

Efficient estimation of a semiparametric dynamic copula model

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30 January 2009
Young Researchers Day

Outline

Introduction

Semi-parametric dynamic copula

- Motivation

- The Model

- Asymptotic theory

- Bandwidth selection

Simulations and applications

- Simulations

- Empirical example

Conclusions

Problems and Solutions

Problems

- ▶ Modeling dependence is critical for financial time series
- ▶ Model volatility of returns of financial assets
- ▶ Different approaches are used to model dynamic correlations
- ▶ **BUT:** return distributions often reveal skewness and lower-tail dependencies



Copula

- ▶ Allow to model nonlinear dependence
- ▶ Look beyond correlation (i.e., linear dependence), which is required for non-elliptical distributions

What are Copulas?

Copulas allow us to model the dependence relationships among r.v. independently of their marginal distributions ¹

Definition:

Function $C : [0, 1]^2 \rightarrow [0, 1]$ such that

$$F(x, y) = C\{F_1(x), F_2(y)\}$$

is a copula

Example: Clayton copula

$$C(u, v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}$$

where $\theta \in (0, \infty)$ - dependence parameter

¹For more details see Embrechts, Lindskog and McNeil (2001)

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Motivation

- ▶ Dependence may vary over time

For $C = \{C_\theta, \theta \in \Theta\}$ we allow θ to be time-varying

- ▶ **Patton (2006)**: dependence parameter is a parametric function of lagged u_t, v_t
- ▶ **Here**: we assume θ to be a smooth function of time

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Modeling of marginal distributions

- ▶ **Assume**

$\{X_t\}$ - a stochastic processes, e.g. $X_t = (X_{1t}, X_{2t})'$

$X_{i,t} | \mathcal{F}_{t-1} \sim N(\mu_{it}, \sigma_{it}^2), i = 1, 2$

- ▶ where

$\mathcal{F}_t = \sigma(X_t, X_{t-1}, \dots, X_0)$

μ_{it} e.g. ARMA

σ_{it}^2 e.g. GARCH

- ▶ **Estimate** the parameter vector

$$\phi = (\phi'_1, \phi'_2)'$$

- ▶ **Standardize**

$$z_{it} = \frac{X_{it} - \mu_{it}}{\sigma_{it}} \sim N(0, 1)$$

Estimating a copula model

- ▶ **Joint distribution of z_{1t}, z_{2t}**

$$F(z_{1t}, z_{2t}) = C(\Phi(z_{1t}; \phi_1), \Phi(z_{2t}; \phi_2); \theta)$$

- ▶ **The joint log-likelihood**

$$\begin{aligned}\mathbb{L}(\theta, \phi) &= \sum_{t=1}^T \ln c(\Phi(z_{1t}; \phi_1), \Phi(z_{2t}; \phi_2); \theta) \\ &\quad + \sum_{t=1}^T \ln \varphi(z_{1t}; \phi_1) + \sum_{t=1}^T \ln \varphi(z_{2t}; \phi_2) \\ &= \mathbb{L}_C(\theta, \phi) + \mathbb{L}_V(\phi)\end{aligned}$$

$(\phi, \theta) = [\phi'_1, \phi'_2, \theta']'$ is the parameter vector to be estimated

$$c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v}$$

Estimating a copula model

Two-step Maximum likelihood (ML)

First step

$$\tilde{\phi} = \arg \max_{\phi \in \Phi} \mathbb{L}_V(\phi)$$

Second step

$$\tilde{\theta} = \arg \max_{\theta \in \Theta} \mathbb{L}_C(\theta, \tilde{\phi})$$

$$\tilde{\theta} = \arg \max_{\theta \in \Theta} \mathbb{L}_C(\theta, \tilde{\phi})$$

$$\tilde{\theta}(\tau) = \arg \max_{\theta} \sum_{t=1}^T \ln c(\Phi_1(\cdot), \Phi_2(\cdot); \theta) \cdot K_h(t/T - \tau)$$

Efficient estimator:

$$\left(\frac{1}{T} \sum_{t=1}^T \text{diag}(\tilde{\phi}_1, \tilde{\phi}_2) \text{diag}(\tilde{\phi}_1, \tilde{\phi}_2) \right)^{-1} \frac{1}{T} \sum_{t=1}^T \text{diag}(\tilde{\phi}_1, \tilde{\phi}_2)$$

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Let

$$\ell_t(\theta) = \ln c(\Phi(z_{1t}; \phi_1), \Phi(z_{2t}; \phi_2); \theta)$$

Define

$$s(\tau) = E [(\ell'_t(\theta))^2 | t/T = \tau]$$

$$J(\tau) = E [\ell''_t(\theta) | t/T = \tau]$$

$$g(\tau) = E [\ell'''_t(\theta) | t/T = \tau]$$

Theorem 1: Under certain assumptions

$$\sqrt{Th} \left(\tilde{\theta}(\tau) - \theta(\tau) - h^2 b(\tau) \right) \rightarrow_d N(0, V_\theta(\tau)),$$

where

$$b(\tau) = \left\{ \frac{\theta''(\tau)}{2} + J(\tau)^{-1} \frac{g(\tau)}{2} \theta'(\tau)^2 \right\} \mu_2(K)$$

$$V_\theta(\tau) = J^{-2}(\tau) s(\tau) \|K\|^2$$

$K_h(\cdot)$ is a Kernel

h is a bandwidth

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Local likelihood estimation

Bandwidth selection

We need to choose bandwidth h to balance the bias and variance.

► **Estimator for MSE is**

$$\widehat{MSE}_p(\tau; h) = \widehat{B}_p^2(\tau; h) + \widehat{V}_p(\tau; h)$$

► **Bandwidth**

$$\widehat{h}_p = \operatorname{argmin}_h \left\{ \int \widehat{MSE}_p(x; h) w(x) dx \right\}$$

where $w(x)$ is a weight function

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Simulations design

Step 1: **Simulate** $\{r_{1,t}, r_{2,t}\}_{t=1}^T$ such that:

$$\begin{aligned}\epsilon_{i,t} &\sim GARCH(1, 1), i = 1, 2 \\ z_{i,t} &\sim N(0, 1) \\ F(z_{1,t}, z_{2,t}) &= C(\Phi(z_{1,t}), \Phi(z_{2,t}); \theta) \\ \theta &= \theta(t/T)\end{aligned}$$

- ▶ # replicas $K = 100$
- ▶ # observations $T = 100, 500$ and 1000

Step 2: **Estimate** $\tilde{\phi}$, $\hat{\theta}_t$ and $\hat{\phi}$

Step 3: Test the performance of the model: **MSE** and dynamic quantile (**DQ**) test of Engle and Manganelli (2004)

Simulations design

Functions for the dependence parameter:

1. **Constant:** $\theta(u) = 0, 1, 2$ and 3
2. **Step:** $\theta(u) = 2 + 1(u > 0.5)$
3. **Slow sine** $\theta(u) = 2 + \sin(50u/3)$
4. **Fast sine** $\theta(u) = 2 + \sin(50u)$
5. **Slow sine, big amplitude** (Slow BA)
 $\theta(u) = 2 + 1.9 \sin(50u/3)$
6. **Fast sine, big amplitude** (Fast BA)
 $\theta(u) = 2 + 1.9 \sin(50u)$

Simulations design

One replica example

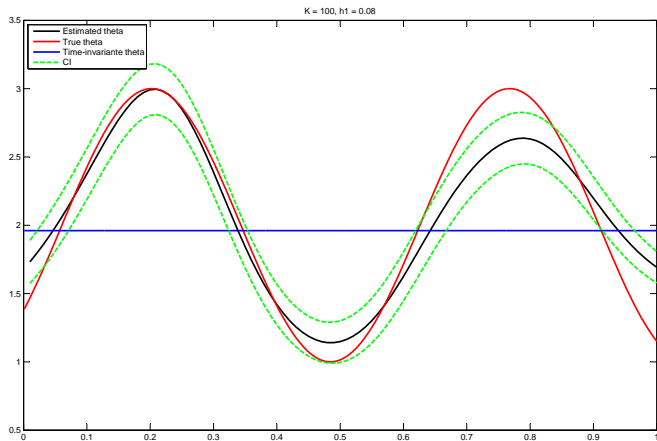


Figure: True θ (red line), time-varying $\hat{\theta}_t$ (black line), time-invariant $\hat{\theta}$ (blue line) and 95% confidence intervals

Simulations results

Constant models

Model	MSE*1000						MSE	
	$\tilde{\omega}$	$\tilde{\alpha}$	$\tilde{\beta}$	$\hat{\omega}$	$\hat{\alpha}$	$\hat{\beta}$	$\tilde{\theta}$	$\hat{\theta}$
$\theta = 0$	$5.1E-8$	0.62	5.46	$5.0E-8$	0.57	5.38	0.01	0.01
$\theta = 1$	$5.9E-8$	0.66	6.08	$4.8E-8$	0.56	5.39	0.05	0.04
$\theta = 2$	$5.0E-8$	0.50	5.09	$3.6E-8$	0.30	4.38	0.11	0.09
$\theta = 3$	$5.1E-8$	0.48	5.25	$2.9E-8$	0.26	3.81	0.21	0.16
$\theta = 2$ to 4	$5.6E-8$	0.71	5.78	$3.6E-8$	0.32	4.55	0.17	0.12

Table: Mean squared error MSE for copula dependency parameter $\hat{\theta}$ and for parameter vectors $\tilde{\phi}$ and $\hat{\phi}$.

Simulations results

Sine models

Model	MSE*1000						MSE	
	$\tilde{\omega}$	$\tilde{\alpha}$	$\tilde{\beta}$	$\hat{\omega}$	$\hat{\alpha}$	$\hat{\beta}$	$\tilde{\theta}$	$\hat{\theta}$
Slow sine	$6.3E-8$	0.66	6.53	$4.1E-8$	0.36	5.49	0.21	0.15
Slow BA sine	$5.2E-8$	0.46	5.23	$3.6E-8$	0.29	4.53	0.47	0.43
Fast sine	$5.6E-8$	0.67	6.35	$4.0E-8$	0.34	5.09	0.60	0.54
Fast BA sine	$5.6E-8$	0.52	5.68	$4.7E-8$	0.39	5.02	2.01	1.96

Table: Mean squared error MSE for copula dependency parameter $\hat{\theta}$ and for parameter vectors $\tilde{\phi}$ and $\hat{\phi}$.

Simulations results

Time increment models

Model	MSE*1000						MSE	
	$\tilde{\omega}$	$\tilde{\alpha}$	$\tilde{\beta}$	$\hat{\omega}$	$\hat{\alpha}$	$\hat{\beta}$	$\tilde{\theta}$	$\hat{\theta}$
T=100	$1.2E-7$	1.40	7.90	$9.0E-8$	0.97	6.93	0.68	0.62
T=500	$6.3E-8$	0.66	6.53	$4.1E-8$	0.36	5.49	0.21	0.15
T=1000	$3.7E-8$	0.43	4.66	$2.0E-8$	0.18	2.85	0.12	0.11

Table: Mean squared error MSE for copula dependency parameter $\hat{\theta}$ and for parameter vectors $\tilde{\phi}$ and $\hat{\phi}$.

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Data set: MSCI Germany and UK

► Data:

- Morgan Stanley Capital International (MSCI) index for Germany and UK
- weekly quotations, 11 October 1989 - 15 October 2008

► Model:

- r_G, r_{UK} - log-returns
- r_G and r_{UK} show presence of autocorrelation

$$r_{G,t} = \underset{(0.03)}{-0.08} r_{G,t-1} + \epsilon_{G,t}$$

$$r_{UK,t} = \underset{(0.03)}{-0.10} r_{UK,t-1} + \epsilon_{UK,t}$$

- for $\hat{\epsilon}_{G,t}$ and $\hat{\epsilon}_{UK,t}$ build GARCH(1,1) with Student errors

$$h_{G,t} = \underset{(0.08E-4)}{0.13E-4} + \underset{(0.03)}{0.10} \epsilon_{G,t-1}^2 + \underset{(0.03)}{0.89} h_{G,t-1},$$

$$\nu_G = \underset{(1.77)}{8.05}$$

$$h_{UK,t} = \underset{(0.24E-4)}{0.32E-4} + \underset{(0.05)}{0.13} \epsilon_{UK,t-1}^2 + \underset{(0.09)}{0.82} h_{UK,t-1},$$

$$\nu_{UK} = \underset{(3.30)}{11.16}$$

Data set: MSCI Germany and UK

Volatilities estimated from univariate GARCH models with Student errors.

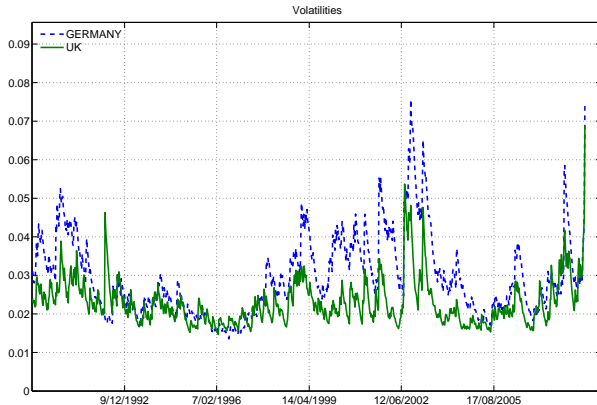
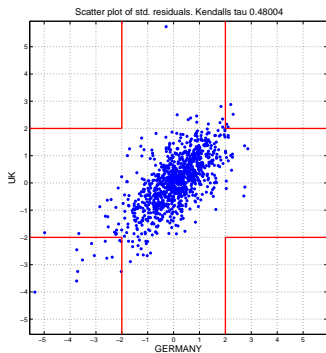


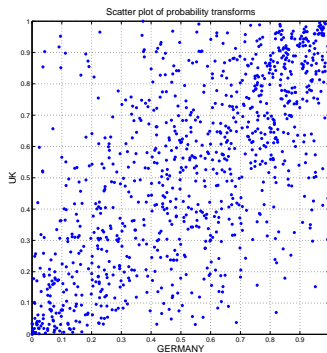
Figure: Germany (blue line) and UK (green line)

Data set: MSCI Germany and UK

Scatter plots



(a) z_G vs. z_{UK}



(b) $F(z_G)$ vs. $F(z_{UK})$

Figure: Scatter plots for the standardized returns r_G versus r_{UK} and for $F(r_G; \nu_G)$ versus $F(r_{UK}; \nu_{UK})$, where $F(\cdot; \nu)$ is a Student distribution with ν d.o.f.

Data set: MSCI Germany and UK

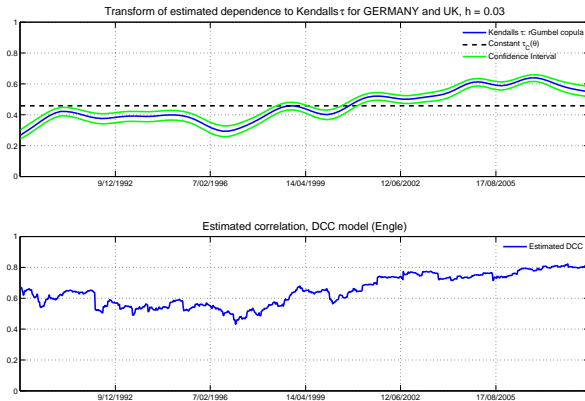


Figure: Estimated dependence $\hat{\theta}(t)$ and 95% confidence intervals (upper panel) transformed to Kendall's tau and estimated correlation via Dynamic Conditional Correlation (DCC) model (lower panel)

Data set: MSCI Germany and UK

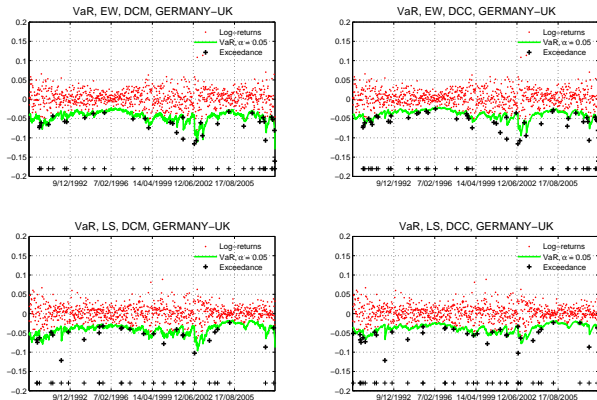


Figure: Value-at-Risk for Dynamic Copula model (left panels) and for Dynamic Conditional Correlation (DCC) model (right panels) for equal weighted (EW) (upper panels) and long-short (LS) (lower panels) portfolios

Finally

- ▶ **What has been done:**

 - Simulation results for different copulas

 - Theory for efficient estimator of ϕ

 - Asymptotic theory for the estimator of θ

- ▶ **In progress:**

 - Goodness-of-fit test for time-varying copulas

 - Extend to higher dimensions

 - Improved DCC model for higher dimensions



For Further Reading I



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