

Adaptive estimation procedure with applications

Olga Reznikova

Institut de Statistique
Université Catholique de Louvain

21 September 2007

Outline

- 1 Introduction
- 2 Theory
 - Modeling of marginal distributions
 - Local Change Point procedure
 - Parametric time variation in the conditional copula
- 3 Simulations and application
 - Simulations
 - Application
- 4 Conclusion

Problems and Solutions

Problems

- Modeling joint dependence is critical for financial time series
- Model volatility of price change of financial assets
- Different approaches are used to model correlations, i.e. MV-GARCH:
 - VEC model (Bollerslev, Engle and Wooldridge, 1988)
 - BEKK model (Baba, Engle, Kraft and Kroner, 1991)
 - DCC model (Engle, 2002) and etc.

but

- “Stocks tend to crash together, but not boom together”
- Need to model asymmetric behavior of the assets
- ⇒ use Copula

What are Copulas?

Copulas allow us to model the dependence relationships among r.v. independently of their marginal distributions ¹

Definition:

Function $C : [0, 1]^2 \rightarrow [0, 1]$ such that

$$F(x, y) = C\{F_1(x), F_2(y)\}$$

is a copula

Example: Clayton copula

$$C(u, v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}$$

where $\theta \in (0, \infty)$ - dependence parameter

¹For more details see Embrechts, Lindskog and McNeil (2001)

Modeling of marginal distributions

Copula models for marginal distributions

Lets define log-differences of 2 different assets as x_t and y_t
 Assume that x_t and y_t follow typical univariate
GARCH(1, 1)

$$\begin{aligned} x_t | \mathcal{F}_t &\sim N(0, h_t) \\ h_t &= \omega_x + \alpha_x x_{t-1}^2 + \beta_x h_{t-1} \\ y_t | \mathcal{F}_t &\sim N(0, v_t) \\ v_t &= \omega_y + \alpha_y y_{t-1}^2 + \beta_y v_{t-1} \end{aligned}$$

After deGARCHing

$$\epsilon_t = \frac{x_t}{\sqrt{h_t}}, \eta_t = \frac{y_t}{\sqrt{v_t}}$$

Then ϵ_t and η_t are cleaned from GARCH effect \implies use
 Copula

$$F_{t-1}(\epsilon_t, \eta_t) = C_\theta (F_x(\epsilon_t), F_y(\eta_t))$$

Local Change Point procedure

- **Approach proposed by** D.Mercurio and V.Spokoiny
“Estimation of time dependent volatility via LCP analysis with applications to VaR” (2004)
- **Idea:** application to copula: \exists intervals of time homogeneity where dependence parameter of a copula is constant

Notations

- I - interval candidate and $\mathcal{I}_I \in I$

$$H_0 : \forall \tau \in \mathcal{I}_I, \theta_t = \theta$$

$$H_1 : \exists \tau \in \mathcal{I}_I : \begin{array}{ll} \theta_t = \theta_1 & \text{for } t \in J \\ \theta_t = \theta_2 & \text{for } t \in J^c \end{array}$$

Estimation

Canonical Maximum Likelihood

Canonical Maximum Likelihood (CLM)

maximize pseudo *log-likelihood* function

$$l(\theta) = \sum_{t=1}^T \log c \left\{ \hat{F}_{X_1}(x_{1,t}), \dots, \hat{F}_{X_d}(x_{d,t}); \theta \right\}$$

with *empirical* marginal distributions

$$\hat{F}_{X_j}(x) = \frac{1}{T+1} \sum_{t=1}^T \mathbb{I}\{X_{j,t} \leq x\}$$

$$\hat{\vartheta}_{CML} = \arg \max_{\theta} l(\theta)$$

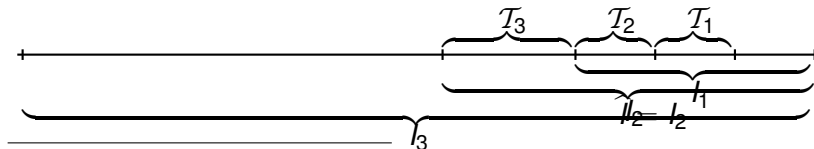
Local Change Point (LCP) procedure ²

Adaptive choice of the *intervals of homogeneity*

Start with $k = 1$

- 1 test $H_{0,k}$ within I_k on \mathcal{I}_k
- 2 if $H_{0,k}$ is not rejected, take I_{k+1} and repeat until homogeneity is rejected
- 3 if $H_{0,k}$ is rejected within I_k , then $\hat{I} = I_{k-1}$
- 4 if the largest possible interval is reached, then $\hat{I} = [0, t_0]$

θ is estimated from observations in \hat{I} , i.e. $\hat{\theta}_t = \tilde{\theta}_t$



²For more details see Giacomini, Härdle, Ignatieva and Spokoiny (2006)

Parametric time variation in the conditional copula

- **Approach proposed by** Andrew J. Patton
“Modeling asymmetric exchange rate dependence” (2006)
- **Idea:** copula dependence parameter is time-varying and follows a parametric model

$$C(u, v) = C_{\theta_t}(u_t, v_t)$$
$$\theta_t \sim \text{ARMA}(p, q) - \text{type}$$

Parametric time variation in the conditional copula

Choice of a copula

- *the Joe-Clayton copula*

$$C(u, v | \tau^U, \tau^L) = 1 - \left(1 - \{ [1 - (1 - u)^\kappa]^{-\gamma} + [1 - (1 - v)^\kappa]^{-\gamma} - 1 \}^{-1/\gamma} \right)$$

$$\kappa = \frac{1}{\log_2(2 - \tau^U)}, \gamma = -\frac{1}{\log_2(\tau^L)}$$

$$\tau_t^i = \Lambda \left(\omega_i + \beta_i \tau_{t-1}^i + \alpha_i \frac{1}{10} \sum_{j=1}^{10} |u_{t-j} - v_{t-j}| \right), i = U, L$$

where $\Lambda(x) \equiv (1 + e^{-x})^{-1}$ logistic transformation to keep τ^U and τ^L in $(0, 1)$ at all times

Parametric modeling of dependence parameter

Choice of a copula

- **Here:** *Clayton copula*

$$C(u, v|\theta) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$$

$$\theta_t = \bar{\Lambda} \left(\omega_\theta + \beta_\theta \theta_{t-1} + \alpha_\theta \frac{1}{10} \sum_{j=1}^{10} |u_{t-j} - v_{t-j}| \right) \quad (1)$$

where $\bar{\Lambda}(x) \equiv |x|$

The log-likelihood is given by

$$l(\theta) = \log(1 + \theta) - (\theta + 1) \log(uv) - \left(\frac{1}{\theta} + 2\right) \log(u^{-\theta} + v^{-\theta} - 1)$$

then

$$\hat{\theta} = \arg \max_{\theta} l(\theta)$$

given that (1) holds

Local Change Point (LCP) procedure

Simulation set

Conducted simulations

Test sensitivity of LCP to changes in

- angle of growth and reduction of θ_t
- height of increase of θ_t
- dimension of data $d = 2, 6$ and 10

$$\theta_t = \begin{cases} 0.1 & \text{if } t \in [1, 100] \\ 0.1 + \frac{1}{\Delta t_{up}} \Delta(t - 100) & \text{if } t \in [101, 100 + \Delta t_{up}] \\ \vartheta & \text{if } t \in [101 + \Delta t_{up}, 200 + \Delta t_{up}] \\ \vartheta - \frac{1}{\Delta t_{dn}} \Delta(t - \Delta t_{up} - 200) & \text{if } t \in [201 + \Delta t_{up}, 200 + \Delta t_{up} + \Delta t_{dn}] \\ 0.1 & \text{if } t \in [201 + \Delta t_{up} + \Delta t_{dn}, 300 + \Delta t_{up} + \Delta t_{dn}] \end{cases}$$

Local Change Point (LCP) procedure

Simulation set: example

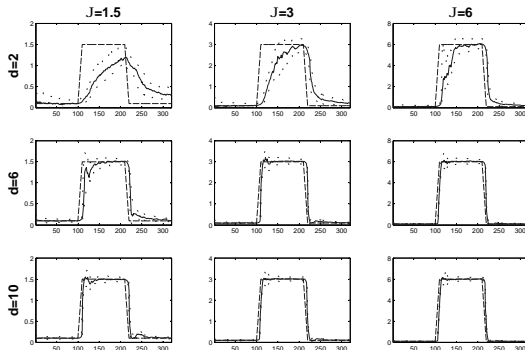


Figure: For $\Delta t_{up} = \Delta t_{dn} = 10$. From top down: 2-, 6- and 10-dim. From left to right: jump in θ from 0.1 to 1.5, 3 and 6

Parametric time variation approach

Simulations

Simulations in progress

Test parametric approach to changes in

- structural changes (sudden jumps in θ)
- smooth functional behavior of θ , i.e. $\theta_t = \sin(t)$
- transformation function $\Lambda(x)$
- number of lags p, q in $\theta \sim ARMA(p, q)$

Parametric time variation approach

Simulation example: Jump from 2 to 4, Clayton copula, 100 paths

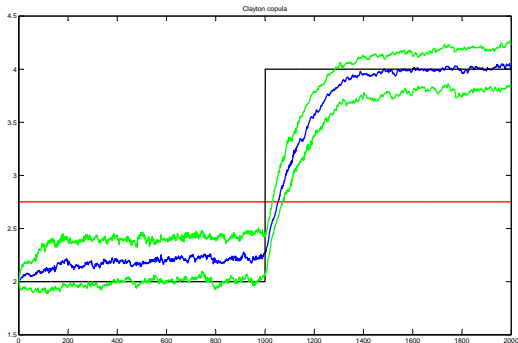


Figure: True θ_t (black), time-invariant θ (red), median of time-varying θ_t (blue), 25% and 75% quantiles.

Parametric time variation approach

Simulation example: $\theta_t = \sin(t)$, Clayton copula, 100 paths

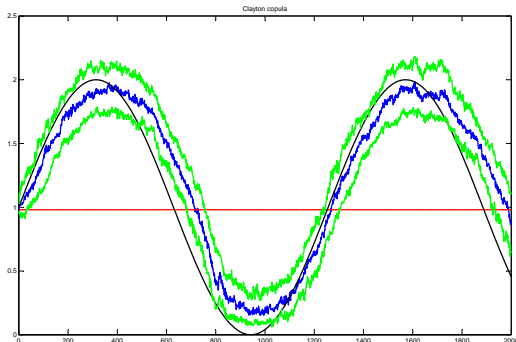


Figure: True θ_t (black), time-invariant θ (red), median of time-varying θ_t (blue), 25% and 75% quantiles.

Data set

Dow Jones Industrial Average and NASDAQ composite

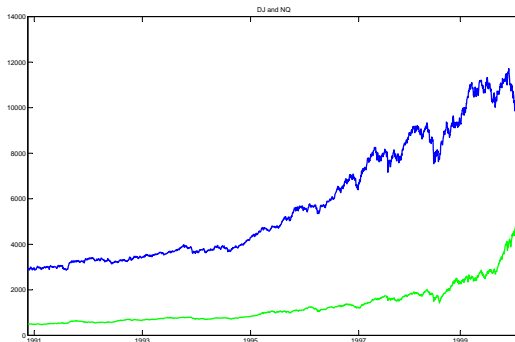


Figure: Dow Jones (blue line) and NQ (green line), daily data: April 1991 - March 2000

Data set

Log returns and estimate of $h_t^{1/2}$ from GARCH(1,1)

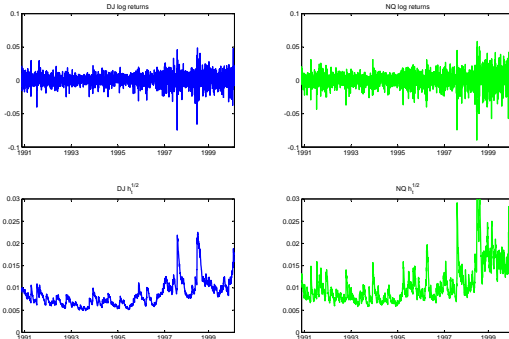


Figure: Dow Jones (blue line) and NQ (green line), daily data: April 1991 - March 2000

Local Change Point procedure

Estimated θ parameter and intervals of time homogeneity via V. Spokoiny

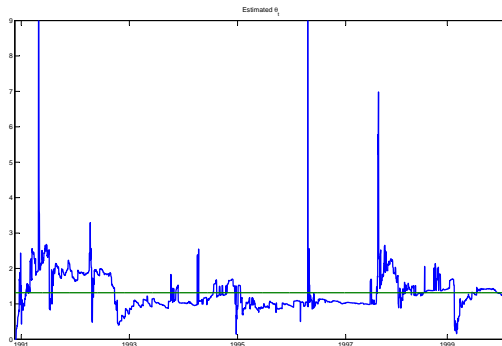


Figure: Estimated θ_t (upper panel) and intervals of time homogeneity (lower panel):
April 1991 - March 2000

Parametric time variation approach

Estimated θ parameter via method of A. Patton: $\theta_t \sim ARMA(1, 10)$

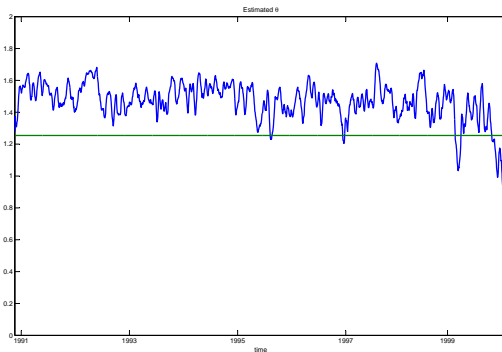


Figure: Time-varying θ_t (blue line) and time-invariant θ (red line): April 1991 - March 2000

Conclusions

- **LCP procedure** does not require any assumptions on Marginal distributions and is more flexible
- **LCP procedure** is computationally more laborious - hard to apply to long time-series.
- **Parametric approach** by Patton is clear and easy to implement
- Need to fine-tune the proper functional form for θ_t and transformation function $\Lambda(x)$
- **Further research:**
 - Run simulations to compare results of LCP procedure vs. Parametric approach
 - Compare both models to one of frequently used correlation models, i.e. DCC, CCC, BEKK and etc.

For Further Reading I



Andrew J.Patton

Modeling asymmetric exchange rate dependence

INTERNATIONAL ECONOMIC REVIEW, Vol. 47, No. 2, May
2006



D.Mercurio and V.Spokoiny

*Estimation of time dependent volatility via LCP analysis with
applications to VaR*

Annals of Statistics, 32:558-602,2004



E.Giacomini, W.Härdle, E.Ignatieva, and V.Spokoiny.

*Inhomogeneous Dependency Modelling with Time Varying
Copulae.*

SFB 649 Discussion Paper 2006-075.