A Galois-theoretic approach to the covering theory of quandles

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Abstract

The purpose of this article is to clarify the relationship between the algebraic notion of quandle covering introduced by M. Eisermann and the categorical notion of covering arising from Galois theory. A crucial role is played by the adjunction between the variety of quandles and its subvariety of trivial quandles.

Keywords : covering theory, quandle, categorical Galois theory, trivial covering.

Introduction

One of the motivations to introduce the notion of *quandle* is that it encrypts the group conjugation, or equivalently, the Reidemeister moves of knot diagrams, and it does so without taking into account the whole group structure. A quandle is thus a suitable structure to study situations where only the group conjugation is needed, as it is the case for the Wirtinger presentation of the knot group, for example. Although this structure encodes a very natural property, it has only been introduced in the early 1980's independently by S. V. Matveev [6], who called it *distributive groupoid*, and by D. Joyce [5] who introduced the term quandle which is still used today. A quandle is a set A together with two binary operations $\triangleleft$ and $\triangleleft^{-1}$ satisfying three identities :

- $q \triangleleft q = q = q \triangleleft^{-1} q$ for all $q \in Q$ (idempotency);
- $(q \triangleleft p) \triangleleft^{-1} p = q = (q \triangleleft^{-1} p) \triangleleft p$ for all $p,\ q \in Q$ (right invertibility);
- $(p \triangleleft q) \triangleleft r = (p \triangleleft r) \triangleleft (q \triangleleft r)$ and $(p \triangleleft^{-1} q) \triangleleft^{-1} r = (p \triangleleft^{-1} r) \triangleleft^{-1} (q \triangleleft^{-1} r)$ for all $p,\ q,\ r \in Q$ (self-distributivity).

In the last years, there have been many developments in the study of this structure, leading M. Eisermann [2] [3] to propose an algebraic covering theory of quandles. A *quandle covering* is defined as a surjective quandle homomorphism $f\colon A \to B$ such that $c \triangleleft a = c \triangleleft b$ whenever $f(a) = f(b)$. In the present paper, we shall call such a covering an E-covering.

On the other hand, a general covering theory for exact categories, thus in particular for varieties of universal algebras, has been introduced in 1993 by G. Janelidze and G.M. Kelly [4]. Given an adjunction between an exact category $\mathcal{C}$ and a suitable full reflective subcategory $\mathcal{H}$ of $\mathcal{C}$,

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "I"] \arrow[r, phantom, "\perp"] & \mathcal{H} \arrow[l, bend left, "H"] \end{tikzcd}$$

where the functor I is the left adjoint of the inclusion functor H, one can define the corresponding notions of *trivial covering* and of *covering*, provided that the subcategory $\mathcal{H}$ is *admissible* with

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respect to $\mathcal{C}$. This admissibility condition of the subcategory depends on a left exactness property of the functor I, that should preserve a certain type of pullbacks (see section 1). When this is the case, trivial coverings are defined as regular epimorphisms $f\colon A \to B$ such that the square

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & HI(A) \arrow[d, "HI(f)"] \\ B \arrow[r, "\eta_B"'] & HI(B) \end{tikzcd}$$

encoding the naturality of the unit of the adjunction is a pullback. A regular epimorphism $f\colon A \to B$ is then a *covering* when there exists another regular epimorphism $p\colon E \to B$ such that the pullback $\pi_1\colon E \times_B A \to E$ of f along p is a trivial covering

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B. \end{tikzcd} \tag{0.1}$$

As show in [4], admissibility is always guaranted whenever the exact category $\mathcal{C}$ is a *Goursat* category, this meaning that $RSR = SRS$ holds for equivalence relations (= congruences in the universal algebraic context) on any object in $\mathcal{C}$. As a matter of fact, most of the examples that can be found in the litterature share this property. The variety of quandles, denoted by Qnd, is not a Goursat category, so that the present paper provides an additional non-trivial example to the covering theory of Janelidze and Kelly.

The goal of this article is precisely to show that the notion of algebraic covering proposed by M. Eisermann is a particular case of the covering theory developed by G. Janelidze and G. M. Kelly. For this, the adjunction between the variety Qnd of quandles and the subvariety of so-called *trivial quandles* Qnd^* is studied, where a quandle A is trivial if it satisfies the additional identity $a \lhd b = a$ for all $a, b \in A$,

$$\begin{tikzcd} \mathrm{Qnd} \arrow[rr, bend left, "\pi_0"] \arrow[rr, phantom, "\perp"] & & \mathrm{Qnd}^* \arrow[ll, bend left, "U"] \end{tikzcd} .$$

The first point will be to show that this adjunction is admissible (Theorem 1). Then we shall characterize trivial coverings algebraically, and this will allow us to show that the notion of E-covering in the sense of Eisermann is equivalent to the categorical notion of covering (Theorem 2). This will be achieved thanks to the existence of a *universal* surjective homomorphism $p\colon E \to B$ as in (0.1) making π_1 trivial. Finally we shall provide an example of a covering that is not a trivial covering.

1 Categorical covering theory

In this section, we briefly recall the main definitions and some known results of the categorical covering theory (also called categorical theory of central extensions). We refer the reader to [4] for more details and examples.

Recall that a finitely complete category $\mathcal{C}$ is regular when any arrow $f\colon A \to B$ has a factorisation $f = i \circ p$ with p a regular epimorphism and i a monomorphism, and any kernel pair has a coequaliser. A regular category $\mathcal{C}$ is exact [1] when, moreover, any (internal) equivalence relation is effective, i.e. a kernel pair. Consider a full replete reflective subcategory $\mathcal{H}$ of an exact category $\mathcal{C}$. We will write $H : \mathcal{H} \to \mathcal{C}$ for the inclusion functor, $I : \mathcal{C} \to \mathcal{H}$ for its left adjoint, and $\eta : 1 \to HI$ for the unit of the adjunction. Since H is a full inclusion, we often suppress it from the notation, writing $\eta_A : A \to IA$ for the A-component of the unit η, for example.

It is well known that $\mathcal{H}$ is closed under subobjects if and only if $\eta_A \colon A \to IA$ is a regular epimorphism for every A in $\mathcal{C}$. Moreover, one calls the reflective full subcategory $\mathcal{H}$ of $\mathcal{C}$ a *Birkhoff subcategory* when it is closed in $\mathcal{C}$ under both subobjects and regular quotients. In particular, when $\mathcal{C}$ is a *variety of universal algebras*, one knows from Birkhoff's classical theorem that the Birkhoff subcategories of $\mathcal{C}$ are precisely its *subvarieties*.

From now on, we shall assume that $\mathcal{H}$ is a Birkhoff subcategory of a given exact category $\mathcal{C}$. Recall that the naturality of η is expressed by the commutativity of

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & IA \arrow[d, "If"] \\ B \arrow[r, "\eta_B"'] & IB \end{tikzcd} \tag{1.1}$$

for each $f : A \to B$ in $\mathcal{C}$. Observe that, since each η_A is a regular epi, it follows that If will be a regular epi in $\mathcal{C}$ whenever f is one. We shall write $\mathcal{H} \downarrow B$ for the full subcategory of the slice category $\mathcal{H}/B$ whose objects are the regular epimorphisms $f : A \to B$. Since $I : \mathcal{C} \to \mathcal{H}$ sends regular epis of $\mathcal{C}$ to regular epis of $\mathcal{H}$, it induces a functor

$$I^B : \mathcal{C} \downarrow B \to \mathcal{H} \downarrow IB$$

for each B in $\mathcal{C}$, sending a regular epimorphism $f \colon A \to B$ to $If \colon IA \to IB$. This functor has a right adjoint

$$H^B : \mathcal{H} \downarrow IB \to \mathcal{C} \downarrow B.$$

Indeed, given a regular epi $\phi : X \to IB$, one takes the pullback of ϕ along η_B

$$\begin{tikzcd} B \times_{IB} X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\phi"] \\ B \arrow[r, "\eta_B"'] & IB \end{tikzcd} \tag{1.2}$$

and $H^B\phi$ is the regular epi π_1 of this pullback. One says that the Birkhoff subcategory $\mathcal{H}$ of $\mathcal{C}$ is *admissible* when each functor $H^B : \mathcal{H} \downarrow IB \to \mathcal{C} \downarrow B$ is fully faithful. This condition can be expressed by asking the preservation of a special kind of pullbacks by the functor I:

Proposition 1. *[4] Given a Birkhoff subcategory $\mathcal{H}$ of an exact category $\mathcal{C}$, the following assertions are equivalent :*

- *the Birkhoff subcategory $\mathcal{H}$ is admissible;*
- *I preserves all pullbacks of the form* (1.2) *with $X \in \mathcal{H}$ and ϕ a regular epimorphism;*

We now recall the different notions of trivial coverings and coverings that will play a central role in this article.

A regular epi $f : A \to B$ is said to be a *trivial covering* (with respect to the admissible subcategory $\mathcal{H}$ of $\mathcal{C}$) when it lies in the image of the fully faithful $H^B : \mathcal{H} \downarrow IB \to \mathcal{C} \downarrow B$. The (A, f)-component of the unit $\eta^B : 1 \to H^B I^B$ of the adjunction $I^B \dashv H^B$ is the comparison arrow $A \to B \times_{IB} IA$ induced by the commutative diagram (1.1) and the universal property of the pullback of η_B and If. So the regular epi (A, f) is a trivial covering if and only if the commutative square (1.1) is a pullback. We shall write $g^* : \mathcal{C} \downarrow B \to \mathcal{C} \downarrow D$ for the restriction of the change-of-base functor $g^\bullet : \mathcal{C}/B \to \mathcal{C}/D$ induced by a morphism $g : D \to B$.

The admissibility property guarantees a desirable property of trivial coverings:

Proposition 2. *[4] Let $\mathcal{H}$ be an admissible Birkhoff subacategory of an exact category $\mathcal{C}$. Then the trivial coverings in $\mathcal{C}$ are stable under pullback.*

Proof. Suppose the left square below is a pullback, with f a trivial covering and $g\colon D \to B$ an arrow in $\mathcal{C}$.

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & A \arrow[r, "\eta_A"] \arrow[d, "f"] & IA \arrow[d, "If"] \\ D \arrow[r, "g"'] & B \arrow[r, "\eta_B"'] & IB \end{tikzcd} \tag{1.4}$$

Since both squares are pullbacks, the exterior rectangle is a pullback too. Accordingly, the exterior of the following diagram is again a pullback:

$$\begin{tikzcd} P \arrow[r, "\eta_P"] \arrow[d, "p_1"'] & IP \arrow[r, "Ip_2"] \arrow[d, "Ip_1"] & IA \arrow[d, "If"] \\ D \arrow[r, "\eta_D"'] & ID \arrow[r, "Ig"'] & IB \end{tikzcd} \tag{1.5}$$

But if we apply I to the exterior rectangle in (1.4), we find the right square of (1.5) which is a pullback by Proposition 1. Since the exterior rectangle and the right square are pullbacks, so is the left square, proving that p_1 is a trivial covering, as desired. □

Let $p : E \to B$ be a regular epi in $\mathcal{C}$, we will say that a regular epi $f : A \to B$ of B is (E, p)*-split* when $p^*(A, f)$ (constructed as the pullback of f along p) is a trivial covering. A regular epi $f : A \to B$ is called a *covering* when it is (E, p)-split for some regular epi $p : E \to B$ in $\mathcal{C}$.

The stability of trivial coverings under pullbacks easily implies the following:

Corollary 1. *[4] Let $\mathcal{H}$ be an admissible Birkhoff subcategory of an exact category $\mathcal{C}$. Then the coverings in $\mathcal{C}$ are stable under pullback.*

Proof. Let $f : A \to B$ be a covering, and $p : E \to B$ a regular epi that "splits" f, so that $\pi_1\colon E \times_B A \to E$ in the diagram

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

is a trivial covering. If $q\colon E' \to B$ is any arrow in $\mathcal{C}$, one can form the pullback

$$\begin{tikzcd} E' \times_B A \arrow[r, "p_2"] \arrow[d, "p_1"'] & A \arrow[d, "f"] \\ E' \arrow[r, "q"'] & B, \end{tikzcd}$$

and we are to prove that p_1 is a covering as well. For this, form the pullback

$$\begin{tikzcd} E \times_B E' \arrow[r, "v"] \arrow[d, "u"'] & E' \arrow[d, "q"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

and observe that $v^*(p_1) = v^*(q^*(f)) = u^*(p^*(f)) = u^*(\pi_1)$ is a trivial covering by Proposition 2. □

If we write $\mathrm{Cov}(B)$ for the category of coverings of B and $\mathrm{Triv}(B)$ for the category of trivial coverings of B, we have the full inclusion

$$\mathrm{Triv}(B) \subset \mathrm{Cov}(B).$$

This inclusion is known to be proper in general (see [4]).

2 Covering theory of quandles

We are now going to focus on the algebraic structure of quandle, as named by David Joyce [5]. We shall prove that the category of trivial quandles is an admissible subcategory of the category of quandles, and then give precise algebraic characterisations of the corresponding trivial coverings and coverings.

Definition 2.1. A *quandle* is a set Q equipped with two binary operations $\lhd$ and $\lhd^{-1}$ such that :

- $q \lhd q = q = q \lhd^{-1} q$ for all $q \in Q$ (idempotency)
- $(q \lhd p) \lhd^{-1} p = q = (q \lhd^{-1} p) \lhd p$ for all $p,\ q \in Q$ (right invertibility)
- $(p \lhd q) \lhd r = (p \lhd r) \lhd (q \lhd r)$ and $(p \lhd^{-1} q) \lhd^{-1} r = (p \lhd^{-1} r) \lhd^{-1} (q \lhd^{-1} r)$ for all $p,\ q,\ r \in Q$ (self-distributivity).

A quandle homomorphism $f\colon Q \to Q'$ is a map preserving both the binary operations $\lhd$ and $\lhd^{-1}$. Quandles form a variety of universal algebras, and the corresponding category of quandles, written Qnd, is then an exact category.

Let us then recall some classical examples of quandles.

Examples 2.1. *(a) Let Q be a set, then define $\lhd = \lhd^{-1}$ as $p \lhd q = p$ for all $p,\ q \in Q$. The set Q equipped with these binary operations $\lhd$ and $\lhd^{-1}$ is a quandle, called a* trivial quandle*. We denote by* Qnd^* *the category of trivial quandles, which is clearly isomorphic to the category of sets.*

(b) Let G be a group, and define $g \lhd h := h^{-1}gh$ and $g \lhd^{-1} h := hgh^{-1}$, for all $g, h \in G$. Then the set G with $\lhd$ and $\lhd^{-1}$ is a quandle. We call it the conjugation quandle *of G, and denote it by* $\mathrm{Conj}(G)$*. It determines a functor* $\mathrm{Conj} : \mathrm{Grp} \to \mathrm{Qnd}$ *from the category* Grp *of groups to the category* Qnd *of quandles.*

Observe that the second and the third quandle axioms guarantee that the right actions $\rho_p : Q \to Q$ defined by $\rho_p(q) = q \lhd p$, for all $q \in Q$, are bijective homomorphisms. We will write $\mathrm{Inn}(Q)$ for the subgroup of $\mathrm{Aut}(Q)$ (the group of all automorphisms of Q) generated by all such ρ_p, with $p \in Q$, which is called the subgroup of *inner automorphisms* of Q.

Definition 2.2. A quandle Q is *connected* if $\mathrm{Inn}(Q)$ acts transitively on Q. A *connected component* of Q is an orbit under the action of $\mathrm{Inn}(Q)$. Two elements p and q are in the same orbit if there exist $q_1,\ q_2, ...,\ q_n \in Q$ such that

$$q \lhd^{\alpha_1} q_1 \lhd^{\alpha_2} q_2 ... \lhd^{\alpha_n} q_n = p,$$

where we write by convention

$$q \lhd^{\alpha_1} q_1 \lhd^{\alpha_2} q_2 ... \lhd^{\alpha_n} q_n := (...((q \lhd^{\alpha_1} q_1) \lhd^{\alpha_2} q_2)...) \lhd^{\alpha_n} q_n$$

with $\lhd^{\alpha_i} \in \{\lhd,\ \lhd^{-1}\}$ for all $1 \le i \le n$.

The set of connected components of a quandle Q is denoted by $\pi_0(Q)$. Note that $\pi_0(Q)$ is a trivial quandle (as any set is). In fact, we have a functor $\pi_0 : \mathrm{Qnd} \to \mathrm{Qnd}^*$, that turns out to be the left adjoint of the inclusion functor $U : \mathrm{Qnd}^* \to \mathrm{Qnd}$:

$$\begin{tikzcd} \mathrm{Qnd} \arrow[r, bend left, "\pi_0"] \arrow[r, phantom, "\perp"] & \mathrm{Qnd}^* \arrow[l, bend left, "U"] \end{tikzcd} . \tag{1}$$

In order to study the coverings in the category Qnd of quandles arising from the better known subvariety Qnd^*, we shall first look for an algebraic characterization of trivial coverings. Recall that $f : A \to B$ is a trivial covering with respect to the adjunction (1) if and only if the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & \pi_0(A) \arrow[d, "\pi_0(f)"] \\ B \arrow[r, "\eta_B"'] & \pi_0(B) \end{tikzcd}$$

is a pullback.

Lemma 1. *Given any surjective homomorphism $f\colon A \to B$, the induced arrow $\phi : A \to B \times_{\pi_0(B)} \pi_0(A)$ defined by $\phi(a) = (f(a), [a])$ is always surjective.*

Proof. Let $(b, [a]) \in B \times_{\pi_0(B)} \pi_0(A)$, then we know that $[b] = [f(a)]$, i.e. there exist $b_i \in B$, for $1 \leq i \leq n$, such that

$$b \lhd^{\alpha_1} b_1 ... \lhd^{\alpha_n} b_n = f(a),$$

with $\lhd^{\alpha_i} \in \{\lhd, \lhd^{-1}\}$. By the surjectivity of $f\colon A \to B$ one can find $a_i \in A$ such that $f(a_i) = b_i$, for all $1 \leq i \leq n$. We thus have

$$b \lhd^{\alpha_1} f(a_1) ... \lhd^{\alpha_n} f(a_n) = f(a)$$

and, by using the second axiom of quandle,

$$b = f(a) \lhd^{-\alpha_n} f(a_n) ... \lhd^{-\alpha_1} f(a_1) = f(a \lhd^{-\alpha_n} a_n ... \lhd^{-\alpha_1} a_1).$$

It then suffices to take $a' = a \lhd^{-\alpha_n} a_n ... \lhd^{-\alpha_1} a_1$, since $f(a') = b$ and $[a'] = [a]$. $\square$

This lemma is useful to see that the subvariety of trivial quandles is an admissible subcategory of the category of quandles.

Theorem 1. *The subvariety* Qnd^* *of the variety* Qnd *of quandles is admissible.*

Proof. Consider the following commutative diagram where $A \in \mathrm{Qnd}^*$, the induced arrow $\psi : \pi_0\left(E \times_{\pi_0(E)} A\right) \to A$ comes from universality of $\eta_{E \times_{\pi_0(E)} A} : E \times_{\pi_0(E)} A \to \pi_0(E \times_{\pi_0(E)} A)$ and $\phi : E \times_{\pi_0(E)} A \to E \times_{\pi_0(E)} \pi_0(E \times_{\pi_0(E)} A)$ is the induced arrow to the pullback of $\pi_0(p_1)$ along η_E.

$$\begin{tikzcd} E \times_{\pi_0(E)} A \arrow[rrr, "p_2"] \arrow[dd, "p_1"'] \arrow[dr, dotted, "\phi"'] \arrow[drr, dotted, "\eta_{E \times_{\pi_0(E)} A}"] & & & A \arrow[dd, "f"] \\ & E \times_{\pi_0(E)} \pi_0(E \times_{\pi_0(E)} A) \arrow[r, "\pi_2"'] \arrow[dl, "\pi_1"'] & \pi_0(E \times_{\pi_0(E)} A) \arrow[ur, dotted, "\psi"] \arrow[dr, "\pi_0(p_1)"] & \\ E \arrow[rrr, "\eta_E"'] & & & \pi_0(E) \end{tikzcd}$$

By the previous proposition, we know that ϕ is a surjection. But $\pi_1 \circ \phi = p_1$ and $\psi \circ \pi_2 \circ \phi = p_2$ and because p_1 and p_2 are jointly monic, ϕ is also injective, thus an isomorphism.

We can then consider the following diagram

$$\begin{tikzcd} E \times_{\pi_0(E)} \pi_0(E \times_{\pi_0(E)} A) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "(1)"] & \pi_0(E \times_{\pi_0(E)} A) \arrow[r, "\psi"] \arrow[d, "\pi_0(p_1)"'] \arrow[dr, phantom, "(2)"] & A \arrow[d, "f"] \\ E \arrow[r, "\eta_E"'] & \pi_0(E) \arrow[r, equal, "1_{\pi_0(E)}"'] & \pi_0(E) \end{tikzcd}$$

where both (1) + (2) and (1) are pullbacks. Since η_E is a regular epi and Qnd is a regular category, it follows that (2) is a pullback (see [4] Proposition 2.7). $\square$

Note that the previous results do not use the first quandle axiom, so it remains true in the slightly more general setting of the variety of racks: recall that a *rack* (or *wrack*) is a set equipped with two binary operations $\lhd$ and $\lhd^{-1}$ satisfying $(Q2)$ and $(Q3)$. Racks have been first studied by J.H. Conway and G.C.Wraith in an unpublished correspondence.

We are now going to show that the surjectivity of $f : A \to \pi_0(E)$ is crucial, the functor π_0 no longer preserves pullbacks of the form (1.2) when $f : A \to \pi_0(E)$ is not surjective.

Counter-example 2.2. *Let $A = \{*\}$ be the trivial quandle on the one-element set, and take for E the quandle having three elements x, y and z with $\lhd = \lhd^{-1}$ defined by the following table:*

$\lhd$	x	y	z
x	$x \lhd x = x$	$x \lhd y = x$	$x \lhd z = y$
y	$y \lhd x = y$	$y \lhd y = y$	$y \lhd z = x$
z	$z \lhd x = z$	$z \lhd y = z$	$z \lhd z = z$

By setting $f(*) = [x] = [y]$ we define a quandle homomorphism that it is not surjective. We thus have

$$E \times_{\pi_0(E)} A = \{(p \lhd q_1 ... \lhd q_n, *) |\, p \in \{x, y\} \, and \, q_i \in \{x, y, z\}, \, \forall i \in \{1, ..., n\}\},$$

so that

$$\phi([(x, *)]) = ([x], *) = ([y], *) = \phi([(y, *)])$$

with $[(x, *)] \neq [(y, *)]$.

Let us now go back to the problem of characterising the notion of trivial covering in algebraic terms. Under the assumption of Proposition 1 one knows that the induced arrow $\phi \colon A \to E \times_{\pi_0(B)} \pi_0(A)$ is always surjective: it suffices then to find an algebraic condition on $f \colon A \to B$ for ϕ to be injective.

Proposition 3. *A surjective homomorphism $f : A \to B$ is a trivial covering if and only if the following condition (T) holds:*

(T): $\forall a, a' \in A$, *if* $f(a) = f(a')$ *and* $[a] = [a']$, *then* $a = a'$.

Proof. This follows directly from the definition of $\phi \colon A \to E \times_{\pi_0(B)} \pi_0(A)$, together with the fact that ϕ is always surjective by Proposition 1. $\square$

We will now focus on coverings. Our goal here is to show that the notion of covering given by M. Eisermann [2] is a particular case of the categorical notion of covering (in the sense of [4]) arising from the adjunction (1).

Definition 2.3. A quandle homomorphism $f : A \to B$ is a *covering in the sense of Eisermann* when it is surjective and $p(a) = p(b)$ implies $c \lhd a = c \lhd b$ for all $a,\ b,\ c \in A$. When this is the case, we shall say that $p \colon A \to B$ is an *E-covering*, for short.

We have seen how to associate a quandle with each group. We can also go the other way round: any quandle naturally gives rise to a group. Indeed, given a quandle Q, one defines the *adjoint group* $\mathrm{Adj}(Q) = \langle Q|R \rangle$ of Q as the quotient group of the group $F(Q) = \langle e_a | a \in Q \rangle$ freely generated by the set Q modulo the induced relations $R = \{e_{(a \lhd b)} e_b^{-1} e_a^{-1} e_b = 1 | a, b \in Q\}$. Since we will only consider members of the adjoint group in what follows, we shall write e_a for an element of the adjoint group instead of $[e_a]$.

Remark that, given an E-covering $f \colon A \to B$, there is an induced action of the adjoint group $\mathrm{Adj}(B)$ of the target quandle B on the source quandle A. For an E-covering $f : A \to B$, we have an action $A \times \mathrm{Adj}(B) \to A$ with $(a, g) \mapsto a^g$ as follows: if $g = e_x$ with $x \in B$ then $a^{e_x} := a \lhd x'$ where $x' \in A$ such that $f(x') = x$. Remark that this action is well defined because of the algebraic condition of an E-covering. In particular we also have the action $A \times \mathrm{Adj}(A) \to A$ of the adjoint group of a quandle A on itself that acts by inner automorphisms.

Definition 2.4. For every quandle A, there exists a unique group homomorphism $\epsilon : \mathrm{Adj}(A) \to \mathbb{Z}$ with $\mathrm{adj}(A) \to \{1\}$ where $\mathrm{adj} : A \to \mathrm{Adj}(A); a \mapsto e_a$. Its kernel $\mathrm{Adj}(A)^\circ = \ker(\epsilon)$ is generated by all products of the form $e_a e_b^{-1}$ with $a, b \in A$.

Let us now see that the algebraic property of being an E-covering behaves well with respect to pullbacks.

Lemma 2. *Consider the following pullback*

$$\begin{tikzcd} E \times_B A \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ E \arrow[r, two heads, "p"'] & B, \end{tikzcd}$$

where p is a regular epi. In this case f is an E-covering if and only if p_1 is an E-covering.

Proof. Let us check that the property of being an E-covering is stable under pullbacks. Let then $(e, a),\ (e', a') \in E \times_B A$ such that $p_1(e, a) = p_1(e', a')$, i.e. $e = e'$. Then we have

$$f(a) = p(e) = p(e') = f(a'),$$

and by assumption we know that $c \lhd a = c \lhd a'$ for all $c \in A$, so that p_1 is a covering.

Now let us see that this property is reflected by pullbacks along regular epis. Suppose that $f(a) = f(a')$: by surjectivity of p, there exists $e \in E$ such that $p(e) = f(a) = f(a')$, so that both (e, a) and (e, a') belong to $E \times_B A$. Moreover, these elements have the same image by p_1 and, by assumption, we know that $(x, y) \lhd (e, a) = (x, y) \lhd (e, a')$ for all $(x, y) \in E \times_B A$. This implies that $y \lhd a = y \lhd a'$ for all $y \in A$ because p_2 is surjective. $\square$

From this lemma we get the following:

Corollary 2. *If $f : A \to B$ is a covering then $f : A \to B$ is an E-covering.*

Proof. If f is a covering then there exists a regular epi p such that $p^*(f) = p_1$ in the diagram

$$\begin{tikzcd} \pi_0(E \times_B A) \arrow[d, "\pi_0(p_1)"'] \arrow[dr, phantom, "(2)"] & E \times_B A \arrow[l, two heads, "\eta_{E \times_B A}"'] \arrow[r, two heads, "p_2"] \arrow[d, "p_1"] & A \arrow[d, "f"] \\ \pi_0(E) & E \arrow[l, two heads, "\eta_E"] \arrow[r, two heads, "p"'] & B, \end{tikzcd}$$

is a trivial covering, so that the square (2) is a pullback. Of course, in Qnd^*, every regular epi is an E-covering. By using the previous lemma twice, one can lift the E-covering property from $\pi_0(p_1)$ to f. $\square$

Let us now introduce the notion of universal E-covering that will be useful to achieve our goal.

Definition 2.5. An E-covering $p \colon \tilde{Q} \to Q$ is *universal* if, for any E-covering $f \colon X \to Q$, there exists a unique quandle homomorphism $\phi \colon \tilde{Q} \to X$ such that $f \circ \phi = p$.

To construct a universal E-covering, we will use the work of M. Eisermann [3]. The idea is the following: given a quandle Q, we take its connected components $\{Q_i\}_{i \in I}$ and we choose a point q_i in each component Q_i. We write (Q, q) for a quandle Q equipped with a given $q \colon I \to Q$ which specifies one base point in each connected component. Moreover, one would like to keep track of the elements of $\mathrm{Adj}(Q)^\circ$ allowing one to send our base point to any other point of the same component. From this, one constructs $\tilde{Q}$ as follows:

Definition 2.6. Let (Q, q) be a quandle with connected components $(Q_i, q_i)_{i\in I}$. Let $\mathrm{Adj}(Q)^\circ$ be the kernel of the group homomorphism $\epsilon : \mathrm{Adj}(Q) \to \mathbb{Z}$ with $\epsilon(\mathrm{adj}(Q)) = 1$. For each $i \in I$, we define

$$\tilde{Q}_i := \{(a, g) \in Q_i \times \mathrm{Adj}(Q)^\circ | a = q_i^g\}, \ \tilde{q}_i := (q_i, 1).$$

We define $\tilde{Q} = \coprod_{i\in I} \tilde{Q}_i$ as the disjoint union of the $\tilde{Q}_i$.

In the following lemma, we define the quandle structure on $\tilde{Q}$, and also the quandle homomorphism $p\colon (\tilde{Q}, \tilde{q}) \to (Q, q)$ which will be proved to be a universal E-covering of (Q, q). The following two results can both be found in [3] (Lemma 7.11 and Proposition 7.12, respectively); we include the proofs to make this article more self-contained.

Lemma 3. *1. The set $\tilde{Q} = \coprod_{i\in I} \tilde{Q}_i$ becomes a quandle by defining :*

$$(i, (a, g)) \lhd (j, (b, h)) = (i, (a \lhd b, ge_a^{-1}e_b))$$
$$(i, (a, g)) \lhd^{-1} (j, (b, h)) = (i, (a \lhd^{-1} b, ge_a e_b^{-1})).$$

2. *The quandle $\tilde{Q}$ is equipped with an action $\tilde{Q} \times \mathrm{Adj}(Q) \to \tilde{Q}$ defined by*

$$(i, (a, g))^h := (i, (a^h, e_{q_i}^{-\epsilon(h)} gh)).$$

There is thus a restricted action of $\mathrm{Adj}(Q)$ on each $\tilde{Q}_i$, defined by

$$(a, g)^h = (a^h, e_{q_i}^{-\epsilon(h)} gh).$$

The subgroup $\mathrm{Adj}(Q)^\circ$ acts freely and transitively on each $\tilde{Q}_i$. As a consequence, the connected components of $\tilde{Q}$ are the sets $\tilde{Q}_i$.

3. *The arrow $p : (\tilde{Q}, \tilde{q}) \to (Q, q)$ defined by $p(i, (a, g)) = a$ is a surjective quandle homomorphism and is equivariant under the action of $\mathrm{Adj}(Q)$.*

Proof. 1. We will only give the proof of one identity per axiom, the other identity being proved in the same way by exchanging the roles of $\lhd$ and $\lhd^{-1}$. The first quandle axiom is obvious: $(i, (a, g)) \lhd (i, (a, g)) = (i, (a \lhd a, ge_a^{-1}e_a)) = (i, (a, g))$. The second one follows from the equality $e_{a\lhd b} = e_b^{-1}e_a e_b$:

$$\begin{aligned}((i, (a, g)) \lhd (j, (b, h))) \lhd^{-1} (j, (b, h)) &= (i, (a \lhd b, ge_a^{-1}e_b)) \lhd^{-1} (j, (b, h)) \\ &= (i, ((a \lhd b) \lhd^{-1} b, ge_a^{-1}e_b e_{a\lhd b} e_b^{-1})) \\ &= (i, (a, ge_a^{-1}e_b e_b^{-1}e_a e_b e_b^{-1})) \\ &= (i, (a, g)).\end{aligned}$$

The third axiom also results from the previous equality:

$$((i, (a, g)) \lhd (j, (b, h))) \lhd (k, (c, l)) = (i, ((a \lhd b) \lhd c, ge_a^{-1}e_a^{-1}e_b e_c)),$$

and

$$\begin{aligned}&((i, (a, g)) \lhd (k, (c, l))) \lhd ((j, (b, h)) \lhd (k, (c, l))) \\ &\quad= (i, (a \lhd c, ge_a^{-1}e_c)) \lhd (j, (b \lhd c, he_b^{-1}e_c)) \\ &\quad= (i, ((a \lhd c) \lhd (b \lhd c), ge_a^{-1}e_c e_{a\lhd c}^{-1} e_{b\lhd c})) \\ &\quad= (i, ((a \lhd b) \lhd c, ge_a^{-1}e_a^{-1}e_b e_c)).\end{aligned}$$

2. Now let us prove that $\mathrm{Adj}(Q)^\circ$ acts transitively via the action defined in the lemma: let (a, g) and (b, h) be in $\tilde{Q}_i$, then $a = \tilde{q}_i^g$ and $b = \tilde{q}_i^h$ where $g,\ h \in \mathrm{Adj}(Q)^\circ$. By taking $g^{-1}h \in \mathrm{Adj}(Q)^\circ$ we see that $(a, g)^{g^{-1}h} = (b, h)$. It also acts freely on each $\tilde{Q}_i$: if $(a, g)^h = (a, g)^k$ for some $h, k \in \mathrm{Adj}(Q)^\circ$, then $(a^h, gh) = (b^k, gk)$ and so $h = k$.

3. It is easy to see that the arrow $p : (\tilde{Q}, \tilde{q}) \to (Q, q)$ is a surjective quandle homomorphism and is equivariant under the action of $\mathrm{Adj}(Q)$.

$\square$

Note that the construction of $\tilde{Q}$ does not really depend on the choice of points $q_i \in Q_i$: another choice of base points would just lead to an isomorphic structure.

Proposition 4. *Let (Q, q) be a quandle and let $(\tilde{Q}, \tilde{q})$ be defined as in Lemma 3. Then the arrow $p : (\tilde{Q}, \tilde{q}) \to (Q, q)$ defined by*

$$p(i, (a, g)) = a \qquad \forall (i, (a, g)) \in (\tilde{Q}, \tilde{q})$$

is a universal E-covering of (Q, q).

Proof. Clearly, p is an E-covering. So we need to prove that if $f : (\hat{Q}, \hat{q}) \to (Q, q)$ is an E-covering then there exists a unique homomorphism $\phi : (\tilde{Q}, \tilde{q}) \to (\hat{Q}, \hat{q})$ such that $f \circ \phi = p$. Define $\phi : (\tilde{Q}, \tilde{q}) \to (\hat{Q}, \hat{q})$ by $\phi(i, (a, g)) = \hat{q}_i^g$ with $f(\hat{q}_i) = q_i$, so that $f(\hat{q}_i^g) = q_i^g = a$. Thus $f \circ \phi = p$ on each connected component, so that this is true in general. It suffices now to show that ϕ is equivariant under $\mathrm{Adj}(Q)^\circ$ in order to prove that it is a quandle homomorphism because

$$\begin{aligned} \phi((i, (a, g)) \lhd (j, (b, h))) &= \phi((i, (a \lhd b, g e_a^{-1} e_b))) \\ &= \phi((i, (a, g))^{e_a^{-1} e_b}) \end{aligned}$$

and

$$\phi((i, (a, g)) \lhd \phi(j, (b, h)) = \phi((i, (a, g)))^{e_a^{-1} e_b}.$$

But indeed, if $h \in \mathrm{Adj}(Q)^\circ = \ker(\epsilon)$, then

$$\begin{aligned} \phi((i, (a, g))^h) &= \phi((i, (a^h, e_q^{-\epsilon(h)} g h)) \\ &= \phi((i, (a^h, g h))) \\ &= \hat{q}_i^{gh} \\ &= \phi((i, (a, g)))^h. \end{aligned}$$

$\square$

Before proving the main result of this article, we shall need one technical lemma.

Lemma 4. *Let Q be a quandle with $a = b^g$ for some $a,\ b \in Q$ and $g \in \mathrm{Adj}(Q)$. Then*

$$e_a = g^{-1} e_b g.$$

In particular, $g e_a^\gamma = e_b^\gamma g$ with $\gamma \in \mathbb{Z}$.

Proof. Since $g \in \mathrm{Adj}(Q)$,

$$g = e_{a_1}^{\alpha_1} e_{a_2}^{\alpha_2} ... e_{a_n}^{\alpha_n}$$

for some $a_i \in Q$ and $\alpha_i \in \{-1, 1\}$. So

$$\begin{aligned} b^g &= b^{e_{a_1}^{\alpha_1} e_{a_2}^{\alpha_2} ... e_{a_n}^{\alpha_n}} \\ &= b \lhd^{\alpha_1} a_1 \lhd^{\alpha_2} a_2 ... \lhd^{\alpha_n} a_n. \end{aligned}$$

And by using the identity $e_{a \lhd b} = e_b^{-1} e_a e_b$, one finds that

$$\begin{aligned} e_a &= e_{b \lhd^{\alpha_1} a_1 \lhd^{\alpha_2} a_2 ... \lhd^{\alpha_n} a_n} \\ &= e_{a_n}^{-\alpha_n} (e_{b \lhd^{\alpha_1} a_1 \lhd^{\alpha_2} a_2 ... \lhd^{\alpha_{n-1}} a_{n-1}}) e_{a_n}^{\alpha_n} \\ &= ... \\ &= e_{a_n}^{-\alpha_n} ... e_{a_2}^{-\alpha_2} e_{a_1}^{-\alpha_1} e_b e_{a_1}^{\alpha_1} e_{a_2}^{\alpha_2} ... e_{a_n}^{\alpha_n} \\ &= g^{-1} e_b g. \end{aligned}$$

$\square$

This lemma allows one to redefine the quandle operations on $\tilde{Q}$ this way:

$$(i,(a,g)) \lhd (j,(b,h)) = (i,(a \lhd b, e_{q_i}^{-1} g e_b))$$
$$(i,(a,g)) \lhd^{-1} (j,(b,h)) = (i,(a \lhd^{-1} b, e_{q_i} g e_b^{-1})).$$

Theorem 2. *$f : X \to Q$ is an E-covering if and only it is a covering.*

Proof. By Corollary 2 one only needs to show that any E-covering is a covering. More precisely we are going to show that any E-covering is split by the universal covering $p : (\tilde{Q}, \tilde{q}) \to (Q, q)$ constructed in Theorem 4. Let us then consider the pullback

$$\begin{tikzcd}
\tilde{Q} \times_Q X \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
\tilde{Q} \arrow[r, "p"'] & Q
\end{tikzcd}$$

and check that the surjective homomorphism p_1 is a trivial covering. For this, suppose that

$$p_1((i,(a,g)), y) = p_1((j,(b,h)), z)$$

and

$$[((i,(a,g)), y)] = [((j,(b,h)), z)]\,,$$

and we have to prove that $((i,(a,g)), y) = ((j,(b,h)), z)$ (by Proposition 3).

The first equality already gives $(i,(a,g)) = (j,(b,h))$.

The second one guarantees the existence of $((i_k,(a_{i_k}, g_{i_k})), y_{i_k}) \in \tilde{Q} \times_Q X$, with $1 \le k \le n$ such that

$$\begin{aligned}((i,(a,g)), y) \lhd^{\alpha_1} ((i_1,(a_{i_1}, g_{i_1})), y_{i_1}) ... \lhd^{\alpha_n} ((i_n,(a_{i_n}, g_{i_n}), y_{i_n}) &= ((j,(b,h)), z) \\ &= ((i,(a,g)), z)\end{aligned}$$

with $\lhd^{\alpha_k} \in \{\lhd, \lhd^{-1}\}$. This implies that

$$((i,(a,g)) \lhd^{\alpha_1} (i_1,(a_{i_1}, g_{i_1})) ... \lhd^{\alpha_n} (i_n,(a_{i_n}, g_{i_n})), y \lhd^{\alpha_1} y_1 ... \lhd^{\alpha_n} y_n) = ((i,(a,g)), z),$$

and one then gets the following equality by using the alternative definitions of the quandle operations mentioned after Lemma 4:

$$((i,(a \lhd^{\alpha_1} a_{i_1} ... \lhd^{\alpha_n} a_{i_n}, e_{q_i}^{-\alpha_n} ... e_{q_i}^{-\alpha_2} e_{q_i}^{-\alpha_1} g e_{a_{i_1}}^{\alpha_1} ... e_{a_{i_n}}^{\alpha_n})), y \lhd^{\alpha_1} y_{i_1} ... \lhd^{\alpha_n} y_{i_n}) = ((i,(a,g)), z)$$

so if we write $\alpha := \sum_{k=1}^{n} \alpha_k$, then

$$((i,(a \lhd^{\alpha_1} a_{i_1} ... \lhd^{\alpha_n} a_{i_n}, e_{q_i}^{-\alpha} g e_{a_{i_1}}^{\alpha_1} ... e_{a_{i_n}}^{\alpha_n})), y \lhd^{\alpha_1} y_{i_1} ... \lhd^{\alpha_n} y_{i_n}) = ((i,(a,g)), z).$$

From this and Lemma 4, one deduces that

$$(i,(a \lhd^{\alpha_1} a_{i_1} ... \lhd^{\alpha_n} a_{i_n}, g e_a^{-\alpha} e_{a_{i_1}}^{\alpha_1} ... e_{a_{i_n}}^{\alpha_n})) = (i,(a,g))$$

and

$$y \lhd^{\alpha_1} y_{i_1} ... \lhd^{\alpha_n} y_{i_n} = z.$$

Accordingly:

$$\begin{aligned}(a,g) &= (a \lhd^{\alpha_1} a_{i_1} ... \lhd^{\alpha_n} a_{i_n}, g e_a^{-\alpha} e_{a_{i_1}}^{\alpha_1} ... e_{a_{i_n}}^{\alpha_n}) \\ &= (a \lhd^{-\alpha} a \lhd^{\alpha_1} a_{i_1} ... \lhd^{\alpha_n} a_{i_n}, g e_a^{-\alpha} e_{a_{i_1}}^{\alpha_1} ... e_{a_{i_n}}^{\alpha_n}) \\ &= (a,g)^{e_a^{-\alpha} e_{a_{i_1}}^{\alpha_1} ... e_{a_{i_n}}^{\alpha_n}}\end{aligned}$$

But because $e_a^{-\alpha} e_{a_{i_1}}^{\alpha_1} ... e_{a_{i_n}}^{\alpha_n} \in \ker(\epsilon) = \mathrm{Adj}(Q)^\circ$ acts freely on $\tilde{Q}_i$, one has that

$$e_a^{-\alpha} e_{a_{i_1}}^{\alpha_1} ... e_{a_{i_n}}^{\alpha_n} = 1,$$

or

$$e_{a_{i_1}}^{\alpha_1} ... e_{a_{i_n}}^{\alpha_n} = e_a^{\alpha}.$$

Since f is an E-covering, there is an action of $\mathrm{Adj}(Q)$ on X, which gives the following:

$$\begin{aligned} z &= y \lhd^{\alpha_1} y_{i_1} ... \lhd^{\alpha_n} y_{i_n} \\ &= y^{e_{a_{i_1}}^{\alpha_1} ... e_{a_{i_n}}^{\alpha_n}} \\ &= y^{e_a^{\alpha}} \\ &= y \lhd^{\alpha} y = y. \end{aligned}$$

□

Remark 2.3. *Note that the arguments used to prove the last Theorem do not apply, at least as such, if we replace the category of quandles by the category of racks. Indeed, we make explicit use of the first axiom quandle.*

To conclude the article we now give an example of a covering that is not a trivial covering.

Example 2.4. *Consider the arrow $f\colon A \to B$ that sends the involutive ($\lhd = \lhd^{-1}$) quandle A having four elements*

$\lhd$	x	y	z	w
x	$x \lhd x = x$	$x \lhd y = x$	$x \lhd z = y$	$x \lhd w = y$
y	$y \lhd x = y$	$y \lhd y = y$	$y \lhd z = x$	$y \lhd w = x$
z	$z \lhd x = z$	$z \lhd y = z$	$z \lhd z = z$	$z \lhd w = z$
w	$w \lhd x = w$	$w \lhd y = w$	$w \lhd z = w$	$w \lhd w = w$

onto the trivial quandle B having three elements a, b and c with $f(x) = a = f(y)$, $f(z) = b$ and $f(w) = c$.

This application is clearly surjective, and also satisfies the algebraic property of being a covering (that can be expressed by the equality of the two first columns in the base quandle). However, f is not a trivial covering; we have indeed $f(x) = f(y)$ and $x \lhd z = y$ (i.e. $[x] = [y]$), but $x \neq y$.

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