SUBADDITIVE QUASI-MEASURES IN LOCALLY COMPACT SPACES

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Abstract. This note yields a contribution to the "Quasi-measure Theory" introduced by J.F. Aarnes in [1] to answer a question rising from the study of quasi-states in C^*-algrebras. We here show the equivalence, for quasi-measures in locally compact spaces, between a subadditivity property and the inclusion-exclusion formula.

Quasi-measures considered below appeared for the first time, in the context of compact topological spaces, in rich papers by J.F. Aarnes (see [1] and [2]), where they solve a problem related to the study of "quasi-states" in the theory of C^*-algebras. They indeed give a negative answer to the following question (formulated in real terms): given a (compact) topological space X, is it true that every (positive) quasi-linear functional on the space $C(X)$ of continuous functions on X be continuous, where "quasi-linear" means "linear on each closed subalgebra of $C(X)$ generated by a single continuous function". In fact, the integral with respect to any given quasi-measure on X which is *not* a Radon measure on X would provide an example of quasi-linear functional on $C(X)$ which is *not* linear on $C(X)$.

Although it is well-known (see Wheeler [9, Proposition 3.1] for example) that every (finitely) subadditive quasi-measure on a given compact space is (we should say: "extends", *cf.* below) a Radon measure on X. D.J. Grubb, on the other hand, has in [5] a simple and elegant argument to show that, in a one-dimensional compact space X, every *signed* quasi-measure satisfying an inclusion-exclusion formula extends to a Radon measure on X.

In this note, we work in the more general context of locally compact spaces (quasi-measures in locally compact spaces have been introduced by J.F. Aarnes in [4] and studied in their relations to quasi-linear functionals by A.B. Rustad in [8]) and show, using simple arguments that "being subadditive" and "satisfying an inclusion-exclusion formula" are equivalent properties for a (nonnegative) quasi-measure, without using the extension result. We then give, as an application of the previous result, a direct proof of the fact that every subadditive quasi-measure is also countably additive, which is not a trivial fact in this context.

1. Basic definitions

Unless otherwise specified, X stands, in this chapter, for a locally compact (Hausdorff) topological space while $\mathcal{P}(X)$ denote the collection of all subsets of X.

A set $E \subseteq X$ will be called *bounded* in case $\bar{E}$ is compact. Given any collection $\mathcal{E} \subseteq \mathcal{P}(X)$, we denote by $\mathcal{E}^*$ the subcollection of all *bounded* sets belonging to $\mathcal{E}$.

We further let $\mathcal{C}(X)$ and $\mathcal{O}(X)$ denote the collection of all closed and open subets of X, respectively, and we define $\mathcal{A}(X) := \mathcal{C}^*(X) \cup \mathcal{O}(X)$ and $\hat{\mathcal{A}}(X) := \mathcal{C}(X) \cup \mathcal{O}(X)$. We also denote by $\mathcal{B}(X)$ the σ-algebra of Borel subsets of X, generated by $\mathcal{O}(X)$.

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We now introduce what we shall call a *quasi-measure* on X. Note that we shall use the notation $\sqcup$ instead of $\cup$ for denoting the union of pairwise disjoint sets.

Definition 1 (Aarnes [4])**.** A *quasi-measure* on X is a set function $\mu : \mathcal{A}(X) \to [0, \infty]$ satisfying the following conditions:

(QM1) if $A, B \in \mathcal{A}(X)$ are two disjoint sets with $A \sqcup B \in \mathcal{A}(X)$, then

$$\mu(A \sqcup B) = \mu(A) + \mu(B);$$

(QM2) given $U \in \mathcal{O}(X)$,

$$\mu(U) = \sup\{\mu(K) : K \in \mathcal{C}^*(X), K \subseteq U\};$$

(QM3) given $K \in \mathcal{C}^*(X)$,

$$\mu(K) = \inf\{\mu(U) : U \in \mathcal{O}(X), U \supseteq K\};$$

(QM4) for each $K \in \mathcal{C}^*(X)$ we have $\mu(K) < \infty$.

Let us immediately deduce from the previous definition a set of basic properties of quasi-measures. We include their short proofs for the sake of clarity.

Observation 2. If μ is a quasi-measure on X, then the following condition hold:

(1) $\mu(\emptyset) = 0$;
(2) given $A, B \in \mathcal{A}(X)$ with $A \subseteq B$, we have $\mu(A) \leqslant \mu(B)$;
(3) if $\{A_i\}_{i\in I} \subseteq \mathcal{A}(X)$ is a *finite* family of pairwise disjoint sets with $\bigsqcup_{i\in I} A_i \in \mathcal{A}(X)$ then

$$\mu\left(\bigsqcup_{i\in I} A_i\right) = \sum_{i\in I} \mu(A_i);$$

(4) $\mu(A) < \infty$ for each $A \in \mathcal{A}^*(X)$;
(5) if I is an ordered set and $\{U_i\}_{i\in I}$ is an increasing family of open sets, then the net $\{\mu(U_i)\}_{i\in I}$ converges increasingly to $\mu\left(\bigcup_{i\in I} U_i\right)$;

if moreover X is compact, then

(6) $\mu(A) < \infty$ for each $A \in \mathcal{A}(X)$;
(7) for any $A \in \mathcal{A}(X)$, $\mu(A) + \mu(X \setminus A) = \mu(X)$;
(8) for a set function satisfying (QM1) and (QM4), (QM2) and (QM3) are equivalent.

Proof. To prove part (1), infer from (QM1) that

$$2\mu(\emptyset) = \mu(\emptyset) + \mu(\emptyset) = \mu(\emptyset \sqcup \emptyset) = \mu(\emptyset).$$

It follows that $\mu(\emptyset) = 0$.

Part (2) follows easily from either (QM2) or (QM3) when A is compact (resp. open) and B is open (resp. compact). In case both A and B are open, the desired inequality follows from (QM2) and the fact that $\mu(K) \leqslant \mu(B)$ holds for each compact set $K \in \mathcal{C}^*(X)$ satisfying $K \subseteq A$. In case both A and B are compact, it suffices to use (QM3) after having observed that $\mu(A) \leqslant \mu(U)$ holds for any open set $U \in \mathcal{O}(X)$ satisfying $U \supseteq B$.

To prove part (3), we proceed by induction on the number of elements n in I; without loss of generality we can suppose $A_i \neq \emptyset$ and $A_i \neq X$ for each $i \in I$. For $n = 2$, the desired equality is implied by (QM1). Now suppose $n > 2$ and assume that the result holds for families of sets having less than n elements. Define

$$I_1 := \{i \in I : A_i \in \mathcal{C}^*(X)\} \quad \text{and} \quad I_2 := I \setminus I_1 \subseteq \{i \in I : A_i \in \mathcal{O}(X)\}.$$

By assumption $I_1 \cap I_2 = \emptyset$ and we have $\bigsqcup_{i\in I_1} A_i \in \mathcal{C}(X)$ and $\bigsqcup_{i\in I_2} A_i \in \mathcal{O}(X)$; moreover, using (QM1) and the induction hypothesis we get for $j = 1, 2$

$$\mu\left(\bigsqcup_{i\in I_j} A_i\right) = \sum_{i\in I_j} \mu(A_i).$$

Finally, (QM1) yields

$$\mu\left(\bigsqcup_{i\in I} A_i\right) = \mu\left(\bigsqcup_{i\in I_1} A_i\right) + \mu\left(\bigsqcup_{i\in I_2} A_i\right) = \sum_{i\in I_j} \mu(A_i).$$

The proof is complete.

Part (4) follows immediately from part (2) for $\bar{A} \in \mathcal{C}^*(X)$ holds for each $A \in \mathcal{A}^*(X)$.

To prove part (5), let $U := \bigcup_{i\in I} U_i$ and observe that $\mu(U_i) \leqslant \mu(U)$ holds for each $i \in I$. On the other hand, given $\varepsilon > 0$, choose according to (QM2) a compact set $K \in \mathcal{C}^*(X)$ satisfying $K \subseteq U$ and $\mu(K) \geqslant \mu(U) - \varepsilon$. Taking $i_0 \in I$ an index for which $K \subseteq U_{i_0}$ we have for any $i \in I$ satisfying $i \geqslant i_0$:

$$\mu(U_i) \geqslant \mu(U_{i_0}) \geqslant \mu(K) \geqslant \mu(U) - \varepsilon;$$

which finishes the proof of part (5).

Part (6) follows immediately from part (4). To prove part (7), apply (QM1) to $A, X \setminus A \in \mathcal{A}(X)$. Part (8) now follows immediately from part (7). □

Topological measures on X extends in a very natural way to the collection $\hat{\mathcal{A}}(X)$ of all closed or open subsets of X.

Observation 3. If μ is a topological measure on X, then the set function $\hat{\mu} : \hat{\mathcal{A}}(X) \to [0, \infty]$ defined by

$$\hat{\mu}(A) = \begin{cases} \mu(A) & \text{if } A \in \mathcal{A}(X), \\ \inf\{\mu(U) : U \in \mathcal{O}(X), U \supseteq A\} & \text{if } A \in \mathcal{C}(X); \end{cases}$$

is well defined and satisfies the following conditions:

(1) given $A, B \in \hat{\mathcal{A}}(X)$ with $A \subseteq B$, we have $\hat{\mu}(A) \leqslant \hat{\mu}(B)$;
(2) **if moreover** X **is normal**, then $\hat{\mu}(F_1 \sqcup F_2) = \hat{\mu}(F_1) + \hat{\mu}(F_2)$ for any disjoint sets $F_1, F_2 \in \mathcal{C}(X)$.

Proof. To show that $\hat{\mu}$ is well defined, fix $F \in \mathcal{C}(X)$ and observe that if we have $F \in \mathcal{C}^*(X)$, then

$$\hat{\mu}(F) := \inf\{\mu(U) : U \in \mathcal{O}(X), U \supseteq F\} = \mu(F),$$

according to (QM3). On the other hand, if $F \in \mathcal{O}(X)$ then it follows from Observation 2, (2) that

$$\hat{\mu}(F) := \inf\{\mu(U) : U \in \mathcal{O}(X), U \supseteq F\} = \mu(F).$$

Observe on the other hand that statement (1) follows easily from the definition of $\hat{\mu}$, Definition 1 and Observation 2.

Assuming moreover X to be normal, fix two disjoint sets $F_1, F_2 \in \mathcal{C}(X)$ and choose disjoint open sets $U_1, U_2 \in \mathcal{O}(X)$ satisfying $F_i \subseteq U_i$, $i = 1, 2$. Observing that if $U \in \mathcal{O}(X)$ satisfies $U \supseteq F_1 \sqcup F_2$, then $U_i' := U_i \cap U \in \mathcal{O}(X)$ contains F_i for $i = 1, 2$, we see that

$$\mu(U) \geqslant \mu(U_1' \sqcup U_2') = \mu(U_1') + \mu(U_2') \geqslant \hat{\mu}(F_1) + \hat{\mu}(F_2);$$

hence $\hat{\mu}(F_1 \cap F_2) \geqslant \hat{\mu}(F_1) + \hat{\mu}(F_1)$. Conversely, given any open set $U_i' \in \mathcal{O}(X)$ satisfying $F_i \subseteq U_i'$, $i = 1, 2$ we see that

$$\hat{\mu}(F_1 \sqcup F_2) \leqslant \mu((U_1' \cap U_1) \sqcup (U_2' \cap U_2)) = \mu(U_1' \cap U_1) + \mu(U_2 \cap U_2') \leqslant \mu(U_1') + \mu(U_2'),$$

and it follows that $\hat{\mu}(F_1 \sqcup F_2) \leqslant \hat{\mu}(F_1) + \hat{\mu}(F_2)$. □

Given a Radon (outer) measure ϕ^* on X (see Mattila [7, Definition 1.5] for a precise definition), it is clear that the restriction $\mu := \phi^* \restriction \mathcal{A}(X)$ of ϕ^* to $\mathcal{A}(X)$ is a quasi-measure on X; we then have, in particular, for any $A_1, A_2 \in \mathcal{A}(X)$:

$$\mu(A_1 \cup A_2) \leqslant \mu(A_1) + \mu(A_2). \tag{1}$$

The latter condition, necessary for a quasi-measure to extend to a Radon measure on X, is not always satisfied as shows the following classical example.

Example 4. Let $D \subseteq \mathbb{C}$ be the closed unit disk in the complex plane, fix $m \in \mathbb{N}^*$ a positive integer and, for $1 \leqslant j \leqslant 2m+1$, define $z_j := e^{2ij\pi/(2m+1)}$ and let C_j be the closed sector of D centered on the segment $[0, z_j]$ with amplitude $2\pi/(2m+1)$. Finally, we define $C_0 := \{z_1, \ldots, z_{2m+1}\}$.

Denote by $\mathcal{C}_s(D)$ the collection of all compact subsets $C \subseteq D$ having the property that C and $D \backslash C$ be connected. It follows from [3, Theorem 5.1] that the set function

$$\mathcal{C}_s(D) \to [0, \infty), \quad C \mapsto \begin{cases} 0 & \text{if } \#(C \cap C_0) \leqslant m, \\ 1 & \text{otherwise} \end{cases}$$

extends to a quasi-measure μ on D; we hence have by definition $\mu(D) = 1$. On the other hand, we have $\mu(C_j) = 0$ for each $1 \leqslant j \leqslant 2m+1$, so that

$$\mu\left(\bigcup_{j=1}^{2m+1} C_j\right) = \mu(D) = 1 > 0 = \sum_{j=1}^{2m+1} \mu(C_j);$$

in particular, μ cannot be extended to a Radon measure on D for it violates equation (1) above.

In fact the *subadditivity* property expressed by equation (1) above is also sufficient for a quasi-measure to extend to a Radon measure (see Wheeler [9, Proposition 3.1] for a weaker statement in compact spaces; for a proof in locally compact spaces, we refer to a forthcoming survey of ours which will appear elsewhere).

2. Subadditivity and the inclusion-exclusion formula

The following proposition give equivalent properties to the subadditivity property expressed in equation (1) above.

Proposition 5. *If μ is a quasi-measure on X, then the following are equivalent:*

(SA1) $\mu(K_1 \cup K_2) \leqslant \mu(K_1) + \mu(K_2)$ *for every* $K_1, K_2 \in \mathcal{C}^*(X)$*;*
(SA2) $\mu(U_1 \cup U_2) \leqslant \mu(U_1) + \mu(U_2)$ *for every* $U_1, U_2 \in \mathcal{O}(X)$*;*
(SA3) $\mu(A_1 \cup A_2) \leqslant \mu(A_1) + \mu(A_2)$ *for every* $A_1, A_2 \in \mathcal{A}(X)$ *for which* $A_1 \cup A_2 \in \mathcal{A}(X)$*.*

Proof. To prove that (SA1) implies (SA2), fix $U_1, U_2 \in \mathcal{O}(X)$ and let $K \in \mathcal{C}^*(X)$ satisfy $K \subseteq U_1 \cup U_2$. Then $K \setminus U_1$ and $K \setminus U_2$ are disjoint compact sets, and we can find disjoint open sets $W_1, W_2 \in \mathcal{O}(X)$ such that $K \setminus U_i \subseteq W_i$, $i = 1, 2$. Let $K_i := K \setminus W_i$, $i = 1, 2$. We have for $i = 1, 2$:

$$K_i = K \setminus W_i \subseteq K \setminus (K \setminus U_i) = K \cap U_i \subseteq U_i.$$

On the other hand,

$$K_1 \cup K_2 = K \setminus (W_1 \cap W_2) = K.$$

In particular, we get using (SA1)

$$\mu(K) = \mu(K_1 \cup K_2) \leqslant \mu(K_1) + \mu(K_2) \leqslant \mu(U_1) + \mu(U_2)$$

and (SA2) follows from (QM2).

To show that (SA2) implies (SA3), fix $A_1, A_2 \in \mathcal{A}(X)$ and assume that they are not both compact or both open, say that $A_1 \in \mathcal{C}^*(X)$ and $A_2 \in \mathcal{O}(X)$. Choose an open set $U \in \mathcal{O}(X)$ satisfying $U \supseteq A_1$ and compute using (SA2) and Observation 2, 2:

$$\mu(A_1 \cup A_2) \leqslant \mu(U \cup A_2) \leqslant \mu(U) + \mu(A_2).$$

The result then follows from (QM3).

It is obvious that statement (SA3) implies (SA1). □

Definition 6. A quasi-measure μ on X is called *subadditive* if it satisfies one of the equivalent statements (SA1)-(SA3).

Observation 7. Conditions (SA1) — (SA3) above are still equivalent to the following two:

(SA1') $\hat{\mu}(F_1 \cup F_2) \leqslant \hat{\mu}(F_1) + \hat{\mu}(F_2)$ for every $F_1, F_2 \in \mathcal{C}(X)$;

(SA3') $\hat{\mu}(A_1 \cup A_2) \leqslant \hat{\mu}(A_1) + \hat{\mu}(A_2)$ for every $A_1, A_2 \in \hat{\mathcal{A}}(X)$ for which $A_1 \cup A_2 \in \hat{\mathcal{A}}(X)$.

Proof. To show that (SA2) implies (SA1'), observe that for any open sets $U_i \in \mathcal{O}(X)$, $i = 1, 2$ satisfying $U_i \supseteq F_i$, $i = 1, 2$ one has $F_1 \cup F_2 \subseteq U_1 \cup U_2$, hence

$$\hat{\mu}(F_1 \cup F_2) \leqslant \mu(U_1 \cup U_2) \leqslant \mu(U_1) + \mu(U_2),$$

according to (SA2), and (SA1) follows from the definition of $\hat{\mu}$.

To infer (SA3') from (SA2), fix $A_i \in \hat{\mathcal{A}}(X)$, $i = 1, 2$ and assume without loss of generality that $A_1 \in \mathcal{C}(X)$ and $A_2 \in \mathcal{O}(X)$ (all other cases have been discussed before). Using the definition of $\hat{\mu}$ in case $A_1 \cup A_2$ is closed, and the equality $\hat{\mu}(A_1 \cup A_2) = \mu(A_1 \cup A_2)$ in case $A_1 \cup A_2$ is open, we see that for any open set $U \in \mathcal{O}(X)$ satisfying $U \supseteq A_1$, we have

$$\hat{\mu}(A_1 \cup A_2) \leqslant \mu(U \cup A_2) \leqslant \mu(U) + \mu(A_2),$$

using (SA2). The result then follows from the definition of $\hat{\mu}$. □

We previously said that a quasi-measure μ on X is subadditive if and only if it extends to a Radon measure on X. On the other hand, D.J. Grubb elegantly showed in [5] that any *signed* measure μ on a compact space extends to a Radon measure if and only if it satisfies the following inclusion-exclusion formula for each open sets U_1, U_2:

$$\mu(U_1 \cup U_2) = \mu(U_1) + \mu(U_2) - \mu(U_1 \cap U_2).$$

Without using the extension result, somewhat "external" to Quasi-measure Theory, we prove below (see Theorem 9) that this inclusion-exclusion formula is equivalent to subadditivity.

To that purpose, we first explore some more properties of subadditive quasi-measures.

Lemma 8. *If μ is a subadditive quasi-measure on X and if $K_1, K_2 \in \mathcal{C}^*(X)$ satisfy $K_1 \subseteq K_2$, then the following conditions hold:*

(1) $\mu(K_2) = \mu(K_1) + \sup\{\mu(K) : K \in \mathcal{C}^*(X), K \subseteq K_2 \setminus K_1\}$;

(2) $\mu(K_2) = \mu(K_1) + \inf\{\mu(U) : U \in \mathcal{O}(X), U \supseteq K_2 \setminus K_1\}$.

Moreover, if $U_1, U_2 \in \mathcal{O}(X)$ satisfy $U_1 \subseteq U_2$, we also have

(3) $\mu(U_2) = \mu(U_1) + \sup\{\mu(K) : K \in \mathcal{C}^*(X), K \subseteq U_2 \setminus U_1\}$;

(4) $\mu(U_2) = \mu(U_1) + \inf\{\mu(U) : U \in \mathcal{O}(X), U \supseteq U_2 \setminus U_1\}$.

Proof. We begin by proving condition 1. First, observe that for any $K \in \mathcal{C}^*(X)$ satisfying $K \subseteq K_2 \setminus K_1$, we have

$$\mu(K) + \mu(K_1) \leqslant \mu(K \sqcup K_1) \leqslant \mu(K_2),$$

hence

$$\mu(K_2) \geqslant \mu(K_1) + \sup\{\mu(K) : K \in \mathcal{C}^*(X), K \subseteq K_2 \setminus K_1\}.$$

In order to show the converse inequality, fix $\varepsilon > 0$ and begin by choosing $V_1 \in \mathcal{O}(X)$ satisfying $V_1 \supseteq K_1$ and $\mu(V_1) \leqslant \mu(K_1) + \varepsilon/2$. Choose also $V_2 \in \mathcal{O}(X)$ for which $V_2 \supseteq K_2 \setminus V_1$ and $\mu(V_2) \leqslant \mu(K_2 \setminus V_1) + \varepsilon/2$. Now we compute

$$\mu(K_1) \geqslant \mu(V_1) + \mu(V_2) - \mu(V_2) - \frac{\varepsilon}{2} \geqslant \mu(V_1 \cup V_2) - \mu(V_2) - \frac{\varepsilon}{2};$$

however we have $K_2 \subseteq V_1 \cup V_2$ so that

$$\begin{aligned}\mu(K_1) &\geqslant \mu(K_2) - \mu(K_2 \setminus V_1) - \varepsilon \\ &\geqslant \mu(K_2) - \sup\{\mu(K) : K \in \mathcal{C}^*(X), K \subseteq K_2 \setminus K_1\} - \varepsilon,\end{aligned}$$

for $K_2 \setminus V_1 \in \mathcal{C}^*(X)$ satisfies $K_2 \setminus V_1 \subseteq K_2 \setminus K_1$. We hence proved equality 1.

To show that condition 2 also holds, observe first that for any $U \in \mathcal{O}(X)$ with $U \supseteq K_2 \setminus K_1$, one has for each $V \in \mathcal{O}(X)$ satisfying $V \supseteq K_1$

$$\mu(K_2) \leqslant \mu(U \cup V) \leqslant \mu(U) + \mu(V),$$

since $K_2 \subseteq U \cup V$. Hence it follows that $\mu(K_2) \leqslant \mu(K_1) + \mu(U)$; in particular we obtain

$$\mu(K_2) \leqslant \mu(K_1) + \inf\{\mu(U) : U \in \mathcal{O}(X), U \supseteq K_2 \setminus K_1\}.$$

To show the converse inequality, fix $\varepsilon > 0$ and choose $V \in \mathcal{O}(X)$ satisfying $V \supseteq K_2$ and $\mu(V) \leqslant \mu(K_2) + \varepsilon$. We then compute

$$\begin{aligned}\mu(K_2) &\geqslant \mu(K_1) + \mu(V) - \mu(K_1) - \varepsilon = \mu(K_1) + \mu(V \setminus K_1) - \varepsilon \\ &\geqslant \mu(K_1) + \inf\{\mu(U) : U \in \mathcal{O}(X), U \supseteq K_2 \setminus K_1\} - \varepsilon,\end{aligned}$$

for $U := V \setminus K_1 \in \mathcal{O}(X)$ satisfies $U \supseteq K_2 \setminus K_1$; hence condition 2 is proved.

In order to prove condition 3, observe that for any $K_1 \in \mathcal{C}^*(X)$ satisfying $K_1 \subseteq U_1$ and any $K \in \mathcal{C}^*(X)$ satisfying $K \subseteq U_2 \setminus U_1$ we have

$$\mu(K_1) + \mu(K) = \mu(K_1 \sqcup K) \leqslant \mu(U_2);$$

hence it follows that

$$\mu(U_1) + \sup\{\mu(K) : K \in \mathcal{C}^*(X), K \subseteq U_2 \setminus U_1\} \leqslant \mu(U_2).$$

To prove the converse inequality, let us distinguish between two cases.
First case: $\mu(U_2) = \infty$. If $\mu(U_1) = \infty$, then there is nothing to prove. Otherwise for any $0 < M < \infty$ such that $\mu(U_1) \leqslant M$, one can find $K_2 \subseteq U_2$ for which $\mu(K_2) > 2M$; observing that $K_2 \subseteq (K_2 \setminus U_1) \cup U_1$ we infer from (SA2) that for any $V \in \mathcal{O}(X)$ with $V \supseteq K_2 \setminus U_1$ we have $\mu(K_2) \leqslant \mu(V) + \mu(U_1)$; hence we get

$$\mu(K_2 \setminus U_1) \geqslant \mu(K_2) - \mu(U_1) > M,$$

since V is arbitrary. Consequently, $\sup\{\mu(K) : K \in \mathcal{C}^*(X), K \subseteq U_2 \setminus U_1\} = \infty$ which proves the desired inequality.

Second case: $\mu(U_2) < \infty$. Fix $\varepsilon > 0$ and find $K_2 \in \mathcal{C}^*(X)$ satisfying $K_2 \subseteq U_2$ and $\mu(K_2) \geqslant \mu(U_2) - \varepsilon$. Observe then that $K_2 \subseteq (K_2 \setminus U_1) \cup U_1$ and compute, using (SA1) and the previous inclusion in (2):

$$\begin{aligned} \mu(U_2) &\leqslant \mu(K_2) + \varepsilon, \\ &= \mu(U_1) + \mu(K_2) - \mu(U_1) + \varepsilon, \\ &\leqslant \mu(U_1) + \mu(K_2 \setminus U_1) + \varepsilon, \\ &\leqslant \mu(U_1) + \sup\{\mu(K) : K \in \mathcal{C}^*(X), K \subseteq U_2 \setminus U_1\} + \varepsilon, \end{aligned} \tag{2}$$

for $K := K_2 \setminus U_1 \in \mathcal{C}^*(X)$ satisfies $K \subseteq U_2 \setminus U_1$. The desired inequality follows again for $\varepsilon > 0$ is arbitrary.

To show 4, observe first that given $U \in \mathcal{O}(X)$ with $U \supseteq U_2 \setminus U_1$ we have $U_2 \subseteq U_1 \cap U$ and hence $\mu(U_2) \leqslant \mu(U_1) + \mu(U)$ according to (SA2). It then follows that

$$\mu(U_2) \leqslant \mu(U_1) + \inf\{\mu(U) : U \in \mathcal{O}(X), U \supseteq U_2 \setminus U_1\},$$

and equality even follows in case $\mu(U_2) = \infty$. We hence show the converse inequality by assuming moreover that $\mu(U_2) < \infty$. Therefore, fix $\varepsilon > 0$ and choose $K_1 \in \mathcal{C}^*(X)$ with $K_1 \subseteq U_1$ and $\mu(K_1) \geqslant \mu(U_1) - \varepsilon/2$ together with $K_2 \in \mathcal{C}^*(X)$ satisfying $K_2 \subseteq U_2 \setminus K_1$ and $\mu(K_2) \geqslant \mu(U_2 \setminus K_1) - \varepsilon/2$. Now observe that $K_1 \sqcup K_2 \subseteq U_2$ and compute

$$\begin{aligned} \mu(U_1) \leqslant \mu(K_1) + \frac{\varepsilon}{2} &= \mu(K_1) + \mu(K_2) - \mu(K_2) + \frac{\varepsilon}{2} \\ &= \mu(K_1 \sqcup K_2) - \mu(K_2) + \frac{\varepsilon}{2} \leqslant \mu(U_2) - \mu(U_2 \setminus K_1) + \varepsilon, \\ &\leqslant \mu(U_2) - \inf\{\mu(U) : U \in \mathcal{O}(X), U \supseteq U_2 \setminus U_1\} + \varepsilon, \end{aligned}$$

for $U := U_2 \setminus K_1 \in \mathcal{O}(X)$ satisfies $U \supseteq U_2 \setminus U_1$. The proof is now complete for $\varepsilon > 0$ is arbitrary. □

We now obtain the equivalence between the subadditivity property and the inclusion-exclusion formula.

Theorem 9. *If μ is a quasi-measure, then the following two properties are equivalent to (SA1)-(SA3):*

(A1) *for any $K_1, K_2 \in \mathcal{C}^*(X)$, $\mu(K_1 \cup K_2) = \mu(K_2) + \mu(K_2) - \mu(K_1 \cap K_2)$;*
(A2) *for any $U_1, U_2 \in \mathcal{O}(X)$, $\mu(U_1 \cup U_2) = \mu(U_1) + \mu(U_2) - \mu(U_1 \cap U_2)$;*

Proof. Let us prove that if μ is sub-additive, then μ satisfies (A1). To that purpose, fix $K_1, K_2 \in \mathcal{C}^*(X)$ and observe that by [Lemma 8, (1)] we have

$$\mu(K_1) = \mu(K_1 \cap K_2) + \sup\{\mu(K) : K \in \mathcal{C}^*(X), K \subseteq K_1 \setminus K_2\}, \tag{3}$$

$$\mu(K_2) = \mu(K_1 \cap K_2) + \sup\{\mu(K) : K \in \mathcal{C}^*(X), K \subseteq K_2 \setminus K_1\}. \tag{4}$$

Now observe that for any $K \in \mathcal{C}^*(X)$ satisfying $K \subseteq K_1 \setminus K_2$ and any $K' \in \mathcal{C}^*(X)$ satisfying $K \subseteq K_2 \setminus K_1$, we have

$$\begin{aligned} &2\mu(K_1 \cap K_2) + \mu(K) + \mu(K') \\ &\quad = \mu(K_1 \cap K_2) + \mu((K_1 \cap K_2) \sqcup K \sqcup K') \leqslant \mu(K_1 \cap K_2) + \mu(K_1 \cup K_2); \end{aligned}$$

hence we get, using (3), (4) and the arbitrariness of K, K':

$$\mu(K_1) + \mu(K_2) \leqslant \mu(K_1 \cap K_2) + \mu(K_1 \cup K_2).$$

On the other hand, we get from [Lemma 8, (2)]:

$$\mu(K_1) = \mu(K_1 \cap K_2) + \inf\{\mu(U) : U \in \mathcal{O}(X), U \supseteq K_1 \setminus K_2\}, \tag{5}$$

$$\mu(K_2) = \mu(K_1 \cap K_2) + \inf\{\mu(U) : U \in \mathcal{O}(X), U \supseteq K_2 \setminus K_1\}, \tag{6}$$

and we observe that given $U \in \mathcal{O}(X)$ with $U \supseteq K_1 \setminus K_2$ and $U' \in \mathcal{O}(X)$ with $U' \supseteq K_2 \setminus K_1$, the subadditivity of μ yields

$$2\mu(K_1 \cap K_2) + \mu(U) + \mu(U') = \mu(K_1 \cap K_2) + [\mu(K_1 \cap K_2) + \mu(U) + \mu(U')] \geqslant \mu(K_1 \cap K_2) + \mu(K_1 \cup K_2);$$

from (5), (6) and the arbitrariness of U, U' we hence get that

$$\mu(K_1) + \mu(K_2) \geqslant \mu(K_1 \cap K_2) + \mu(K_1 \cup K_2);$$

which proves (A1). On the other hand, it is obvious that (A1) implies (SA1).

One prove in a similar way, using [Lemma 8, (3) and (4)] instead, that (SA2) implies (A2). It is again obvious that (A2) implies (SA2). The proof is thus complete. □

3. An application: the countable additivity of subadditive quasi-measures

We first state that any subadditive quasi-measure in X is also *infinitely subadditive* on open sets. The reader will easily provide a proof of this fact using property (QM2) and an usual compactness argument.

Proposition 10. *Let μ be a subadditive quasi-measure on X, and fix a family $\{U_i\}_{i\in I} \subseteq \mathcal{O}(X)$. We have*

$$\mu\left(\bigcup_{i\in I} U_i\right) \leqslant \sum_{i\in I} \mu(U_i).$$

In fact, subadditive quasi-measures are *countably subadditive* in the following sense:

Proposition 11. *Assume that $A \in \mathcal{A}(X)$ and $\{A_k\}_{k\in\mathbb{N}} \subseteq \mathcal{A}(X)$ are such that $A \subseteq \bigcup_{k\in\mathbb{N}} A_k$. Then for any subadditive quasi-measure μ on X we have*

$$\mu(A) \leqslant \sum_{k=0}^{\infty} \mu(A_k).$$

Proof. Fix $\varepsilon > 0$ and begin by choosing for each $k \in \mathbb{N}$ an open set $U_k \in \mathcal{O}(X)$ satisfying $U_k \supseteq A_k$ (we might of course take $U_k = A_k$ in case A_k is open, and use (QM3) in case A_k is compact) satisfying $\mu(U_k) \leqslant \mu(A_k) + \frac{\varepsilon}{2^{k+1}}$; observe now that $A \subseteq \bigcup_{k\in\mathbb{N}} U_k$; we hence get using Proposition 10:

$$\mu(A) \leqslant \mu\left(\bigcup_{k\in\mathbb{N}} U_k\right) \leqslant \sum_{k=0}^{\infty} \mu(U_k) \leqslant \varepsilon + \sum_{k=0}^{\infty} \mu(A_k).$$

The proof is complete. □

Subadditive quasi-measures are also countably *additive*.

Proposition 12. *Assume that $\{A_k\}_{k\in\mathbb{N}} \subseteq \mathcal{A}(X)$ is a collection of pairwise disjoint sets for which $\bigsqcup_{k\in\mathbb{N}} A_k \in \mathcal{A}(X)$; then for any subadditive quasi-measure μ on X we have*

$$\mu\left(\bigsqcup_{k\in\mathbb{N}} A_k\right) = \sum_{k=0}^{\infty} \mu(A_k).$$

Proof. Let $A := \bigsqcup_{k\in\mathbb{N}} A_k \in \mathcal{A}(X)$, define $K_1 := \{k \in \mathbb{N} : A_k \in \mathcal{O}(X)\}$ and

$$K_2 := \mathbb{N} \setminus K_1 \subseteq \{k \in \mathbb{N} : A_k \in \mathcal{C}(X)\}.$$

We observe that $A^{(1)} := \bigsqcup_{k\in K_1} A_k \in \mathcal{O}(X)$ and $A^{(2)} := \bigcup_{k\in K_2} A_k = A \setminus A^{(1)}$. We find for each $n \in \mathbb{N}$:

$$\sum_{\{0\leqslant k\leqslant n : k\in K_1\}} \mu(A_k) = \mu\left(\bigsqcup_{\{0\leqslant k\leqslant n : k\in K_1\}} A_k\right) \leqslant \mu\left(A^{(1)}\right),$$

and

$$\sum_{\{0\leqslant k\leqslant n : k\in K_2\}} \mu(A_k) = \mu\left(\bigsqcup_{\{0\leqslant k\leqslant n : k\in K_2\}} A_k\right) \leqslant \sup\left\{\mu(K) : K \in \mathcal{C}^*(X), K \subseteq A \setminus A^{(1)}\right\},$$

from which it follows that

$$\sum_{k=0}^{n} \mu(A_k) = \sum_{i=1}^{2} \sum_{\{1\leqslant k\leqslant n : k\in K_i\}} \mu(A_k) \leqslant \mu\left(A^{(1)}\right) + \sup\left\{\mu(K) : K \in \mathcal{C}^*(X), K \subseteq A \setminus A^{(1)}\right\} = \mu(A),$$

using either (QM1) in case A is compact (for this implies that $A^{(2)} = A \setminus A^{(1)}$ is also compact) or [Lemma 8, (3)] if A is open. Letting n go to ∞ in the previous equation, we get $\sum_{k=0}^{\infty} \mu(A_k) \leqslant \mu(A)$. The proof is now complete for Proposition 11 yields the converse inequality. □

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