Stochastic Frontier Analysis of Efficiency of Moroccan Municipalities

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Efficiency analysis

- Basic idea
  - Comparison between the Decision Making Units DMU (firms, for example) in order to know how the inputs are used to produce outputs.
    - Nonparametric Data Envelopment Analysis (DEA) or the Free Disposal Hull (FDH),
    - Stochastic Parametric Frontier Analysis (FSA)

    …

Stochastic Frontier Analysis (SFA)

- Error term is divided in two **independent** terms,

$$\varepsilon_i = v_i - u_i$$

i : Decision Making unit number i (DMUi)

- v_i reflects the pure randomness or the usual statistic noise, $v_i \sim N(0, \sigma_v^2)$
- u_i reflects the technical inefficiency,

  and u is a non-negative error term $(u_i \geq 0)$

  $u_i = 0$ for a technically efficient decision unit

Stochastic Frontier Analysis (SFA)

- Some possible distribution of u_i
  - Half-Normal

    $u_i \sim iidN^{+}\left(0, \sigma_u^2\right)$
  - Trancated Normal

    $u_i \sim iidN\left(\mu, \sigma_u^2\right)$ truncated at zero
  - Exponentiel

    $u_i \sim iid\mathrm{E}xp\left(\frac{1}{\sigma_u}\right)$
  - Gamma …

Stochastic Frontier Analysis (SFA)

- Stochastic model (Production Function) with cross-sectional data

$$y_i = f(x_i, \beta) + v_i - u_i$$

Where

$f(x_i, \beta)$:The production technology, it is assumed either a Cobb-Douglas or a translog function (less restrictive).

y_i : The observed output for observation i (one logged output);

x_i : A vector of inputs for observation i (logged);

β : A vector of parameters to be estimated;

Estimation of the stochastic model

- Estimation can be made using
  - Maximum Likelihood (ML) method
  - Corrected Ordinary Least Squares (COLS) method. Greene (1980) proposes to correct the bias by shifting β_0 ,

$$\hat{\beta}_0^* = \hat{\beta}_0 + \varepsilon^*$$

where $$\varepsilon^* = \max(\varepsilon)$$

Estimation of the stochastic model

- How to separate the error term into two components v and u
  - ε_i can be estimated as $\hat{\varepsilon}_i = y_i - f\left(x_i, \hat{\beta}\right)$
  - Jondrow, Knox Lovell, Materov and Schmidt (1981) have proposed a decomposition by considering the expected value of u , conditional on $\varepsilon = v - u$

$$E\left(u \big/ \varepsilon\right)$$

Estimation of the stochastic model

- The half-normal distribution (For example)

$$h(u)=\frac{2}{\sqrt{2\pi}\sigma_u}\exp\left\{-\frac{1}{2\sigma_u^2}u^2\right\}, \qquad u \geq 0$$

$$g(v,u)= f(v).h(u)=\frac{1}{\pi\sigma_v\sigma_u}\left[\exp\left(-\frac{1}{2\sigma_v^2}v^2-\frac{1}{2\sigma_u^2}u^2\right)\right]$$

Replace $v=u+\varepsilon$ to obtain $g(u,\varepsilon)$

$$g(\varepsilon)=\int_0^{+\infty}g(u,\varepsilon)du=\frac{2}{\sqrt{2\pi}\sigma}(1-\Phi)\left[\exp\left(-\frac{1}{2\sigma^2}\varepsilon^2\right)\right]$$

$$g\left(u/\varepsilon\right)=\frac{g(u,\varepsilon)}{g(\varepsilon)} \qquad \Rightarrow \qquad E\left(u/\varepsilon\right)=\mu_*+\sigma_*\frac{\phi\left(-\frac{\mu_*}{\sigma_*}\right)}{1-\Phi\left(-\frac{\mu_*}{\sigma_*}\right)}$$

Estimation of the stochastic model

- The half-normal distribution (cont.)
  - $\hat{u} = E\left(u / \hat{\varepsilon}\right)$

avec $\mu_* = -\dfrac{\sigma_u^2 \varepsilon}{\sigma^2}$, $\sigma_*^2 = \dfrac{\sigma_v^2 \sigma_u^2}{\sigma^2}$, $\sigma^2 = \sigma_v^2 + \sigma_u^2$

and $\lambda = \dfrac{\sigma_u}{\sigma_v} \geq 0$ is a measure of asymmetry

(Skewness) of the disturbance term ε .

Estimation of the stochastic model

- The maximum likelihood estimator (MLE) of $\varphi = (\sigma, \lambda, \beta)$ is

$$\hat{\varphi}_{ML} = \arg\max_{\varphi} \; l(\varphi)$$

and $\widehat{TE} = exp(-\hat{u})$

Estimation of the stochastic model

- Estimation methods
  - Analytical estimation (Not usually possible)

    Alternative:
  - Numerical estimation;
  - Monte Carlo Simulations.

Application

- DMUs : 91 (1298) municipalities (DMUs)
- One input : Recipe of functioning (Urban tax, tax on the collection of the waste, subsidies…)
- One output : Financial autonomy

Estimation of the stochastic model

- The half-normal distribution of u_i (a and b)
  $v_i \sim iidN(0, \sigma_v^2)$ and $u_i \sim iidN^+(0, \sigma_u^2)$

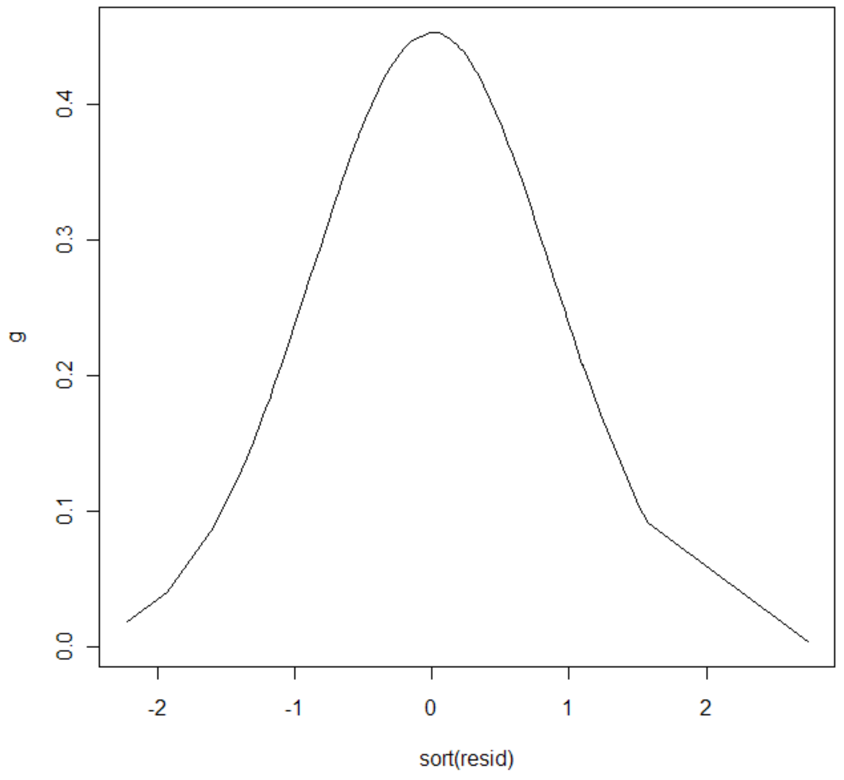
- The truncated-normal distribution of u_i (c and d)
  $v_i \sim iidN(0, \sigma_v^2)$ and $u_i \sim iidN(\mu, \sigma_u^2)$ truncated at zero

- $u_i \sim iidN(m_i, \sigma_u^2)$ truncated at zero where $m_i = z_i\delta$
  (e and f)

Results

- N=91 : does not provides avalid TE_i due to the positive Skewness of the distribution of ε

Fig.1: Density of mle residuals



Results

Table1: Efficiencies of the six models

Obs	Eff. Half-N. Cobb D. (a)	Eff. Half-N. Translog (b)	Eff. Trunc-N. Cobb D. (c)	Eff. Trunc-N. Translog (d)	Eff. $m_i = z_i\delta$ Cobb D. (e)	Eff. $m_i = z_i\delta$ Translog (f)
1	0,097	0,092	0,122	0,121	0,133	0,132
2	0,094	0,089	0,117	0,116	0,128	0,126
3	0,095	0,090	0,119	0,117	0,129	0,128
4	0,092	0,088	0,114	0,113	0,124	0,123
5	0,087	0,084	0,108	0,107	0,117	0,116
6	0,550	0,544	0,641	0,642	0,671	0,669
⋮	⋮	⋮	⋮	⋮	⋮	⋮
244	0.919	0.917	0.916	0.915	1,000	1,000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
1296	0,435	0,432	0,538	0,541	0,577	0,575
1297	0,691	0,693	0,743	0,747	0,761	0,761
1298	0,257	0,241	0,339	0,338	0,380	0,376
-logL	-1771.50	-1771.165	-1764.131	-1764.117	-1763.299	-1763.286
mean	0.478	0.475	0.558	0.560	0.587	0.586

Results

- Table1 and Fig.2 indicate
  - Efficiencies are different according to distribution of u_i;
  - Efficiencies are not more different according to Cobb-Douglas or translog functions;
  - The two models with the half-normal distribution (a and b) provide smaller technical efficiencies than the four others;
  - $\lambda = 3.537 \neq 0$, no problem of skewness.

Results

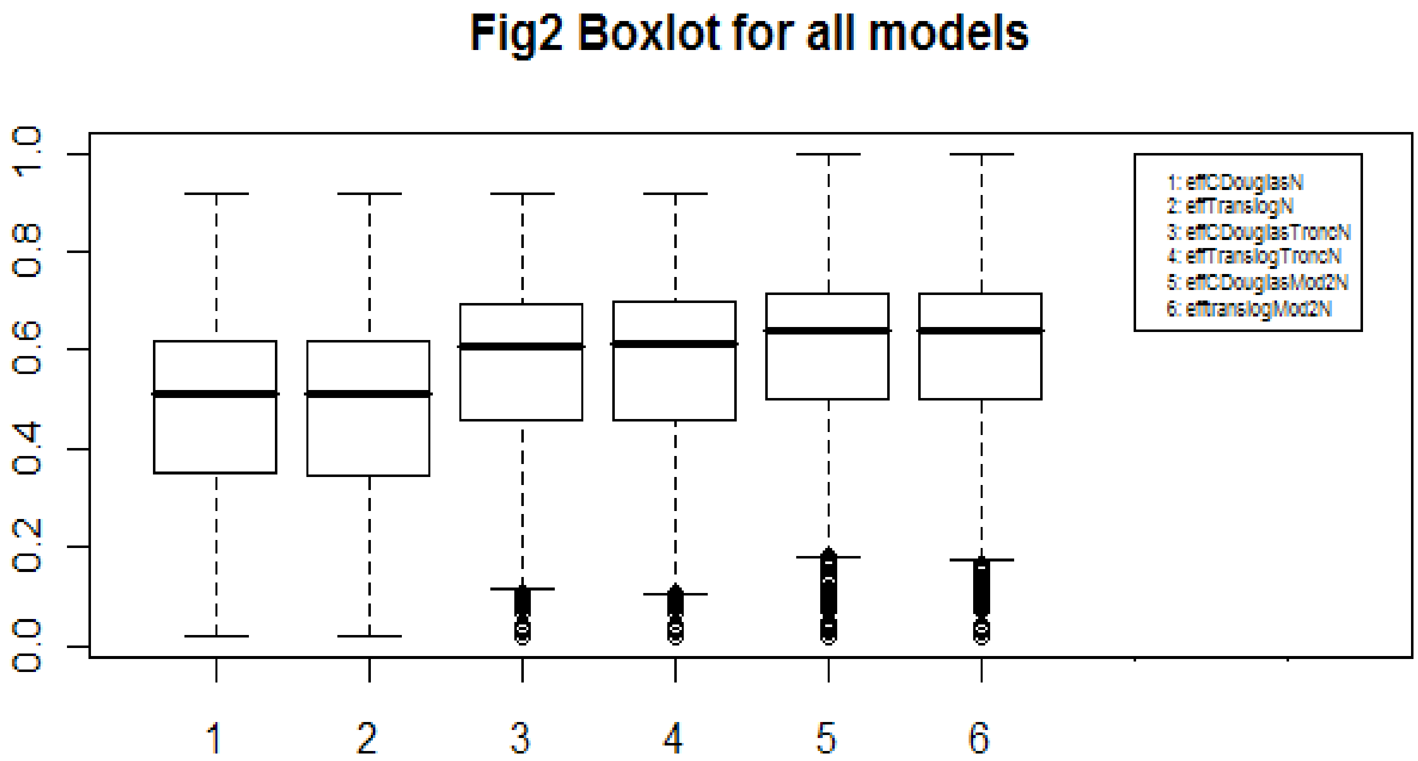


Fig2 Boxlot for all models
0.0
0.2
0.4
0.6
0.8
1.0
1
2
3
4
5
6
1: effCDouglasN
2: effTranslogN
3: effCDouglasTroncN
4: effTranslogTroncN
5: effCDouglasMod2N
6: efftranslogMod2N

Include the Instrumental Variable in the estimation

- When $\operatorname{cov}(x,\varepsilon)\neq 0$ and $E(v/Z)=0$

  Estimation in two Stages

  – $x = Z'\Pi + v$ is estimated by OLS

  – $y = \hat{x}'\beta + \varepsilon$ is estimated by OLS (ML)

$$\hat{\beta}_{IV} = \left(x' P_Z x\right)^{-1} x' P_Z y \quad \text{with} \quad P_Z = Z\left(Z'Z\right)^{-1} Z'$$

Include the Copula in the estimation

- If $U \not\perp V$ it is recommended to find the joint density $g(u,v)$ with copula

Then

$$G(u,v) = C_\theta\left(F_1(u), F_2(v)\right)$$

or
$$g(u,v) = f_1(u).f_2(v).c_\theta\left(F_1(u), F_2(v)\right)$$

Include the Copula in the estimation

- Farlie-Gumbel-Morganstern (FGM) copula

$$g_\theta(u,v) = f_1(u).f_2(v).c_\theta(F_1(u),F_2(v))$$

$$= f_1(u).f_2(v).[1+\theta-2\theta F_1(u)-2\theta F_2(v)+4\theta F_1(u)F_2(v)]$$

$$= \left(\frac{2}{\sigma_u}\phi\left(\frac{u}{\sigma_u}\right)\right).\phi\left(\frac{v}{\sigma_v}\right)$$

$$.\left[1+\theta-2\theta\left(\frac{2}{\sigma_u}\Phi\left(\frac{u}{\sigma_u}\right)\right)-2\theta\Phi\left(\frac{v}{\sigma_v}\right)+4\theta\left(\frac{2}{\sigma_u}\Phi\left(\frac{u}{\sigma_u}\right)\right)\Phi\left(\frac{v}{\sigma_v}\right)\right].$$