Heavy tailed functional linear processes

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Overview

- Introduction
- Linear processes in function spaces
- Extremal properties
- Conclusion

Introduction

- A measurable function $f : [a, \infty) \to \mathbb{R}_+$ is *regularly varying with index* $\alpha \in \mathbb{R}$ if

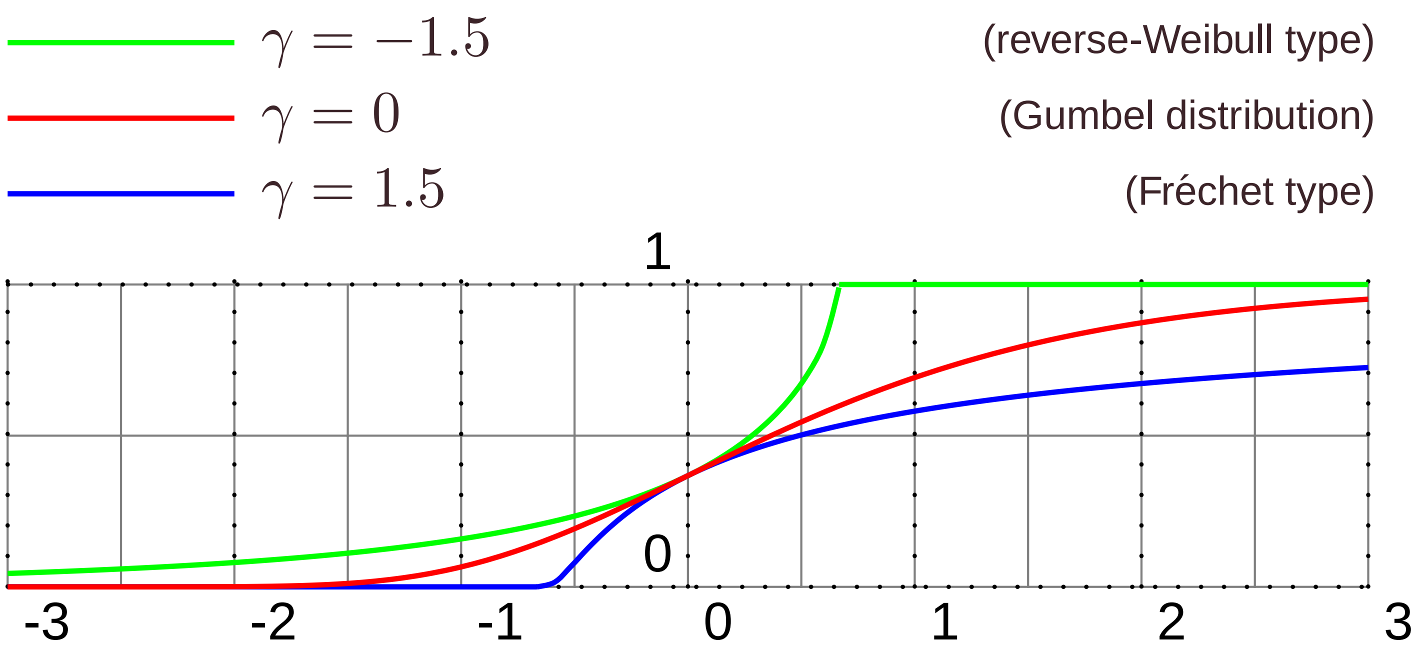
$$\lim_{t \to \infty} \frac{f(xt)}{f(t)} = x^\alpha.$$

This means that, at infinity, the behavior of f is like the one of x^α.

- Basic link with *extreme value theory*:

Given $\gamma > 0$, *a positive random variable* X *belongs to the domain of attraction of* $G_\gamma \Leftrightarrow 1 - F_X$ *is regularly varying with index* $-1/\gamma$.

- Graphics of G_γ for different values of $\gamma \in \mathbb{R}$:



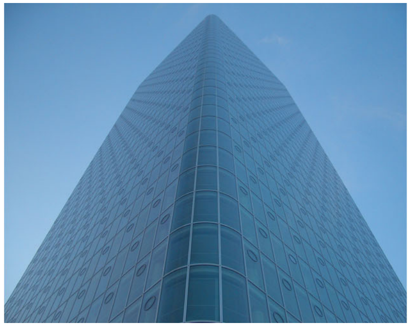
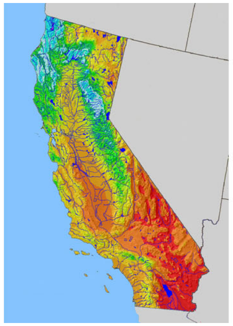
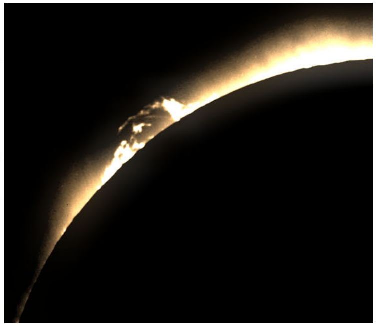
- Heavy tail = regularly varying behavior
- Two distinct cases:
  - *the observed process itself is heavy tailed*
    (finance, telecommunication, environment ...)
  - *a transformation of observed process is heavy tailed*
    (to have unit-Fréchet margins, to study the extremal properties)

- A random *variable* represents a single observation.
- A random *vector* in $\mathbb{R}^d$ represents d observations, for instance at d meteorological stations.
- Sometimes it may be better to consider the information *everywhere*.

For these reasons we are led to consider *random functions*.

Examples of random functions: *spatial phenomena*.

- sea level along a dike
- windspeed along the facade of a building
- precipitation in a geographical region
- protuberances on the surface of the sun

- Functions are *infinite dimensional objects*.

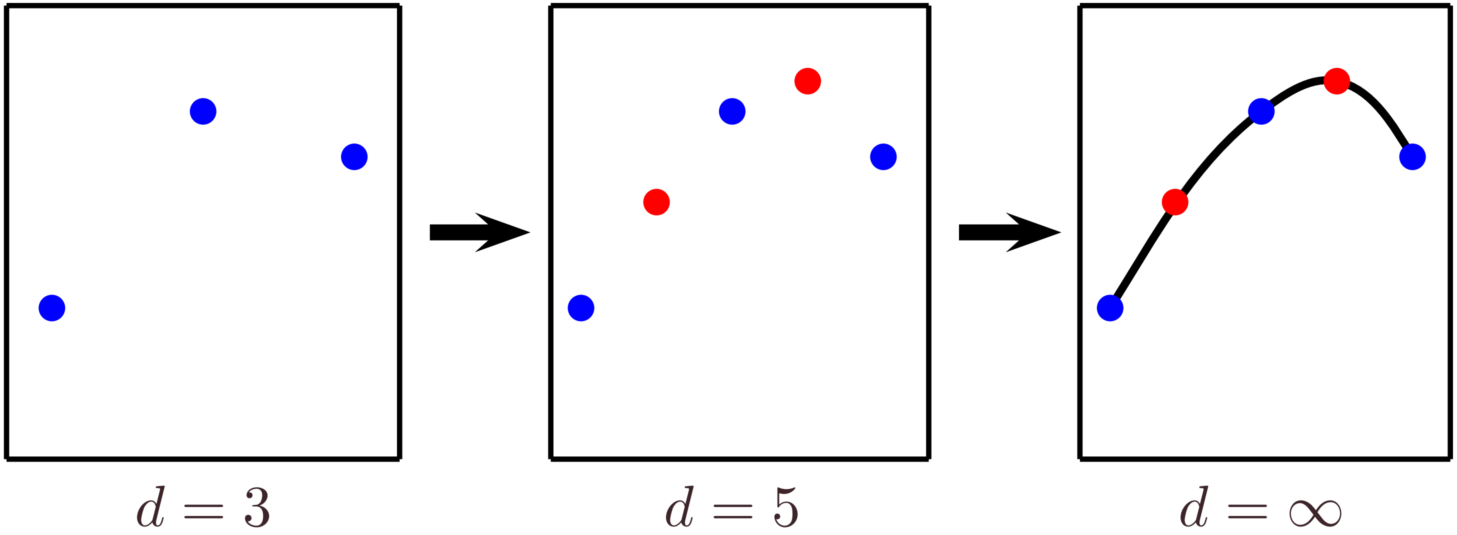


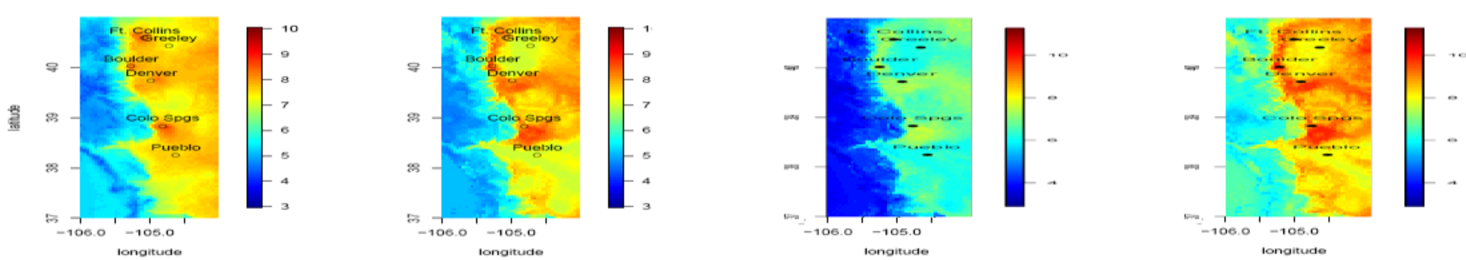
Figure 1: Functional observations $\Leftrightarrow$ "∞ measurement points".

- Sets of functions: $C([0,1]^d, \mathbb{R})$, $D([0,1]^d, \mathbb{R})$, $USC([0,1]^d)$ ...
- In this talk we focus on $C([0,1]^d, \mathbb{R})$.

- A sequence of observations of spatial phenomena is a *time series of function-valued data*.
- Goal: to deal with *space and time dependencies for extremes*.

Space Cross-sectional *tail dependence*

Time Temporal *clusters of extremes*



- *Functional linear processes*: definition and motivation
- For stationary processes in $C([0,1]^d, \mathbb{R})$:
  - *spectral measure*
  - *spectral process*
- Illustration of these concepts on linear processes

Linear processes

- Let C_1 and C_2 be two spaces of continuous functions.
- A *linear process* $(X_t)_{t\in\mathbb{Z}}$ is of the form

$$X_t = \sum_{i\in\mathbb{Z}} T_i(Z_{t-i}),\ t \in \mathbb{Z},$$

where

- Z_t are iid regularly varying *innovations in C_1*,
- $T_i : C_1 \to C_2$ are linear and continuous.

- If there exists $0 < p < \min(1, \alpha)$

$$\sum_{i \in \mathbb{Z}} \|T_i\|^p < \infty$$

then the sum

$$X_t = \sum_{i \in \mathbb{Z}} T_i(Z_{t-i})$$

is almost surely convergent and the process $(X_t)_{t \in \mathbb{Z}}$ is stationary.

- The expression

$$X_t = \sum_{i \in \mathbb{Z}} T_i(Z_{t-i}),\ t \in \mathbb{Z},$$

generalizes real-valued $MA(\infty)$ *processes*.

- If $C_1 = C_2$: *autoregressive processes*

$$X_t = T_1(X_{t-1}) + \ldots + T_n(X_{t-n}) + Z_t$$

are linear processes.

- Main novelty:

  Allow *integral operators* aimed at modelling spatio-temporal transformation, such as

$$T_i(f) = x \mapsto \int_{y \in [0,1]^d} K_i(x,y) f(y) dy.$$

Extremal properties

- Are linear processes *jointly* *regularly varying*?

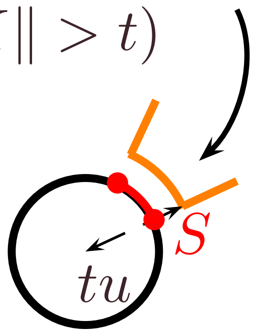
  $\rightarrow$ *Regular variation* in $C([0,1]^d, \mathbb{R})$?
  Answer: *spectral measure*.

  $\rightarrow$ *Joint regular variation* for the series ?
  Answer: *spectral process*.

Multivariate case

- Let X be a random vector in $\mathbb{R}^d$.
- Denote $\mathbb{S} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ the unit ball of $\mathbb{R}^d$.
- If, given $S \in \mathcal{B}_{\mathbb{S}}$ with $\lambda(\partial S) = 0$, as $t \to \infty$,

$$\frac{P\left(\|X\| > tu, X/\|X\| \in S\right)}{P(\|X\| > t)} \to \frac{1}{u^\alpha}\lambda(S)$$



with a probability measure λ, then λ is the *spectral measure* of X.

- This define the *regular variation* of X in $\mathbb{R}^d$:

Functional case

- The construction with spheres makes sense in $C([0,1]^d, \mathbb{R})$.
- Put $\mathbb{S} = \{x \in C([0,1]^d, \mathbb{R}) : \|x\| = 1\}$.
- **Definition**: A random continous function X is *regularly varying* if it possesses a *spectral measure* in the sense

$$\frac{P\left(\|X\| > tu, X/\|X\| \in \cdot\,\right)}{P(\|X\| > t)} \xrightarrow{w} \frac{1}{u^\alpha}\lambda(\,\cdot\,)$$

as $t \to \infty$, for a probability measure λ on $\mathbb{S}$.

- For *linear processes* $X_t = \sum_{i \in \mathbb{Z}} T_i(Z_{t-i})$ we have

$$\frac{P\left(\|X_0\| > tu, X_0/\|X_0\| \in \cdot\right)}{P(\|X_0\| > t)} \xrightarrow{w} \frac{1}{u^\alpha}\lambda(\,\cdot\,),$$

where

$$\lambda(f) = \int f d\lambda = \frac{\sum_{i \in \mathbb{Z}} E\left[f\left(\frac{T_i(\Theta^Z)}{\|T_i(\Theta^Z)\|}\right)\|T_i(\Theta^Z)\|^\alpha\right]}{\sum_{i \in \mathbb{Z}} E[\|T_i(\Theta^Z)\|^\alpha]},$$

with Θ^Z distributed according to the spectral measure of Z.

- After transformation, the spectral measure λ also can be given by the limit

$$\mathcal{L}\left(X_0/\|X_0\| \mid \|X_0\| \geqslant x \right) \xrightarrow{d} \lambda(\,\cdot\,).$$

- Can we replace X_0 by $(X_t)_{t\in\mathbb{Z}}$ to obtain a *limit process* $(\Theta_t)_{t\in\mathbb{Z}}$ such that

$$\mathcal{L}\left((X_t/\|X_0\|)_{t\in\mathbb{Z}} \,\Big|\, \|X_0\| \geqslant x \right) \xrightarrow{d} \mathcal{L}\left((\Theta_t)_{t\in\mathbb{Z}} \right) ?$$

- Then the limit distribution $(\Theta_t)_{t\in\mathbb{Z}}$ would include all features about the joint tail distribution of $(X_t)_{t\in\mathbb{Z}}$.

- With this definition $\mathcal{L}(\Theta_0)$ is still equal to λ.

- This construction makes sense for special cases.
  - *Independent time series* $(X_t)_{t\in\mathbb{Z}}$:

$$\mathcal{L}(\Theta_t) = 0 \qquad \text{if } t \neq 0.$$

  - *Fully dependent series* $(X_t)_{t\in\mathbb{Z}}$:

$$\mathcal{L}(\Theta_t) = \lambda \qquad \forall t \in \mathbb{Z}.$$

- For *linear processes* ,

$$\mathcal{L}\left(\left(X_t/\|X_0\|\right)_{t\in\mathbb{Z}} \;\middle|\; \|X_0\| \geqslant x\right) \xrightarrow{d} \mathcal{L}\left((\Theta_t)_{t\in\mathbb{Z}}\right)$$

where

$$E\left[f\left(\Theta_{-s},\dots,\Theta_t\right)\right] = \frac{\sum_{i\in\mathbb{Z}} E\left[f\left(\frac{T_{-s+i}(\Theta^Z)}{\|T_i(\Theta^Z)\|},\dots,\frac{T_{t+i}(\Theta^Z)}{\|T_i(\Theta^Z)\|}\right)\|T_i(\Theta^Z)\|^\alpha\right]}{\sum_{i\in\mathbb{Z}} E[\|T_i(\Theta^Z)\|^\alpha]},$$

with Θ^Z distributed according to the spectral measure of Z.

- Given a stationary process $(X_t)_{t\in\mathbb{Z}}$, if there exists $(\Theta_t)_{t\in\mathbb{Z}}$ such that

$$\mathcal{L}\left((X_t/\|X_0\|)_{t\in\mathbb{Z}} \;\middle|\; \|X_0\| \geqslant x\right) \xrightarrow{d} \mathcal{L}\left((\Theta_t)_{t\in\mathbb{Z}}\right),$$

  we call $(\Theta_t)_{t\in\mathbb{Z}}$ the *spectral process* of $(X_t)_{t\in\mathbb{Z}}$.

- **General theorem**: A stationary process $(X_t)_{t\in\mathbb{Z}}$ in $C([0,1]^d,\mathbb{R})$ admits a *spectral process* $(\Theta_t)_{t\in\mathbb{Z}}$ if and only if $(X_t)_{t\in\mathbb{Z}}$ is *jointly regularly varying* in the sense of, for every k,

$$(X_0,\ldots,X_k)$$

  is regularly varying with the same index.

- *Extremal index* θ of the series $(\|X_t\|)_{t\in\mathbb{Z}}$:

$$\theta = \lim_{t\to\infty} \lim_{x\to\infty} P(\max_{i=1,\ldots,t} \|X_i\| \leqslant x \mid \|X_0\| > x)$$

(this equality holds under additional conditions).

- Explicit expression thanks to the spectral process:

$$\theta = E[\sup_{i\geqslant 0} \|\Theta_i\|^\alpha - \sup_{i\geqslant 1} \|\Theta_i\|^\alpha].$$

- Causal linear process ($i \geqslant 0$):

$$\theta = \frac{E[\sup_{t\geqslant 0} \|T_t(\Theta^Z)\|^\alpha]}{\sum_{t\geqslant 0} E[\|T_t(\Theta^Z)\|^\alpha]}.$$

Conclusion

Conclusion

- Theory of joint regular variation for spatial time series
- Example: Functional linear processes

- Other implications for functional processes:
  - Point processes limit for spatial extremes
  - Tail dependance quantities / extremograms

- All is actually valid in separable Banach spaces
- Other ideas: semi-continuous functions...