$[K, L, X] \leqslant [K, L]^{\mathrm{Huq}}.$$

Proof. Conditions (i) and (ii) are equivalent by definition. The equivalence between (ii) and (iii) is Remark 2.4 in [53], but may also be obtained using Lemma 4.5. Now suppose that (iii) holds and consider normal subobjects K and L of X. Divide out their Huq commutator

$$0 \longrightarrow [K, L]^{\mathrm{Huq}} \rightarrowtriangle X \xrightarrow{q} Q \longrightarrow 0$$

and write $q(K)$, $q(L) \leqslant Q$ for the direct images of K and L along q. By Proposition 2.18.ii we obtain a diagram

$$\begin{tikzcd} & & {[K,L,X]} \arrow[r] \arrow[d, tail] \arrow[dl, dotted] & {[q(K),q(L),Q]} \arrow[d, tail] & \\ 0 \arrow[r] & {[K,L]^{\mathrm{Huq}}} \arrow[r] & X \arrow[r, "q"'] & Q \arrow[r] & 0 \end{tikzcd}$$

and a factorisation of $[K, L, X]$ over $[K, L]^{\mathrm{Huq}}$. Indeed, $[q(K), q(L), Q]$ is zero by Theorem 4.4, as $[q(K), q(L)] = q[K, L] = 0$. Finally, (iv) $\Rightarrow$ (ii) is again a consequence of Theorem 4.4. $\square$

This at once yields a new class of examples.

Example 4.7. A **nilpotent category of class** 2 is a semi-abelian category whose identity functor is **quadratic**, i.e., it has a trivial triple cross-effect [35]. Hence, almost by definition, any such category satisfies (SH). In particular, the *Smith is Huq* condition holds for modules over a square ring, and specifically for algebras over a nilpotent algebraic operad of class two [3].

Example 4.8. When K, L, M are normal subgroups of a group G,

$$[K, L, M] = [K, [L, M]] \vee [L, [M, K]] \vee [M, [K, L]]$$

by a result in [35]. Hence in Gp all triple commutator words are essentially of the shape considered in Example 2.4.

This of course also gives (SH). So far it is not clear which categories allow a similar decomposition of their triple commutators.

For instance, the semi-abelian variety Loop of loops and loop homomorphisms forms a counterexample. We show that it does not satisfy the *Smith is Huq* condition, which also implies that this category is neither action accessible nor strongly protomodular.

Example 4.9. A **loop** is a quasigroup with unit, an algebra

$$(A, \cdot, \backslash, /, 1)$$

of which the multiplication $\cdot$ and the left and right division $\backslash$ and $/$ satisfy the axioms

$$y = x \cdot (x\backslash y) \qquad\qquad y = x\backslash(x \cdot y)$$
$$x = (x/y) \cdot y \qquad\qquad x = (x \cdot y)/y$$

and 1 is a unit for the multiplication, $x \cdot 1 = x = 1 \cdot x$. We shall sometimes write xy for $x \cdot y$. The variety Loop of loops is semi-abelian (as mentioned for instance in [8]). Loops are "non-associative groups", and indeed an associative loop is the same thing as a group. It is easily seen that the abelian objects in Loop are precisely the abelian groups—which are not to be confused with the objects in the variety of commutative loops, which have a commutative, but possibly non-associative, multiplication.

The **associator** of three elements x, y, z of a loop X is the unique element $[\![x, y, z]\!]$ of X such that $(xy)z = [\![x, y, z]\!] \cdot x(yz)$. Hence $[\![x, y, z]\!]$ is equal to $((xy)z)/(x(yz))$. Given three normal subloops K, L and M of X, we write $[\![K, L, M]\!]$ for the **associator subloop** of X determined by K, L and M: the normal subloop of $K \vee L \vee M$ generated by the elements $[\![x, y, z]\!]$, where either (x, y, z) or any of its permutations is in $K \times L \times M$.

It is clear that the object $[\![K, L, M]\!]$ is a subloop of the triple commutator $[K, L, M]$, as for any associator element $[\![x, y, z]\!]$, the associators $[\![1, y, z]\!]$, $[\![x, 1, z]\!]$ and $[\![x, y, 1]\!]$ are trivial (Example 2.5). In general the triple commutator $[K, L, M]$ is bigger though: otherwise the category of groups would be quadratic—which it is not, as there exist examples of groups that are not 2-step nilpotent.

In order to prove that the category Loop does not satisfy the *Smith is Huq* condition, it suffices to give an example of a loop X with an abelian normal subloop A of X such that $[A, A, X]$ is non-trivial. Then by Theorem 4.4 the denormalisation R_A of A does not Smith/Pedicchio-commute with itself, even though $[A, A] = 0$. (This situation is further analysed in Theorem 6.4.) In fact, in our example, already the associator $[\![A, A, X]\!]$ is non-trivial.

We take X to be the well-known (and historically important) loop of order eight occurring in relation with the hyperbolic quaternions: the set

$$\{1, -1, i, -i, j, -j, k, k\}$$

with multiplication determined by the rules

$$\begin{aligned} ij &= k = -ji \\ jk &= i = -kj \qquad\qquad ii = jj = kk = 1 \\ ki &= j = -ik \end{aligned}$$

and the expected behaviour for -1. The subset $\{1,-1,j,-j\}$ of L forms a normal subloop A of index two, isomorphic to the Klein four-group $V \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Now $j(ji) = j(-k) = -i$ while $(jj)i = i$, so

$$1 \neq [\![j,j,i]\!] \in [\![A,A,X]\!] \leqslant [A,A,X].$$

4.10. **Decomposition of the Smith/Pedicchio commutator.** The above Theorem 4.4 leads to a formula for the Smith/Pedicchio commutator of two equivalence relations in terms of binary and ternary commutators of their normalisations: Theorem 4.14.

Lemma 4.11 (cf. Remark 2.12)**.** *For any K, $L \leqslant X$ in a semi-abelian category, the join $[K,L,X] \vee [K,L]$ is normal in X.*

Proof. Consider first the quotient q of X by $[K,L,X]$, then the direct image of $[K,L]$ along q.

$$\begin{tikzcd}
 & & {[K,L]} \arrow[r, two heads] \arrow[d, tail] & {[q(K),q(L)]} \arrow[d, tail] & \\
0 \arrow[r] & {[K,L,X]} \arrow[r, tail] & X \arrow[r, two heads, "q"'] & Q \arrow[r] & 0
\end{tikzcd}$$

Note that $[K,L,X]$ is normal in X by Proposition 2.18.vii. To prove our claim we only need to show that the commutator $[q(K),q(L)]$ is normal in $Q = q(X)$. But

$$[[q(K),q(L)],q(X)] \leqslant [q(K),q(L),q(X)] = q[K,L,X] = 0$$

by Proposition 2.18 so that the result follows from Proposition 2.16. $\square$

Remark 4.12. When, to the above situation, we add $M \leqslant X$ such that $K \vee L \vee M$ is X,

$$[K,L,M] \vee [K,L] = [K,L,X] \vee [K,L].$$

Indeed, freely using the rules from Proposition 2.18, we see that

$$\begin{aligned}
&[K,L,K \vee L \vee M] \\
&\qquad = [K,L,K,L,M] \vee [K,L,K,L] \vee [K,L,L,M] \\
&\qquad\qquad \vee [K,L,K,M] \vee [K,L,K] \vee [K,L,L] \vee [K,L,M] \\
&\qquad \leqslant [K,L,M] \vee [K,L] \vee [K,L,M] \\
&\qquad\qquad \vee [L,K,M] \vee [L,K] \vee [K,L] \vee [K,L,M] \\
&\qquad = [K,L,M] \vee [K,L],
\end{aligned}$$

while the other inclusion is obvious.

Remark 4.13. If K, $L \lhd X$ are such that $K \vee L = X$ then $[K,L] = 0$ suffices for the respective denormalisations R and S of K and L to commute in the Smith/Pedicchio-sense [32]. In other words, when $[K,L]$ is trivial, the ternary commutator $[K,L,X]$ is trivial as well. By Remark 4.12 this also follows from

$$[K,L,X] \vee [K,L] = [K,L,0] \vee [K,L] = [K,L].$$

Theorem 4.14. *In a semi-abelian category, given equivalence relations R and S on X with respective normalisations K, $L \lhd X$, the Smith/Pedicchio commutator $[R,S]^{\mathrm{S}}$ is the left hand side equivalence relation*

$$\begin{tikzcd}[column sep=large]
([K,L,X] \vee [K,L]) \rtimes_{\gamma} X \arrow[r, shift left=2, "\left\langle\begin{smallmatrix} 0 \\ 0 \\ 1_X \end{smallmatrix}\right\rangle"] \arrow[r, shift right=2, "\left\langle\begin{smallmatrix} [k,l,1_X] \\ {[k,l]} \\ 1_X \end{smallmatrix}\right\rangle"'] & X \arrow[l, "s_{\gamma}" description]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
{[K,L]} \rtimes_{\gamma} X \arrow[r, shift left=2, "\left\langle\begin{smallmatrix} 0 \\ 1_X \end{smallmatrix}\right\rangle"] \arrow[r, shift right=2, "\left\langle\begin{smallmatrix} {[k,l]} \\ 1_X \end{smallmatrix}\right\rangle"'] & X \arrow[l, "s_{\gamma}" description]
\end{tikzcd}$$

where γ is the conjugation action of X on $[K, L, X] \vee [K, L]$. When, in addition, $K \vee L = X$ then $[R, S]^{\mathrm{S}}$ simplifies to the above right hand side equivalence relation.

Proof. The equivalence relation in the statement above is the denormalisation of the normal subobject $[K, L, X] \vee [K, L]$ of X considered in Lemma 4.11. By Theorem 4.4 it satisfies the same universal property as $[R, S]^{\mathrm{S}}$, thus it coincides with it. The further refinement is just Remark 4.13. □

4.15. **An application to homology.** One situation where expressing the Smith/Pedicchio commutator in terms of cross-effects yields immediate results is in semi-abelian (co)homology. For instance, according to [31] the Hopf formula for the third homology object $\mathrm{H}_3(Z, \mathsf{ab})$ of an object Z with coefficients in the abelianisation functor

$$\mathsf{ab}\colon \mathcal{A} \to \mathsf{Ab}(\mathcal{A})\colon A \mapsto A/[A, A]^{\mathrm{Huq}}$$

depends on a characterisation of the double central extensions in $\mathcal{A}$. Such a characterisation was given in [60] in terms of the Smith/Pedicchio commutator: a double extension such as (**N**) below is central if and only if

$$[\mathrm{R}[d], \mathrm{R}[c]]^{\mathrm{S}} = \Delta_X = [\mathrm{R}[d] \wedge \mathrm{R}[c], \nabla_X]^{\mathrm{S}}.$$

Here ∇_X is the largest equivalence relation on X, the denormalisation of 1_X. When (SH) holds this condition may be reformulated in terms of the Huq commutator, and when $\mathcal{A}$ has enough projectives this makes it possible to express $\mathrm{H}_3(Z, \mathsf{ab})$ as a quotient of commutators. So far, however, it was unclear how to obtain a similar explicit formula in categories that do not satisfy (SH).

Proposition 4.16. *Given a double extension*

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "d"'] & C \arrow[d, "g"] \\ D \arrow[r, "f"'] & Z \end{tikzcd} \qquad (\mathbf{N})$$

in a semi-abelian category, write $K = \mathrm{Ker}(c)$ and $L = \mathrm{Ker}(d)$. Then (**N**) *is central if and only if*

$$[K, L, X] = [K, L] = [K \wedge L, X] = 0.$$

Proof. Via Theorem 4.4 this is an immediate consequence of Theorem 2.8 in [60]. □

Recall that a **double presentation** of an object Z is a double extension such as (**N**) in which the objects X, D and C are (regular epi)-projective.

Theorem 4.17. *Let $\mathcal{A}$ be a semi-abelian category with enough projectives. Let Z be an object in $\mathcal{A}$ and* (**N**) *a double presentation of Z with $K = \mathrm{Ker}(c)$ and $L = \mathrm{Ker}(d)$. Then*

$$\mathrm{H}_3(Z, \mathsf{ab}) = \frac{K \wedge L \wedge [X, X]}{[K, L, X] \vee [K, L] \vee [K \wedge L, X]}.$$

When, moreover, $\mathcal{A}$ is monadic over Set, *these homology groups are comonadic Barr–Beck homology* [2] *with respect to the canonical comonad on $\mathcal{A}$.*

Proof. This follows from Proposition 4.16 and the main result of [30]; see also [31]. Note that by Lemma 4.11 and Proposition 2.18.vii, the denominator is indeed normal in X so that the formula makes sense. □

5. Internal crossed modules

Now we turn to the study of crossed modules from the viewpoint of the definition of actions in terms of cross-effects. It turns out that this literally generalises the classical definition as in the case of groups—except for a higher coherence condition which does not appear in any of the usual categories where crossed modules have been considered, such as groups, Lie algebras and associative algebras. It expresses the property (SH) needed to extend a star-multiplication to an internal category structure in arbitrary semi-abelian categories, or even finitely cocomplete homological ones—see [41, 53].

5.1. **Internal categories and internal groupoids, star-multiplicative graphs and Peiffer graphs.** The analysis of the *Smith is Huq* condition in terms of higher-order commutators yields new conditions for an internal reflexive graph to be an internal category (or, equivalently, an internal groupoid).

Theorem 5.2. *Consider an internal reflexive graph* (R, G, d, c, e).

$$\begin{tikzcd} R \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & G \arrow[l, "e" description] \end{tikzcd} \qquad d\circ e = c\circ e = 1_G \qquad (\mathbf{O})$$

The following are equivalent:

(i) (R, G, d, c, e) *is an internal category;*
(ii) $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = 0 = [\mathrm{Ker}(d), \mathrm{Ker}(c), R]$*;*
(iii) $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = 0 = [\mathrm{Ker}(d), \mathrm{Ker}(c), \mathrm{Im}(e)]$*;*
(iv) *the conjugation action* $c^{A,R}\colon (A|R) \to A$ *of* R *on* A *factors through the morphism* $(1_A|c)\colon (A|R) \to (A|G)$*;*
(v) $c^{A,R} = (e\circ c)^*(c^{A,R})$.

Proof. Theorem 4.4 implies that(i) and (ii) are equivalent, because the given reflexive graph is a groupoid if and only if the kernel pairs $\mathrm{R}[d]$ and $\mathrm{R}[c]$ of d and c Smith/Pedicchio-commute [57]. It is clear that (ii) implies (iii), while the equivalence between (i) and (iii) may be obtained via Lemma 4.5. In fact (ii) also follows from (iii) by a direct commutator calculation using Proposition 2.18, as $R = A \vee \mathrm{Im}(e)$.

The equivalence between (iii) and (iv) is a consequence of Proposition 2.33. Finally, if $c^{A,R} = c^*(\varphi)$ then

$$e^*(c^{A,R}) = e^*(c^*(\varphi)) = (c\circ e)^*(\varphi) = \varphi,$$

so that $c^{A,R} = c^*(e^*(c^{A,R})) = (e\circ c)^*(c^{A,R})$. □

Condition (ii) on commuting kernels says that a reflexive graph (R, G, d, c, e) with a multiplication $m\colon \mathrm{Ker}(d) \times \mathrm{Ker}(c) \to R$ defined locally around 0 as in

$$\begin{tikzcd} & 0 \arrow[dl, "\beta"'] & \\ \cdot & & \cdot \arrow[ul, "\alpha"'] \arrow[ll, dotted, "\gamma"] \end{tikzcd} \qquad m(\beta, \alpha) = \gamma$$

such that $m\circ\langle 1_{\mathrm{Ker}(d)}, 0\rangle = \ker(d)$ and $m\circ\langle 0, 1_{\mathrm{Ker}(c)}\rangle = \ker(c)$ extends to a globally defined multiplication (i.e., an internal category structure) if and only if the obstruction $[\mathrm{Ker}(d), \mathrm{Ker}(c), R]$ vanishes. Similar "local to global" properties were studied in [49, 53] after they appeared naturally in [41]. Since both are relevant in what follows, we briefly recall their definition; see [49, 53] and Remark 5.8 for more details and a proof that the structures are equivalent.

Consider a reflexive graph (R, G, d, c, e) and the pullback

$$\begin{tikzcd}
R \times_G \mathrm{Ker}(d) \arrow[r, "\pi_{\mathrm{Ker}(d)}"] \arrow[d, "\pi_R"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ker}(d) \arrow[d, "\partial = c \circ \ker(d)"] \\
R \arrow[r, "d"'] & G.
\end{tikzcd}$$

The reflexive graph is **star-multiplicative** [41] when there is a (necessarily unique) morphism $\varsigma\colon R \times_G \mathrm{Ker}(d) \to \mathrm{Ker}(d)$ such that the conditions $\varsigma \circ \langle \ker(d), 0 \rangle = 1_{\mathrm{Ker}(d)}$ and $\varsigma \langle e \circ \partial, 1_{\mathrm{Ker}(d)} \rangle = 1_{\mathrm{Ker}(d)}$ hold.

$$\begin{tikzcd}[column sep=small]
& \cdot \arrow[dl, "\beta"'] & \\
\cdot & & 0 \arrow[ul, "\alpha"'] \arrow[ll, dotted, "\gamma"]
\end{tikzcd}
\qquad\qquad \zeta(\beta, \alpha) = \gamma$$

It is **Peiffer** [49] when there is a (necessarily unique) $\omega\colon \mathrm{Ker}(d) \times \mathrm{Ker}(d) \to R$ such that $\omega \circ \langle 1_{\mathrm{Ker}(d)}, 0 \rangle = \ker(d)$ and $\omega \circ \langle 1_{\mathrm{Ker}(d)}, 1_{\mathrm{Ker}(d)} \rangle = e \circ c \circ \ker(d)$.

$$\begin{tikzcd}[column sep=small]
& 0 \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\
\cdot & & \cdot \arrow[ll, dotted, "\gamma"]
\end{tikzcd}
\qquad\qquad \omega(\beta, \alpha) = \gamma$$

5.3. **Precrossed modules and crossed modules.** A precrossed module is a normalisation of a reflexive graph, while a crossed module is a normalisation of an internal groupoid. We describe these structures in terms of internal actions.

Definition 5.4. A **precrossed module** in a semi-abelian category $\mathcal{A}$ is a quadruple (G, A, μ, ∂) where G and A are objects in $\mathcal{A}$, $\mu\colon (A|G) \to A$ is an action of G on A, and $\partial\colon A \to G$ is a G-equivariant morphism with respect to the action μ and the conjugation action of G on itself, respectively. In other words, the following square commutes.

$$\begin{tikzcd}
(A|G) \arrow[r, "\mu"] \arrow[d, "(\partial|1_G)"'] & A \arrow[d, "\partial"] \\
(G|G) \arrow[r, "c^{G,G}"'] & G
\end{tikzcd} \tag{$\mathbf{P}$}$$

Together with the obvious morphisms, the precrossed modules in $\mathcal{A}$ form a category $\mathsf{PXMod}(\mathcal{A})$.

Proposition 5.5. *The category* $\mathsf{PXMod}(\mathcal{A})$ *is equivalent to* $\mathsf{RG}(\mathcal{A})$.

Proof. This is an extension of the equivalence between actions and split epimorphisms. Given a precrossed module (G, A, μ, ∂), the action μ corresponds to a split exact sequence

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\ker(d)"] & R \arrow[r, shift left=2, "d"] \arrow[r, dotted, shift right=2, "c"'] & G \arrow[l, "e"'] \arrow[r] & 0
\end{tikzcd} \tag{$\mathbf{Q}$}$$

where $R = A \rtimes_\mu G$. Proposition 3.5 gives a unique morphism $c\colon R \to G$ such that $\partial = c \circ \ker(d)$ and $c \circ e = 1_G$ precisely when $(\mathbf{P})$ commutes. $\square$

In particular, since the category $\mathsf{PXMod}(\mathcal{A})$ is equivalent to a category of diagrams in $\mathcal{A}$, it is homological or semi-abelian when so is $\mathcal{A}$.

Definition 5.6. A precrossed module (G, A, μ, ∂) is a **crossed module** if its associated reflexive graph is an internal category. This gives us the full reflective [57] subcategory $\mathsf{XMod}(\mathcal{A})$ of $\mathsf{PXMod}(\mathcal{A})$.

Janelidze analysed this concept of crossed module using internal actions in semi-abelian categories [41]. Our actions are different, and thus we obtain a different characterisation, valid in finitely cocomplete homological categories:

Theorem 5.7. *A precrossed module (G, A, μ, ∂) in a finitely cocomplete homological category is a crossed module if and only if it satisfies the following two additional conditions:*

(i) *the conjugation action of A on itself coincides with the pullback of μ along ∂, i.e., $c^{A,A} = \partial^*(\mu)$ so that the square*

$$\begin{tikzcd} (A|A) \arrow[r, "c^{A,A}"] \arrow[d, "(1_A|\partial)"'] & A \arrow[d, equal] \\ (A|G) \arrow[r, "\mu"'] & A \end{tikzcd} \tag{$\mathbf{R}$}$$

commutes;

(ii) *the square*

$$\begin{tikzcd} (A|A|G) \arrow[r, "\mu_{2,1}"] \arrow[d, "(1_A|\partial|1_G)"'] & A \arrow[d, equal] \\ (A|G|G) \arrow[r, "\mu_{1,2}"'] & A \end{tikzcd} \tag{$\mathbf{S}$}$$

commutes.

Proof. Using Proposition 2.8, we decompose the object R in such a way that the fifth condition of Theorem 5.7 falls apart in three distinct statements. One of those is the commutativity of ($\mathbf{R}$), a second one is the commutativity of ($\mathbf{S}$), and a third is trivially satisfied.

Indeed, $R = A \rtimes_\mu G$, so that we may consider the following pair of parallel morphisms.

$$((A|A|G) \rtimes (A|A)) \rtimes (A|G) \longrightarrow (A|A+G) \xrightarrow{(1_A|q)} (A|A \rtimes_\mu G) \underset{(e\circ c)^*(c^{A,R})}{\overset{c^{A,R}}{\rightrightarrows}} A$$

On $(A|G)$ these morphisms coincide, as $q\circ i_G = e\colon G \to A \rtimes_\mu G = R$ by definition of e, and

$$\begin{aligned} (e\circ c)^*(c^{A,R})\circ(1_A|e) &= e^*((e\circ c)^*(c^{A,R})) = (e\circ c\circ e)^*(c^{A,R}) \\ &= e^*(c^{A,R}) = c^{A,R}\circ(1_A|e). \end{aligned}$$

On $(A|A)$ they coincide if and only if the square ($\mathbf{R}$) commutes. To see this, recall that $q = \left\langle \begin{smallmatrix} \ker(d) \\ e \end{smallmatrix} \right\rangle\colon A + G \to A \rtimes_\mu G = R$, so that $q\circ i_A$ is the monomorphism $\ker(d)\colon A \to R$. Then

$$\ker(d)\circ c^{A,R}\circ(1_A|\ker(d)) = \ker(d)\circ c^{A,A}$$

by naturality of conjugation actions (Proposition 3.4), and

$$\begin{aligned} \ker(d)\circ(e\circ c)^*(c^{A,R})\circ(1_A|\ker(d)) &= \ker(d)\circ c^{A,R}\circ(1_A|e\circ c)\circ(1_A|\ker(d)) \\ &= \ker(d)\circ c^{A,R}\circ(1_A|e)\circ(1_A|c\circ\ker(d)) \\ &= \ker(d)\circ\mu\circ(1_A|\partial). \end{aligned}$$

Hence $c^{A,A} = \mu\circ(1_A|\partial)$ if and only if $c^{A,R}$ and $(e\circ c)^*(c^{A,R})$ coincide on $(A|A)$.

Similarly, $c^{A,R}$ and $(e{\circ}c)^*(c^{A,R})$ coincide on $(A|A|G)$ precisely when $(\mathbf{S})$ commutes. For a proof, consider the commutative diagrams

$$\begin{tikzcd}
(A|A|G) \arrow[r, "(1_A|1_A|e)"] \arrow[d, "\iota^{(A|-)}_{A,G}\circ\iota''_2"'] \arrow[dr, "\iota_{A,A,G}"] & (A|A|R) \arrow[dr, "\iota_{A,A,R}"] & \\
(A|A+G) \arrow[r, "\iota_{A,A+G}"'] \arrow[d, "(1_A|q)"'] & A+A+G \arrow[r, "1_A+1_A+e"] \arrow[d, "1_A+q"] & A+A+R \arrow[dl, "\left\langle\begin{smallmatrix} i_A \\ i_R\circ\ker(d) \\ i_R\end{smallmatrix}\right\rangle"'] \arrow[ddl, bend left, "\left\langle\begin{smallmatrix} \ker(d) \\ \ker(d) \\ 1_R\end{smallmatrix}\right\rangle"'] \arrow[dd, "\ker(d)+\ker(d)+1_R"] \\
(A|R) \arrow[r, "\iota_{A,R}"] \arrow[d, "c^{A,R}"'] & A+R \arrow[d, "\left\langle\begin{smallmatrix} \ker(d) \\ 1_R\end{smallmatrix}\right\rangle"'] & \\
A \arrow[r, tail, "\ker(d)"'] & R & R+R+R \arrow[l, "\nabla^3_R"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A|A|G) \arrow[r, "(1_A|1_A|e)"] \arrow[d, "S^{A,G}_{2,1}"] \arrow[dd, bend right, "\mu_{2,1}"'] & (A|A|R) \arrow[d, "S^{A,R}_{2,1}"] \arrow[drr, bend left, "\iota_{A,A,R}"] & & \\
(A|G) \arrow[r, "(1_A|e)"] \arrow[d, "\mu"] & (A|R) \arrow[r, "\iota_{A,R}"] \arrow[d, "c^{A,R}"] & A+R \arrow[d, "\left\langle\begin{smallmatrix} \ker(d) \\ 1_R\end{smallmatrix}\right\rangle"'] & A+A+R \arrow[l, "\nabla_A+1_R"'] \arrow[d, "\ker(d)+\ker(d)+1_R"] \\
A \arrow[r, equal] & A \arrow[r, tail, "\ker(d)"'] & R & R+R+R \arrow[l, "\nabla^3_R"]
\end{tikzcd}$$

which show that $\mu_{2,1} = c^{A,R}{\circ}(1|q){\circ}\iota^{(A|-)}_{A,G}{\circ}\iota''_2$. Similar diagrams show that

$$\mu_{1,2}{\circ}(1_A|\partial|1_G) = (e{\circ}c)^*(c^{A,R}){\circ}(1|q){\circ}\iota^{(A|-)}_{A,G}{\circ}\iota''_2,$$

and these two equalities together are precisely what we need to prove our claim. □

Alternatively, in this proof we could have used Sequence $(\mathbf{H})$ as in the proof of Lemma 4.5.

Remark 5.8. Condition (i) could be called the **Peiffer condition**. It means that the reflexive graph induced by (G, A, μ, ∂) is a Peiffer graph: the commutativity of $(\mathbf{R})$ gives us a morphism of split short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, tail, "\langle 1_A, 0\rangle"] \arrow[d, equal] & A\times A \arrow[r, shift left, "\pi_2"] \arrow[d, "\omega"'] & A \arrow[l, shift left, "\langle 1_A, 1_A\rangle"] \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A \arrow[r, tail, "\ker(d)"'] & R \arrow[r, shift left, "d"] & G \arrow[l, shift left, "e"] \arrow[r] & 0
\end{tikzcd}$$

as in Example 3.3. The conditions $\ker(d) = \omega{\circ}\langle 1_A, 0\rangle$ and $e{\circ}\partial = \omega{\circ}\langle 1_A, 1_A\rangle$ tell us that ω is a Peiffer structure on (R, G, d, c, e). By Proposition 3.7 in [53] this is equivalent to the reflexive graph being star-multiplicative in the sense of [41], or—when $\mathcal{A}$ is semi-abelian—the condition that $\ker(d)$ and $\ker(c)$ commute.

The star-multiplication on (R, G, d, c, e) may also be obtained directly from the commutativity of $(\mathbf{R})$. Indeed, via the co-universal property of semi-direct products (Proposition 3.5) we see that the needed morphism

$$\zeta\colon A \rtimes_{\partial^*(\mu)} A = R \times_G A \to A$$

exists if and only if $\partial^*(\mu) = c^{A,A}$.

Hence a semi-abelian category satisfies (SH) if and only if the coherence condition (ii) always comes for free: any precrossed module that satisfies the Peiffer condition is a crossed module. This happens, for instance, in all of the examples considered below in 5.9.

In a non-exact context this is not quite true. As explained in the last paragraph of [53], in order that (SH) be equivalent to the condition "all star-multiplications come from internal category structures", a slight strengthening of the definitions of star-multiplicative graph and of Peiffer graph imposes itself. Thus asking that (ii) always follows from (i) in a finitely cocomplete homological category seems formally stronger than assuming (SH), as the Peiffer condition (i) only gives "weak" Peiffer graphs.

Examples 5.9. When $\mathcal{A}$ is the category of groups, the above Definition 5.6 of a crossed module is equivalent with the classical definition, because the coherence condition (ii) follows from (i). In the case of augmented (i.e., non-unitary) associative algebras we obtain the definition due to Dedecker and Lue [26, 47] and Baues [4], and in the case of Lie algebras the one considered by Kassel and Loday [45].

Example 5.10 (Kernels)**.** In all classical algebraic examples, any kernel is a crossed module. This is of course true in general. Given a short exact sequence ($\mathbf{A}$), the quadruple $(X, A, c^{A,X}, a)$ is a precrossed module by naturality of the conjugation action, and it satisfies the Peiffer condition by Example 3.11 and the coherence condition ($\mathbf{S}$) by Example 3.15. The internal category corresponding to $(X, A, c^{A,X}, a)$ is the kernel pair $\mathrm{R}[p]$ of p. In a semi-abelian category, any crossed module (G, A, μ, ∂) where ∂ is monic is of this shape, as follows either from the next example or from Proposition 2.16 and the commutativity of the square ($\mathbf{P}$).

Example 5.11 (Ideals)**.** In a non-exact setting, however, the existence of the action μ is not enough to guarantee that ∂ is a kernel. Actually, when (G, A, μ, ∂) is a crossed module in a finitely cocomplete homological category, ∂ being a monomorphism is equivalent to the corresponding internal category being an equivalence relation—but there is no reason why this equivalence relation would be effective. The kernel of ∂ is precisely

$$\mathrm{Ker}(d) \wedge \mathrm{Ker}(c) = \mathrm{Ker}(\langle d, c\rangle \colon R \to G \times G),$$

which is zero if and only if d and c are jointly monic.

In any case, a monic precrossed module is the same thing as an ideal (Subsection 4.2). Furthermore, any monic *pre*crossed module is automatically a crossed module, since in a Mal'tsev category any reflexive relation is an equivalence relation. In other words, when ∂ is a monomorphism, the commutativity of ($\mathbf{P}$) and the naturality of conjugation actions (Proposition 3.4) imply that also ($\mathbf{R}$) and ($\mathbf{S}$) commute.

Some classical properties of crossed modules of groups easily generalise to homological or semi-abelian categories. Note that here we do not yet use the coherence condition ($\mathbf{S}$); it is needed, however, in a refinement of property (ii) given in Proposition 7.18 below.

Proposition 5.12. *Let (G, A, μ, ∂) be a crossed module in a finitely cocomplete homological category $\mathcal{A}$. Then the following properties hold:*

(i) *K is central in A, so that in particular K is abelian;*
(ii) *K is stable under μ: there is a unique action κ such that the square*

$$\begin{tikzcd}
(K|G) \arrow[r, dotted, "\kappa"] \arrow[d, "(k|1_G)"'] & K \arrow[d, "k"] \\
(A|G) \arrow[r, "\mu"'] & A
\end{tikzcd}$$

commutes;

(iii) *when $\mathcal{A}$ is semi-abelian, the morphism ∂ is proper; we thus have an exact sequence*

$$0 \longrightarrow K \rhd\!\xrightarrow{k} A \xrightarrow{\partial} G \xrightarrow{p}\!\rhd Q \longrightarrow 0 \qquad (\mathbf{T})$$

where $k = \ker(\partial)$ *and* $p = \operatorname{coker}(\partial)$.

Proof. Via Example 3.7, to obtain (i) must prove that the conjugation action of A on K is trivial. By Proposition 3.4 we have that $k \circ c^{K,A} = c^{A,A} \circ (k|1_A)$, so that the result follows from the Peiffer condition ($\mathbf{R}$) and the symmetry of cross-effects. Indeed, since $\nabla_A \circ \mathrm{tw}_A = \nabla_A \colon A + A \to A$ where tw_A denotes the twisting isomorphism $\left\langle \begin{smallmatrix} r_2 \\ r_1 \end{smallmatrix} \right\rangle \colon A + A \to A + A$, we have

$$\begin{aligned} k \circ c^{K,A} &= c^{A,A} \circ (k|1_A) = c^{A,A} \circ \mathrm{tw}' \circ (k|1_A) \\ &= \mu \circ (1_A|\partial) \circ \mathrm{tw}' \circ (k|1_A) = \mu \circ (1_A|\partial) \circ (1_A|k) \circ \mathrm{tw}'' = 0, \end{aligned}$$

where the tw' and tw'' are the obviously induced twistings of the cross-effect.

Statement (ii) follows from the precrossed module condition ($\mathbf{P}$), which implies $\partial \circ \mu \circ (k|1_G) = c^{G,G} \circ (\partial|1_G) \circ (k|1_G) = 0$, so that $\mu \circ (k|1_G)$ factors uniquely over the kernel k of ∂. The resulting morphism κ is an action by Proposition 3.8.

Statement (iii) again follows from Proposition 2.16 since the square ($\mathbf{P}$) commutes. (In general homological categories we only know that $\mathrm{Im}(\partial)$ is an ideal, as essentially explained in Example 5.11.) □

6. Extensions with abelian kernel vs. abelian extensions

There is a subtle difference between the concept of **extension with abelian kernel**—any short exact sequence

$$0 \longrightarrow A \rhd\!\xrightarrow{a} X \xrightarrow{p}\!\rhd G \longrightarrow 0 \qquad (\mathbf{U})$$

where the kernel A is abelian—and the notion of **abelian extension**, a regular epimorphism $p \colon X \to G$ which is an abelian object in the slice category $\mathcal{A}/G$. Since "abelian object" here means that p admits an internal Mal'tsev operation, this amounts to the condition $[\mathrm{R}[p], \mathrm{R}[p]] = \Delta_X$ (see, for instance, the analysis made in [21]). We write $\mathsf{AbExt}(\mathcal{A})$ the full subcategory of $\mathsf{Ext}(\mathcal{A})$ determined by the abelian extensions in $\mathcal{A}$.

As, for some purposes in homological algebra, one needs extensions with abelian kernel to be abelian extensions—see for instance Section 8—it is worth exploring this instance of the *Smith is Huq* property in more detail. Certainly, any abelian extension has abelian kernel, while unlike what happens for groups, in an arbitrary semi-abelian category an extension with abelian kernel need not be an abelian extension (see [7, 16]; in fact also Example 4.9 gives a counterexample, as follows easily from Theorem 6.4). In the present section we investigate the problem from the point of view of internal actions and ternary commutators.

Lemma 6.1. *Suppose that $\mathcal{A}$ is semi-abelian. Let $\psi \colon (A|X) \to A$ be an action in $\mathcal{A}$ and $p \colon X \to G$ a regular epimorphism such that there exists a (necessarily unique) factorisation*

$$\begin{tikzcd} (A|X) \arrow[r, "\psi"] \arrow[d, "(1_A|p)"'] & A \arrow[d, equal] \\ (A|G) \arrow[r, "\varphi"'] & A \end{tikzcd}$$

of ψ. Then φ is an action of G on A.

Proof. This can most conveniently be proved using the extension of ψ to an algebra structure $\xi\colon A\flat X \to A$, cf. [36] and [20, 41], and using the fact that p induces a regular epimorphism $1_A\flat p\colon A\flat X \to A\flat G$. This allows to check the algebra conditions for the factorisation $\zeta\colon A\flat G \to A$ of ξ by precomposing with $1_A\flat p$ and using the obvious commutative diagrams. $\square$

Example 6.2 (Central extensions)**.** The conjugation $c^{A,X}$ of a given short exact sequence (**U**) induces the trivial action $\psi_p = 0\colon (A|G) \to A$ if and only if $c^{A,X}$ itself is trivial, which by Example 3.7 means that (**U**) is a **central extension**: the kernel A of $p\colon X \to G$ is central in X. (Since the denormalisation of A is the kernel pair $\mathrm{R}[p]$ of p, by Example 3.6 this also means that $\mathrm{R}[p]$ is a product of A and X, cf. [18].)

Note that $[A, X]$ being trivial immediately implies that also $[A, A, X]$ is zero by Proposition 2.18.

In the semi-abelian case this gives another classical "extreme" instance of a crossed module (cf. Examples 5.10 and 5.11), the situation where the arrow ∂ is a regular epimorphism. Indeed, when p is a central extension, the quadruple (G, X, μ, p) where $\mu\colon (X|G) \to X$ is the factorisation of $c^{X,X}$ over the morphism $(1_X|p)\colon (X|X) \to (X|G)$ satisfies the three crossed module conditions. First note that such a factorisation exists by Corollary 2.32 since

$$\mathrm{Im}(c^{X,X}\circ S^{X,X}_{1,2}\circ(1_X|a|1_X)) = \mathrm{Im}(c^X_3\circ(1_X|a|1_X)) = [X, A, X] \subset [A, X] = 0$$

as $A = \mathrm{Ker}(p)$ is central. Moreover, μ is an action by Lemma 6.1 and satisfies the Peiffer condition by its very definition. The precrossed module condition now follows by naturality of conjugation actions (Proposition 3.4) since $(1_X|p)$ is a regular epimorphism. Finally, the square corresponding to (**S**) commutes by Lemma 3.15:

$$\begin{aligned}\mu_{1,2}\circ(1_X|p|1_G)\circ(1_X|1_X|p) &= \mu\circ S^{X,G}_{1,2}\circ(1_X|p|p) = \mu\circ(1_X|p)\circ S^{X,X}_{1,2} = c^{X,X}\circ S^{X,X}_{1,2}\\ &= c^{X,X}\circ S^{X,X}_{2,1} = \mu\circ(1_X|p)\circ S^{X,X}_{2,1} = \mu\circ S^{X,G}_{2,1}\circ(1_X|1_X|p)\\ &= \mu_{2,1}\circ(1_X|1_X|p),\end{aligned}$$

while $(1_X|1_X|p)$ is a regular epimorphism by Proposition 2.23.

In fact, as shown in [18], central extensions in a semi-abelian category are precisely normalisations of internal **connected** groupoids, i.e., groupoids (**O**) where $\langle d, c\rangle\colon R \to G \times G$ is regular epic.

Lemma 6.3. *Consider a short exact sequence* (**U**).

(i) *If p is split by s the conjugation action $c^{A,X}$ of X on A admits a factorisation*

$$\begin{tikzcd} (A|X) \arrow[r, "c^{A,X}"] \arrow[d, "(1_A|p)"'] & A \arrow[d, equal] \\ (A|G) \arrow[r, dotted, "\psi_p"'] & A \end{tikzcd}$$

if and only if A is abelian and $c^{A,X}_{2,1}\circ(1_A|1_A|s) = 0$.

(ii) *Suppose that p is arbitrary but $\mathcal{A}$ is semi-abelian. Then the conjugation action $c^{A,X}$ factors through $(1_A|p)$ if and only if A is abelian and $c^{A,X}_{2,1} = 0$. Moreover, when this happens, ψ_p is an action of G on A.*

Proof. We only treat (ii), the proof of which may easily be adapted to (i) using Proposition 2.33 instead of Corollary 2.32. The latter tells us that for semi-abelian $\mathcal{A}$ the action $c^{A,X}$ factors through the morphism $(1_A|p)$ if and only if

$$c^{A,X}\circ S^{A,X}_{1,2}\circ(1_A|a|1_X) \qquad \text{and} \qquad c^{A,X}\circ(1_A|a)$$

are trivial. But $c^{A,X}\circ(1_A|a) = a\circ c^{A,A}$ by naturality of the conjugation action, and $c^{A,X}_{1,2}\circ(1_A|a|1_X) = c^{A,X}_{2,1}$ by Lemma 3.15. Lemma 6.1 now says that (A,ψ_p) is an action. □

The next result is an immediate consequence of Theorem 4.4 and Lemma 6.3.

Theorem 6.4. *For an extension with abelian kernel* (**U**) *the following are equivalent:*

(i) *p is an abelian extension;*
(ii) $[\mathrm{R}[p],\mathrm{R}[p]] = \Delta_X$*;*
(iii) $[A,A,X] = 0$*;*
(iv) $c^{A,X}_{2,1} = 0$.

When $\mathcal{A}$ is semi-abelian, these properties are equivalent to:

(v) *the conjugation action $c^{A,X}$ of X on A factors through a (necessarily unique) action ψ_p of G on A;*
(vi) *$c^{A,X} = p^*(\psi_p)$ for some morphism $\psi_p\colon (A|G)\to A$.* □

Corollary 6.5. *If $\mathcal{A}$ is semi-abelian the inclusion $\mathsf{AbExt}(\mathcal{A})\to\mathsf{Ext}(\mathcal{A})$ has a left adjoint*

$$\mathsf{ab}\colon \mathsf{Ext}(\mathcal{A})\to\mathsf{AbExt}(\mathcal{A})$$

which takes an extension $p\colon X\to G$ and maps it to its induced quotient

$$\mathsf{ab}(p)\colon \frac{X}{[A,A]\vee[A,A,X]}\to G.$$

Proof. This follows from Lemma 4.11 which says that $[A,A]\vee[A,A,X]$ is normal in X. □

7. Beck modules

Where abelian extensions are abelian objects in a slice category $\mathcal{A}/G$, Beck modules [6, 2] are abelian groups in $\mathcal{A}/G$ or, equivalently, abelian objects in the category of points $\mathsf{Pt}_G(\mathcal{A})$. We here provide several equivalent characterisations of Beck modules in finitely cocomplete homological categories, again in terms of internal actions and (higher-order) cross-effects.

7.1. **Beck modules.** Given an object G of a finitely cocomplete homological category $\mathcal{A}$, a **G-module** or **Beck module over** G is an abelian group in the slice category $\mathcal{A}/G$. Thus a G-module (p,m,s) consists of a morphism $p\colon X\to G$ in $\mathcal{A}$, equipped with a multiplication m and a unit s as in the commutative triangles

$$\begin{tikzcd} X\times_G X \arrow[r, "m"] \arrow[dr, "p_\times"'] & X \arrow[d, "p"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "s"] \arrow[dr, equal] & X \arrow[d, "p"] \\ & G \end{tikzcd}$$

satisfying the usual axioms. (Here we write $X\times_G X$ for the kernel pair $\mathrm{R}[p]$ of p, and we put $p_\times = p\circ m = p\circ\pi_1 = p\circ\pi_2$.) In particular we obtain a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\ker(p)"] & X \arrow[r, shift left, "p"] & G \arrow[l, shift left, "s"] \arrow[r] & 0 \end{tikzcd} \qquad (\mathbf{V})$$

where A is an abelian object in $\mathcal{A}$ and p is split by s. Furthermore, since as an abelian extension it carries an internal Mal'tsev operation, the morphism p satisfies $[\mathrm{R}[p],\mathrm{R}[p]]^{\mathrm{S}} = \Delta_X$. Conversely, given the splitting s of p, this latter condition makes it possible to recover the multiplication m. Hence, for split epimorphisms in $\mathcal{A}$, "being a Beck module" is a property; the entire module structure is contained in the splitting. Using the equivalence between split epimorphisms and internal actions, we can replace X with a semi-direct product $A \rtimes_{\psi} G$. By the above, modules are "abelian actions". We write $\mathsf{Mod}_G(\mathcal{A})$ for the category $\mathsf{Ab}(\mathcal{A}/G) = \mathsf{Mal}(\mathsf{Pt}_G(\mathcal{A}))$ of G-modules in $\mathcal{A}$.

Examples 7.2. [6] In the category Gp, a Beck module over G is the same thing as a classical module over the group-ring $\mathbb{Z}G$. In the category $\mathsf{Alg}_{\mathbb{K}}$ of associative (non-unitary) algebras over a commutative ring $\mathbb{K}$, a Beck module over G is a G-G-bimodule. On the other hand, when $\mathcal{A}$ is an additive category, the kernel functor determines an equivalence $\mathsf{Mod}_G(\mathcal{A}) \simeq \mathcal{A}$.

Theorem 7.3. *Let A be an abelian object in $\mathcal{A}$ endowed with an internal G-action $\psi\colon (A|G) \to A$. Then the following are equivalent:*

(i) *(A, ψ) is a G-module;*
(ii) *$(G, A, \psi, 0)$ is a crossed module;*
(iii) *$\psi_{2,1}\colon (A|A|G) \to A$ is trivial.*

Proof. Let $(\mathbf{V})$ be the split short exact sequence induced by ψ. Then (A, ψ) is a G-module if and only if the reflexive graph

$$\begin{tikzcd} X \arrow[r, shift left=2, "p"] \arrow[r, shift right=2, "p"'] & G \arrow[l, "s" description] \end{tikzcd}$$

is an internal category. Since $p \circ \ker(p) = 0$, this proves that (i) and (ii) are equivalent.

Since A is abelian, already $[A, A] = 0$. So Theorem 5.7 tells us that Condition (ii) holds precisely when $\psi_{2,1} = \psi_{1,2} \circ (1_A|0|1_G) = 0$, i.e., when (iii) holds. □

Remark 7.4. Condition (iii) is equivalent with requiring that $\psi_{p,q} = 0$ for all $p \geqslant 2$ since these morphisms $\psi_{p,q}$ clearly factor through $\psi_{2,1}$.

Corollary 7.5. *Suppose that $\mathcal{A}$ satisfies (SH). Then any abelian object in $\mathcal{A}$ endowed with an action of an object G is a G-module.* □

Example 7.6. The situation considered in Example 4.9 is actually a loop action of the cyclic group of order two $\mathbb{Z}_2$ on the Klein four-group $V \cong A$ which is not a module structure. Indeed, the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, tail] & X \arrow[r, shift left, two heads] & \{1, i\} \arrow[l, shift left, dotted] \arrow[r] & 0 \end{tikzcd}$$

is split by the inclusion of $\mathbb{Z}_2 \cong \{1, i\}$ in X. (But the subloop $\{1, i\}$ is not normal in X, as $(ij)j = kj = -i \notin \{1, i\}$ although $(1j)j = 1$.) Hence $X \cong V \rtimes_{\psi} \mathbb{Z}_2$ for some action $\psi\colon (V|\mathbb{Z}_2) \to V$ in the category of loops. Now (V, ψ) cannot be a $\mathbb{Z}_2$-module, as we know that $[R_A, R_A]^{\mathrm{S}} \neq \Delta_X$; so $\psi_{2,1}$ must be non-trivial—and indeed, $\psi_{2,1}[\![j, j, i]\!] = -1$.

Corollary 7.7. *Suppose that $\mathcal{A}$ is semi-abelian. Then for any object G in $\mathcal{A}$ the forgetful functor*

$$\mathsf{Mod}_G(\mathcal{A}) \to \mathsf{Act}_G(\mathcal{A})$$

has a left adjoint $\mathsf{ab}_G\colon \mathsf{Act}_G(\mathcal{A}) \to \mathsf{Mod}_G(\mathcal{A})$, determined by the natural exact sequence

$$\begin{tikzcd}[column sep=large] (A|A) + (A|A|G) \arrow[r, "\left\langle \substack{c^{A,A} \\ \psi_{2,1}} \right\rangle"] & A \arrow[r, two heads, "\eta^G_A"] & \mathsf{ab}_G(A, \psi) \arrow[r] & 0 \end{tikzcd}$$

where η^G_A is a cokernel of the left-hand morphism. Moreover, the natural transformation η^G defined in this way is the unit of the adjunction.

Proof. We must show that $\mathsf{ab}_G(A,\psi)$ carries a G-module structure such that η^G is G-equivariant. We first check that the morphism $k_\psi \circ \left\langle\begin{smallmatrix} c^{A,A} \\ \psi_{2,1} \end{smallmatrix}\right\rangle$ represents a normal subobject of $X = A \rtimes_\psi G$: in fact,

$$\begin{aligned} k_\psi \circ \psi_{2,1} &= k_\psi \circ \psi \circ S_{2,1}^{A,G} = k_\psi \circ c^{A,X} \circ (1_A | s_\psi) \circ S_{2,1}^{A,G} = c^{X,X} \circ (k_\psi | s_\psi) \circ S_{2,1}^{A,G} \\ &= c^{X,X} \circ S_{2,1}^{X,X} \circ (k_\psi | k_\psi | s_\psi) = c_3^X \circ (k_\psi | k_\psi | s_\psi), \end{aligned}$$

whence $\mathrm{Im}(k_\psi \circ \psi_{2,1}) = [A,A,X]$. Thus $\mathrm{Im}(k_\psi \circ \left\langle\begin{smallmatrix} c^{A,A} \\ \psi_{2,1} \end{smallmatrix}\right\rangle) = [A,A] \vee [A,A,X]$ which is normal in X by Lemma 4.11. Now consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "k_\psi", tail] \arrow[d, "\eta_A^G"'] & X \arrow[r, "p_\psi", two heads] \arrow[d, "\eta'"'] & G \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{Coker}(\left\langle\begin{smallmatrix} c^{A,A} \\ \psi_{2,1} \end{smallmatrix}\right\rangle) \arrow[r, tail, "\overline{k_\psi}"'] & \mathrm{Coker}(k_\psi \circ \left\langle\begin{smallmatrix} c^{A,A} \\ \psi_{2,1} \end{smallmatrix}\right\rangle) \arrow[r, two heads, "\overline{p_\psi}"'] & G \arrow[r] & 0 \end{tikzcd}$$

with $\eta' = \mathrm{coker}(k_\psi \circ \left\langle\begin{smallmatrix} c^{A,A} \\ \psi_{2,1} \end{smallmatrix}\right\rangle)$. Both rows are exact; for the bottom row this follows from the Noether isomorphism theorem. Furthermore, $\overline{s_\psi} = \eta' \circ s_\psi$ is a section of $\overline{p_\psi}$. It follows that $(\overline{p_\psi}, \overline{s_\psi})$ is a point in $\mathcal{A}$, giving rise to an action $\overline{\psi}$ of G on $\mathsf{ab}_G(A,\psi)$; moreover, $\eta' \colon (p_\psi, s_\psi) \to (\overline{p_\psi}, \overline{s_\psi})$ is a morphism of points, so that $\eta_A^G \colon (G,A,\psi) \to (G,A,\overline{\psi})$ is G-equivariant. It remains to show that $\overline{\psi}$ is a G-module structure; by Theorem 7.3 it suffices to show that $\overline{\psi}_{2,1} = 0$. But this easily follows from naturality of the morphisms $S_{2,1}$ and the fact that $(\eta_A^G | \eta_A^G | 1_G)$ is a (regular) epimorphism by Proposition 2.23. □

Example 7.8. In a semi-abelian variety of algebras $\mathcal{V}$, consider an abelian object A and an internal G-action $\psi \colon (A|G) \to A$. Then the coherence condition $\psi_{2,1} = 0$ which must hold for ψ to be a module structure may be expressed as follows (cf. Example 2.5):

$$\begin{cases} t(a_1, \ldots, a_k, a_{k+1}, \ldots, a_{k+l}, 0, \ldots, 0) = 0 & \text{in } A + A \\ t(a_1, \ldots, a_k, 0, \ldots, 0, g_1, \ldots, g_m) = 0 & \text{in } A + G \\ t(0, \ldots, 0, a_{k+1}, \ldots, a_{k+l}, g_1, \ldots, g_m) = 0 & \text{in } A + G \end{cases}$$

$$\Rightarrow$$

$$\psi(t(a_1, \ldots, a_{k+l}, g_1, \ldots, g_m)) = 0,$$

for any term t of arity $k+l+m$ in the theory of $\mathcal{V}$ and all $a_1, \ldots, a_{k+l} \in A$ and $g_1, \ldots, g_m \in G$. We believe this is a basic condition; certainly it is of the same level of complexity as for instance the characterisation of ideals due to Ursini [63], valid in semi-abelian varieties [43].

Theorem 7.9 (cf. Theorem 6.4)**.** *Let A be an abelian object in $\mathcal{A}$ endowed with an internal G-action $\psi \colon (A|G) \to A$. Then (A,ψ) is a G-module if and only if the conjugation action of $A \rtimes_\psi G$ on A factors through the G-action on A via the projection $p_\psi \colon A \rtimes_\psi G \to G$. In other words,*

$$c^{A, A \rtimes_\psi G} = \psi \circ (1_A | p_\psi) = p_\psi^*(\psi).$$

Proof. We pass via Condition (iii) in Theorem 7.3. Recall that $X = A \rtimes_\psi G$. Applying Lemma 6.3 to the split extension (**I**) shows that the action

$$c^{A,X} \colon (A|X) \to A$$

factors through $(1_A|p_\psi)$ if and only if $c^{A,X}_{2,1}\circ(1_A|1_A|s_\psi)=0$. However,

$$\begin{aligned} c^{A,X}_{2,1}\circ(1_A|1_A|s_\psi) &= c^{A,X}\circ S^{A,X}_{2,1}\circ(1_A|1_A|s_\psi) \\ &= c^{A,X}\circ(1_A|s_\psi)\circ S^{A,G}_{2,1} \\ &= \psi\circ S^{A,G}_{2,1} = \psi_{2,1}. \end{aligned}$$

Now suppose that $c^{A,X}$ does factor as a composite morphism

$$(A|X) \xrightarrow{(1_A|p_\psi)} (A|G) \xrightarrow{\overline{c}} A;$$

then $\overline{c}=\overline{c}\circ(1_A|p_\psi)\circ(1_A|s_\psi)=c^{A,X}\circ(1_A|s_\psi)=\psi$, which proves our claim. □

Remark 7.10. This directly leads to the (known) result that in an action representable category [9, 10], any action on an abelian object is a module structure. Indeed, it was shown in [22] that any semi-abelian category $\mathcal{A}$ for which the functors

$$\mathsf{Act}(-,A)\colon \mathcal{A}\to \mathsf{Set}_*$$

are representable satisfies (SH). Hence Corollary 7.5 yields the claimed result. Let us now prove this in a different way, passing via Theorem 7.9.

Recall that the functor $\mathsf{Act}(-,A)$ assigns to $G\in \mathrm{Ob}(\mathcal{A})$ the pointed set of actions of G on A. We shall only assume that for any *abelian* object A, the functor $\mathsf{Act}(-,A)\colon \mathcal{A}\to \mathsf{Set}_*$ is representable. This means that there exists an object $[A]$ in $\mathcal{A}$ together with a natural equivalence

$$\alpha\colon \mathsf{Act}(-,A)\Rightarrow \mathcal{A}(-,[A]).$$

Since the functor $\mathcal{A}(-,[A])$ preserves split short exact sequences, any internal action $\psi\colon (A|G)\to G$ induces a short exact sequence

$$0 \longrightarrow \mathsf{Act}(G,A) \underset{\mathsf{Act}(s_\psi,A)}{\overset{\mathsf{Act}(p_\psi,A)}{\rightleftarrows}} \mathsf{Act}(A\rtimes_\psi G,A) \xrightarrow{\mathsf{Act}(k_\psi,A)} \mathsf{Act}(A,A) \longrightarrow 0$$

of pointed sets. (Recall that the category $\mathsf{Set}_*^{\mathrm{op}}$ is semi-abelian [17].) Now consider the action $c^{A,A\rtimes_\psi G}\in \mathsf{Act}(A\rtimes_\psi G,A)$. We have

$$\mathsf{Act}(k_\psi,A)(c^{A,A\rtimes_\psi G}) = c^{A,A\rtimes_\psi G}\circ(1_A|k_\psi) = k_\psi\circ c^{A,A}=0$$

by naturality of conjugation actions (Proposition 3.4) and by Remark 2.13 as A is abelian. Hence $c^{A,A\rtimes_\psi G}$ is in the image of $\mathsf{Act}(p_\psi,A)=p_\psi^*(-)$, so that already $c^{A,A\rtimes_\psi G}=p^*_\psi(\varphi)$ for some action φ of G on A. Now

$$\varphi = s^*_\psi(p^*_\psi(\varphi)) = s^*_\psi(c^{A,A\rtimes_\psi G}) = \psi$$

by Example 3.12, and ψ is a G-module by Theorem 7.9.

7.11. **The biproduct of two modules.** We now work towards Theorem 7.13 which characterises modules in even more elementary terms. To do so, we shall express the biproduct in the additive category $\mathsf{Mod}_G(\mathcal{A})$ as a product of actions.

Lemma 7.12. *The biproduct $(A,\psi)\oplus(B,\varphi)$ in $\mathsf{Mod}_G(\mathcal{A})$ has as underlying G-action the product $(A,\psi)\times(B,\varphi)$ in $\mathsf{Act}_G(\mathcal{A})$, which is the object $A\times B$ in $\mathcal{A}$ equipped with the **diagonal action***

$$\psi\oplus\varphi\colon (A\times B|G) \xrightarrow{\langle(\pi_A|1_G),(\pi_B|1_G)\rangle} (A|G)\times(B|G) \xrightarrow{\psi\times\varphi} A\times B.$$

Proof. Let

$$0\longrightarrow A \overset{a}{\rightarrowtail} X \underset{s}{\overset{p}{\rightleftarrows}} G \longrightarrow 0$$

be the split short exact sequence corresponding to (A,ψ) and

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "b", tail] & Y \arrow[r, shift left, "q"] & G \arrow[l, shift left, "t"] \arrow[r] & 0 \end{tikzcd}$$

the one corresponding to (B,φ); then the biproduct $(A,\psi)\oplus(B,\varphi)$ corresponds to the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A\times B \arrow[r, "a\times b", tail] & X\times_G Y \arrow[r, shift left] & G \arrow[l, shift left, "\langle s,t\rangle"] \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{A}$. By naturality of the conjugation action, the squares

$$\begin{tikzcd} (A\times B|X\times_G Y) \arrow[r, "c^{A\times B,X\times_G Y}"] \arrow[d, "(\pi_A|\pi_X)"'] & A\times B \arrow[d, "\pi_A"] \\ (A|X) \arrow[r, "c^{A,X}"'] & A \end{tikzcd} \qquad \begin{tikzcd} (A\times B|X\times_G Y) \arrow[r, "c^{A\times B,X\times_G Y}"] \arrow[d, "(\pi_B|\pi_X)"'] & A\times B \arrow[d, "\pi_B"] \\ (B|Y) \arrow[r, "c^{B,Y}"'] & B \end{tikzcd}$$

commute, so that the conjugation action of $X\times_G Y$ on $A\times B$ decomposes as

$$(A\times B|X\times_G Y)\xrightarrow{\langle(\pi_A|\pi_X),(\pi_B|\pi_Y)\rangle}(A|X)\times(B|Y)\xrightarrow{c^{A,X}\times c^{B,Y}}A\times B.$$

The asserted decomposition of the diagonal action, which by Example 3.12 is equal to $\langle s,t\rangle^*(c^{A\times B,X\times_G Y})=c^{A\times B,X\times_G Y}\circ(1_{A\times B}|\langle s,t\rangle)$, now follows, as the diagram

$$\begin{tikzcd} (A\times B|G) \arrow[r, "\langle(\pi_A|1_G),(\pi_B|1_G)\rangle"] \arrow[d, "(1_{A\times B}|\langle s,t\rangle)"'] & (A|G)\times(B|G) \arrow[r, "\psi\times\varphi"] \arrow[d, "(1_A|s)\times(1_B|t)"'] & A\times B \arrow[d, equal] \\ (A\times B|X\times_G Y) \arrow[r, "\langle(\pi_A|\pi_X),(\pi_B|\pi_Y)\rangle"'] & (A|X)\times(B|Y) \arrow[r, "c^{A,X}\times c^{B,Y}"'] & A\times B \end{tikzcd}$$

commutes. □

It is clear that any G-module (p,m,s) corresponds to a morphism of split short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A\times A \arrow[r, "a_\times", tail] \arrow[d, "+"'] & X\times_G X \arrow[r, shift left, "p_\times"] \arrow[d, "m"'] & G \arrow[l, shift left, "s_\times"] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "a"', tail] & X \arrow[r, shift left, "p"] & G \arrow[l, shift left, "s"] \arrow[r] & 0 \end{tikzcd}$$

where $+\colon A\times A\to A$ is the abelian group structure on A, the morphism $a_\times$ is $a\times a\colon A\times A\to X\times_G X$ and $s_\times=\langle s,s\rangle$. Hence via Lemma 7.12, the correspondence between actions and points gives us

Theorem 7.13. *Let A be an abelian object in $\mathcal{A}$ endowed with an internal G-action $\psi\colon(A|G)\to A$. Then (A,ψ) is a G-module if and only if the sum $+\colon A\times A\to A$ is G-equivariant with respect to the diagonal action of G on $A\times A$, i.e., if and only if the diagram*

$$\begin{tikzcd} (A\times A|G) \arrow[r, "\langle(\pi_1|1_G),(\pi_2|1_G)\rangle"] \arrow[d, "(+|1_G)"'] & (A|G)\times(A|G) \arrow[r, "\psi\times\psi"] & A\times A \arrow[d, "+"] \\ (A|G) \arrow[rr, "\psi"'] & & A \end{tikzcd} \qquad (\mathbf{W})$$

commutes. □

Example 7.14. Note the parallel with the equality $g\cdot(a+b) = g\cdot a+g\cdot b$ which holds in the case of groups. As explained in Example 3.2, this latter condition is automatically fulfilled, as any action is already a G-group. Theorem 7.13 expresses the precise *internal* sense in which the same property should hold: the rectangle ($\mathbf{W}$) must commute on all of $(A \times A|G)$—which it always does in Gp, as a consequence of the *Smith is Huq* property.

Example 7.15. Let us come now back to Example 7.6 with this viewpoint in mind. We already know that the action (V, ψ) is not a $\mathbb{Z}_2$-module. As a matter of fact, we can prove directly that the function

$$m\colon X \times_{\mathbb{Z}_2} X \to X\colon (x, y) \mapsto x \cdot y$$

is not a loop homomorphism, by simply taking into account that

$$m((j, 1) \cdot (i, i)) = m(-k, i) = -ki = -j$$

even though

$$m(j, 1) \cdot m(i, i) = j1 \cdot ii = j.$$

Of course also the diagram corresponding to ($\mathbf{W}$) should fail to commute, so let us confirm this with a concrete example. Note that the expression

$$(j, j) \cdot ((j, 1)i \cdot (1, j))i$$

determines an element of the formal commutator $(V \times V|\mathbb{Z}_2)$. Now $\psi\circ(+|1_{\mathbb{Z}_2})$ of it is $jj \cdot (ji \cdot j)i = (-kj)i = ii = 1$, while going around the rectangle ($\mathbf{W}$) the other way gives

$$(j \cdot (ji)i) \cdot (j \cdot (ij)i) = j(-ki) \cdot (j \cdot ki) = j(-j) \cdot jj = -1.$$

7.16. **The module in a crossed module.** We add another statement to Proposition 5.12, but to do so we first need a refinement of Proposition 3.8 with respect to module structures.

Lemma 7.17. *Under the hypotheses and with the notation of Proposition 3.8 the following properties hold.*

(i) *If ψ is a G-module structure then φ is a H-module structure.*
(ii) *Suppose that m is an isomorphism and h is a regular epimorphism. If φ is a H-module structure then ψ is a G-module structure.*

Proof. This is an easy consequence of Theorem 7.3 using the commutative diagram

$$\begin{tikzcd}
(M|M|H) \arrow[r, "S_{2,1}^{M,H}"] \arrow[d, "(m|m|h)"'] & (M|H) \arrow[r, "\varphi"] \arrow[d, "(m|h)"'] & M \arrow[d, tail, "m"] \\
(A|A|G) \arrow[r, "S_{2,1}^{A,G}"'] & (A|G) \arrow[r, "\psi"'] & A
\end{tikzcd}$$

and the fact that in (ii) the morphism $(m|m|h)$ is a regular epimorphism by Proposition 2.23. $\square$

Proposition 7.18. *Let (G, A, μ, ∂) be a crossed module in a semi-abelian category and consider the induced exact sequence* ($\mathbf{T}$). *The action κ of G on K constructed in Proposition 5.12 induces a unique action ρ of Q on K such that the square*

$$\begin{tikzcd}
(K|G) \arrow[r, "\kappa"] \arrow[d, two heads, "(1_K|p)"'] & K \arrow[d, equal] \\
(K|Q) \arrow[r, dotted, "\rho"'] & K
\end{tikzcd}$$

commutes. Moreover, ρ is a Q-module structure on K.

Proof. The first claim is an immediate consequence of Lemma 6.1 once we can prove that the factorisation exists. We obtain it via Corollary 2.32: we have to show that $\kappa \circ (1_K|\partial)$ and $\kappa \circ S_{1,2}^{K,G} \circ (1_K|\partial|1_G)$ are trivial. To prove this for $\kappa \circ (1_K|\partial)$, compose with the monomorphism k and use the Peiffer condition. In fact,

$$k \circ \kappa \circ (1_K|\partial) = \mu \circ (k|\partial) = \mu \circ (1_A|\partial) \circ (k|1_A) = c^{A,A} \circ (k|1_A) = k \circ c^{K,A},$$

which is zero since K is central in A by Proposition 5.12.i. To prove that the morphism $\kappa \circ S_{1,2}^{K,G} \circ (1_K|\partial|1_G)$ is trivial consider the following diagram where τ is the symmetry isomorphism $(1\,2)_{1_A}$ of the ternary cross-effect induced by the transposition $(1\,2)$, cf. Proposition 2.6.

$$\begin{tikzcd}
(A|K|G) \arrow[d, "(1_A|k|1_G)"'] & (K|A|G) \arrow[l, "\tau"'] \arrow[r, "(1_K|\partial|1_G)"] \arrow[d, "(k|1_A|1_G)"'] & (K|G|G) \arrow[r, "S_{1,2}^{K,G}"] \arrow[d, "(k|1_G|1_G)"'] & (K|G) \arrow[r, "\kappa"] \arrow[d, "(k|1_G)"'] & K \arrow[d, tail, "k"'] \\
(A|A|G) \arrow[d, "S_{2,1}^{A,G}"'] & (A|A|G) \arrow[l, "\tau"] \arrow[r, "(1_A|\partial|1_G)"'] \arrow[d, "S_{2,1}^{A,G}"'] & (A|G|G) \arrow[r, "S_{1,2}^{A,G}"'] & (A|G) \arrow[r, "\mu"'] & A \arrow[d, equal] \\
(A|G) \arrow[r, equal] & (A|G) \arrow[rrr, "\mu"'] & & & A
\end{tikzcd}$$

The diagram commutes; for the lower rectangle this is condition (**S**). Thus

$$\begin{aligned}
k \circ \kappa \circ S_{1,2}^{K,G} \circ (1_K|\partial|1_G) &= \mu \circ S_{2,1}^{A,G} \circ (1_A|k|1_G) \circ \tau \\
&= \mu \circ S_{1,2}^{A,G} \circ (1_A|\partial|1_G) \circ (1_A|k|1_G) \circ \tau \\
&= \mu \circ S_{1,2}^{A,G} \circ (1_A|0|1_G) \circ \tau = 0.
\end{aligned}$$

It remains to show that ρ is a Q-module structure. By Lemma 7.17.ii it suffices to show that κ is a G-module structure, i.e., that $\kappa_{2,1} = 0$. But

$$\begin{aligned}
k \circ \kappa_{2,1} &= k \circ \kappa \circ S_{2,1}^{K,G} = \mu \circ (k|1_G) \circ S_{2,1}^{K,G} \\
&= \mu \circ S_{2,1}^{A,G} \circ (k|k|1_G) = \mu \circ S_{1,2}^{A,G} \circ (1_A|\partial|1_G) \circ (k|k|1_G) \\
&= \mu \circ S_{1,2}^{A,G} \circ (k|0|1_G) = 0
\end{aligned}$$

as desired. □

8. An application to cohomology

The above Lemma 7.17 may also be used in the study of semi-abelian cohomology. In the present article we shall limit ourselves to the lowest-dimensional case, and extend the interpretation given in [34]—of the second cohomology group $\mathrm{H}^2(G, A)$ of an object G with coefficients in a trivial module A in terms of central extensions—to arbitrary modules, and make the link with the torsor theories established in [21] and [27] explicit. In fact, the article [21] already contains some form of Theorem 8.7, based on properties of points rather than a calculus of internal actions. However, we believe the techniques developed here clarify the connections between several approaches to the same problem, while they may also be used to extend the analysis of the higher cohomology groups developed in [61] to arbitrary coefficients.

We work towards Theorem 8.7 which gives an isomorphism

$$\mathrm{H}^2(G, (A, \psi)) \cong \mathrm{Opext}[G, (A, \psi)]$$

between the second cohomology group of G with coefficients in (A, ψ) and the group of equivalence classes of extensions of G by (A, ψ). Of course, when the action ψ is trivial, those extensions are precisely the central extensions of G by A, and we regain Theorem 6.3 in [34].

According to Lemma 7.17, any abelian extension ($\mathbf{U}$) of an object G by an object A gives rise to a unique module structure ψ_p of G on A through which the conjugation action of X on A factors. This defines a functor which is crucial in the *directions approach* to cohomology, see [15, 23, 59, 60, 61].

Throughout this section we shall work in a semi-abelian category $\mathcal{A}$.

Definition 8.1. Given an abelian extension ($\mathbf{U}$), the induced G-module structure on A is called the **direction** of p and denoted $\mathsf{d}_G(p) \colon (A|G) \to A$. This defines a functor

$$\mathsf{d}_G \colon \mathsf{AbExt}_G(\mathcal{A}) \to \mathsf{Mod}_G(\mathcal{A}) \colon p \mapsto (\mathrm{Ker}(p), \psi_p)$$

called the **direction functor**.

The fibre $\mathsf{d}_G^{-1}(A, \psi)$ of d_G over a given G-module (A, ψ) is the category

$$\mathrm{Opext}(G, (A, \psi))$$

of all **extensions of** G **by** (A, ψ).

Remark 8.2. The direction functor d_G is completely determined by a pullback/ pushout property as in [15], where the concept was originally introduced. Indeed, as essentially explained in Remark 3.10, an abelian extension ($\mathbf{U}$) has direction (A, ψ) if and only if the downward-pointing square in the induced morphism of points

$$\begin{tikzcd}
\mathrm{R}[p] \arrow[r, "1_A \rtimes p"] \arrow[d, shift right, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \rtimes_\psi G \arrow[d, shift right, "p_\psi"'] \\
X \arrow[r, "p"'] \arrow[u, shift right] & G \arrow[u, shift right, "s_\psi"'] \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is both a pullback and a pushout.

Let us make this somewhat more explicit. As recalled in Example 5.10, via Remark 3.10 the conjugation action $c^{A,X}$ of X on A corresponds to the kernel pair projection $p_1 \colon \mathrm{R}[p] \to X$. Hence giving a pullback/pushout square as above amounts to giving a morphism of actions

$$\begin{tikzcd}
(A|X) \arrow[r, "(1_A|p)"] \arrow[d, "c^{A,X}"'] & (A|G) \arrow[d, "\psi"] \\
A \arrow[r, equal] & A.
\end{tikzcd}$$

This means that p has direction (A, ψ) if and only if p_1 is a pullback of p_ψ along p.

This allows us to interpret extensions of an object G by a G-module (A, ψ) as certain torsors in the sense of Duskin and Glenn [27, 28, 33]. Given an object G and a G-module (A, ψ) in a semi-abelian category $\mathcal{A}$, a **one-torsor of** G **by** (A, ψ) is a $\mathbb{K}((A, \psi), 1)$-torsor in the slice category $\mathcal{A}/G$, i.e., a diagram

$$\begin{tikzcd}
\mathrm{R}[p] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_2"'] \arrow[d] & X \arrow[l] \arrow[r, "p"] \arrow[d, "p"'] & G \arrow[d, equal] \\
A \rtimes_\psi G \arrow[r, shift left=2, "p_\psi"] \arrow[r, shift right=2, "p_\psi"'] & G \arrow[l] \arrow[r, equal] & G
\end{tikzcd}$$

in $\mathcal{A}$, where the squares on the left are pullbacks—see [27] or the analysis given in [61]. Morphisms of such torsors are defined as in the slice category over the bottom line of this diagram, and thus the category $\mathrm{Tors}^1(G,(A,\psi))$ is obtained. From Remark 8.2 we now easily obtain the following:

Proposition 8.3. *Let G be an object and (A,ψ) a G-module in a semi-abelian category $\mathcal{A}$. Then there is a category equivalence* $\mathrm{Tors}^1(G,(A,\psi))\simeq \mathrm{Opext}(G,(A,\psi))$. □

It is explained in [27] that the set $\mathrm{Tors}^1[G,(A,\psi)]$ of connected components of the category $\mathrm{Tors}^1(G,(A,\psi))$ comes with a suitable abelian group structure which may be considered as the cohomology group $\mathrm{H}^2(G,(A,\psi))$. Furthermore, when $\mathcal{A}$ is monadic over Set, this cohomology group is isomorphic to the second Barr-Beck comonadic cohomology group [2] of G with coefficients in (A,ψ) relative to the canonically induced comonad on $\mathcal{A}$. We shall now prove that the connected components $\mathrm{Opext}[G,(A,\psi)]=\pi_0(\mathsf{d}_G^{-1}(A,\psi))$ of the category $\mathrm{Opext}(G,(A,\psi))$ form an abelian group isomorphic to $\mathrm{Tors}^1[G,(A,\psi)]$, and thus to $\mathrm{H}^2(G,(A,\psi))$.

Proposition 8.4. *For any object G, the direction functor d_G preserves finite products.*

Proof. The terminal object of $\mathsf{AbExt}_G(\mathcal{A})$ is 1_G, of which the direction is 0, considered as a G-module. Hence d_G preserves terminal objects.

Now we show that d_G preserves binary products. On one hand, Lemma 7.12 says that the biproduct $(A,\psi)\oplus(B,\varphi)$ in $\mathsf{Mod}_G(\mathcal{A})$ has as underlying G-action the product $(A,\psi)\times(B,\varphi)$ in $\mathsf{Act}_G(\mathcal{A})$, which is the object $A\times B$ in $\mathcal{A}$ equipped with the morphism

$$\psi\oplus\varphi\colon (A\times B|G)\xrightarrow{\langle(\pi_A|1_G),(\pi_B|1_G)\rangle}(A|G)\times(B|G)\xrightarrow{\quad\psi\times\varphi\quad}A\times B.$$

On the other hand, given two abelian extensions $p\colon X\to G$ and $q\colon Y\to G$ of G with respective kernels A and B, their product in $\mathsf{AbExt}_G(\mathcal{A})$ is the pullback $p\times q\colon X\times_G Y\to G$, of which the kernel is $A\times B$. It remains to check that this kernel carries $(A,\psi)\times(B,\varphi)$ as G-module structure, but this we may see by taking into account Remark 8.2. Indeed, if p_1 is a pullback of p_ψ and q_1 is a pullback of p_φ, then $(p\times q)_1$ is a pullback of $p_{\psi\oplus\varphi}$. □

Proposition 8.5 (d_G is a fibration)**.** *Given an abelian extension p and a G-module morphism f as in the diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, tail, "a"] \arrow[d, "f"'] & X \arrow[r, two heads, "p"] \arrow[d, dotted, "g"] & G \arrow[r] \arrow[d, dotted, equal] & 0 \\
0 \arrow[r, dotted] & B \arrow[r, tail, dotted, "b"'] & Y \arrow[r, two heads, dotted, "q"'] & G \arrow[r, dotted] & 0
\end{tikzcd}$$

where A carries the direction of p as G-module structure, there exists an abelian extension q which completes the diagram in such a way that $\mathsf{d}_G(q)$ is the given action of G on B.

Proof. We generalise the proof of Corollary 3.3 in [34]: we split the problem in two separate cases (f is split monic, f is regular epic in $\mathcal{A}$) by factoring the morphism f as

$$(A,\psi)\xrightarrow{\langle 1_{(A,\psi)},0\rangle}(A,\psi)\oplus(B,\varphi)\xrightarrow{\left\langle\begin{smallmatrix} f\\ 1_{(B,\varphi)}\end{smallmatrix}\right\rangle}(B,\varphi)$$

in $\mathsf{Mod}_G(\mathcal{A})$. Here ψ and φ denote the given G-module structures.

The first step involves the product in $\mathsf{AbExt}_G(\mathcal{A})$ of p with $p_\varphi\colon B \rtimes_\varphi G \to G$, which gives us the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, tail, "a"] \arrow[d, shift left, "\langle 1_A, 0\rangle"] & X \arrow[r, "p"] \arrow[d, dotted, shift left] & G \arrow[r] \arrow[d, dotted, equal] & 0 \\
0 \arrow[r, dotted] & A \times B \arrow[r, tail, dotted] \arrow[u, shift left, "\pi_A"] & X \times_G (B \rtimes_\varphi G) \arrow[r, dotted, "p \times p_\varphi"'] \arrow[u, shift left, "\pi_X"] & G \arrow[r, dotted] & 0
\end{tikzcd}$$

with short exact rows in $\mathcal{A}$. Proposition 8.4 says that the direction of the lower extension is the biproduct G-module $(A, \psi) \oplus (B, \varphi)$.

For the second step, assume that f is a regular epimorphism, and consider its kernel k as in the following diagram.

$$\begin{tikzcd}
& K \arrow[r, equal] \arrow[d, tail, "k"'] & K \arrow[d, "a\circ k"] & & \\
0 \arrow[r] & A \arrow[r, tail, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\texttt{(i)}"] & X \arrow[r, "p"] \arrow[d, dotted, "g"] & G \arrow[r] \arrow[d, dotted, equal] & 0 \\
0 \arrow[r, dotted] & B \arrow[r, tail, dotted, "b"'] & Y \arrow[r, dotted, "q"'] & G \arrow[r, dotted] & 0
\end{tikzcd}$$

If we can prove that the monomorphism $a\circ k$ is normal, we can take g to be its cokernel. Then the thus arising square `(i)` is a pullback and a pushout by Lemma 1.5, which gives the rest of the diagram and also implies that b is a monomorphism, hence a kernel as a direct image of a kernel. Now K is indeed normal in X by Proposition 2.16.ii, because $c^{A,X}\circ(k|1_X)$ factors through the kernel k of f, as

$$f\circ c^{A,X}\circ(k|1_X) = f\circ\psi\circ(1_A|p)\circ(k|1_X) = \varphi\circ(f|1_X)\circ(1_A|p)\circ(k|1_X) = 0.$$

Also $(f|g)\colon (A|X) \to (B|Y)$ is a regular epimorphism and, by Proposition 3.4,

$$\varphi\circ(1_B|q)\circ(f|g) = \varphi\circ(f|1_G)\circ(1_A|p) = f\circ\psi\circ(1_A|p) = f\circ c^{A,X} = c^{B,Y}\circ(f|g).$$

Hence $c^{B,Y}$ factors through $\varphi\colon (B|G) \to B$, which finishes the proof. □

Corollary 8.6. *For any G in $\mathcal{A}$, the application*

$$(A, \psi) \mapsto \mathrm{Opext}[G, (A, \psi)]$$

defines a finite product-preserving functor $\mathrm{Opext}[G, -]\colon \mathsf{Mod}_G(\mathcal{A}) \to \mathsf{Set}$.

Proof. This may be proved as in [34, Proposition 6.1] using Proposition 8.5. □

Since $\mathsf{Mod}_G(\mathcal{A}) = \mathsf{Ab}(\mathcal{A}/G)$ is a category of internal abelian groups, this implies that the sets $\mathrm{Opext}[G, (A, \psi)]$ carry a natural abelian group structure.

Theorem 8.7. *Let G be an object and (A, ψ) a G-module in a semi-abelian category $\mathcal{A}$.*

(i) *We have a group isomorphism*

$$\mathrm{H}^2(G, (A, \psi)) = \mathrm{Tors}[G, (A, \psi)] \cong \mathrm{Opext}[G, (A, \psi)].$$

(ii) *If $\mathcal{A}$ is monadic over* Set *then these cohomology groups are comonadic Barr–Beck cohomology with respect to the canonical comonad on $\mathcal{A}$.*

(iii) *If (SH) holds in $\mathcal{A}$ then every extension with abelian kernel occurs in some cohomology class.*

Proof. By Corollary 8.6, to obtain (i) we only have to prove that the abelian groups $\mathrm{Tors}[G, (A, \psi)]$ and $\mathrm{Opext}[G, (A, \psi)]$ have the same underlying sets. This, however, follows immediately from Proposition 8.3. Statement (ii) follows from [27] and (iii) from Theorem 6.4. □

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Contents

Université Lille Nord de France, F-59044 Lille, France
UVHC, LAMAV and FR CNRS 2956, Laboratoire de Mathématiques et de leurs Applications, Université de Valenciennes, F-59313 Valenciennes, France
E-mail address: `Manfred.Hartl@univ-valenciennes.fr`

Centro de Matemática, Universidade de Coimbra, 3001–454 Coimbra, Portugal
Institut de recherche en mathématique et physique, Université catholique de Louvain, chemin du cyclotron 2, B-1348 Louvain-la-Neuve, Belgium
E-mail address: `tim.vanderlinden@uclouvain.be`