Extreme-Value Copulas

Gordon Gudendorf and Johan Segers

Université Catholique de Louvain

February, 5 , 2010

- Let $\mathbf{X}_i = (X_{i1}, \ldots, X_{id})_{\{1,\ldots,n\}}$ i.i.d. random vectors with continuous distribution function F and marginal distribution functions $F_1, \ldots, F_d$ and copula C_F.

$$F(x_1, \ldots, x_d) = C_F(F_1(x_1), \ldots, F_d(x_d)) \tag{1}$$

- Vector of componentwise maxima:

$$\mathbf{M}_n = (M_{n,1}, \ldots, M_{n,d}), \qquad \text{where } M_{n,j} = \bigvee_{i=1}^{n} X_{ij}, \tag{2}$$

with '$\bigvee$' denoting maximum, $F_{(n)}$ the distribution function of $\mathbf{M}_n$ and $F_{(n)1}, \ldots, F_{(n)d}$ the marginals associated to $M_{n,1}, \ldots, M_{n,d}$

The copula $C_{(n)}$ associated to $\mathbf{M}_n$ is given by

$$C_{(n)}(u_1, \ldots, u_d) = C_F(u_1^{1/n}, \ldots, u_d^{1/n})^n, \qquad (u_1, \ldots, u_d) \in [0,1]^d.$$

Proof.

$$\begin{aligned} C_{(n)}(F_{(n)1}(x_1), \ldots, F_{(n)d}(x_1)) &= \mathrm{P}(M_{n,1} \leqslant x_1, \ldots, M_{n,d} \leqslant x_d) \\ &= \mathrm{P}(X_{11} \leqslant x_1, \ldots, X_{1d} \leqslant x_d)^n \\ &= C_F(F_1(x_1), \ldots, F_d(x_d))^n \\ &= C_F(F_{(n)1}(x_1)^{1/n}, \ldots, F_{(n)d}(x_d)^{1/n})^n \end{aligned}$$

Definition 1

A copula C is called an extreme-value copula if there exists a copula C_F such that

$$C_F(u_1^{1/n}, \dots, u_d^{1/n})^n \to C(u_1, \dots, u_d) \qquad (n \to \infty) \tag{3}$$

for all $(u_1, \dots, u_d) \in [0,1]^d$. The copula C_F is said to be in the domain of attraction of C.

Definition 2

A d-variate copula C is max-stable if it satisfies the relationship

$$C(u_1, \dots, u_d) = C(u_1^{1/m}, \dots, u_d^{1/m})^m \tag{4}$$

for every integer $m \geqslant 1$ and all $(u_1, \dots, u_d) \in [0,1]^d$.

Theorem 3

A copula is an extreme-value copula if and only if it is max-stable.

Proof.

$\Leftarrow$: Trivial

$\Rightarrow$:for fixed integer $m \geqslant 1$ and for $n = mk$, write

$$C_{(n)}(u_1, \dots, u_d) = C_F(u_1^{1/n}, \dots, u_d^{1/n})^n = C_F(u_1^{1/mk}, \dots, u_d^{1/mk})^{mk} = C_F((u_1^{1/m})^{1/k}, \dots, (u_d^{1/m})^{1/k})^{mk} = C_{(k)}(u_1^{1/m}, \dots, u_d^{1/m})^m$$

□

Theorem 4

A d-variate copula C is an extreme-value copula if and only if there exists a finite Borel measure H on Δ_{d-1}, called spectral measure, such that $(u_1, \ldots, u_d) \in (0,1]^d$,

$$C(u_1, \ldots, u_d) = \exp\big(-\ell(-\log u_1, \ldots, -\log u_d)\big),$$

where the tail dependence function $\ell : [0,\infty)^d \to [0,\infty)$ *is given by*

$$\ell(x_1, \ldots, x_d) = \int_{\Delta_{d-1}} \bigvee_{j=1}^{d} (w_j x_j) \, dH(w_1, \ldots, w_d), \qquad (5)$$

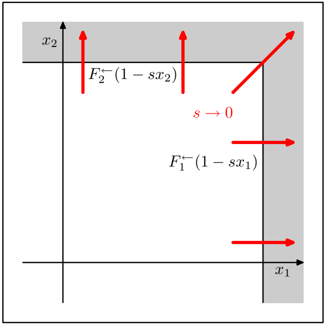
with $(x_1, \ldots, x_d) \in [0,\infty)^d$*. The spectral measure H is arbitrary except for the d moment constraints*

$$\int_{\Delta_{d-1}} w_j \, dH(w_1, \ldots, w_d) = 1, \qquad j \in \{1, \ldots, d\}. \qquad (6)$$

For $(x_1, \ldots, x_d) \in [0, \infty)^d$:

$$\lim_{t \downarrow 0} t^{-1}\big(1 - C_F(1 - tx_1, \ldots, 1 - tx_d)\big)$$
$$= \ell(x_1, \ldots, x_d),$$

see for instance
Drees & Huang (1998).



Definition 5 (Pickands (1981))

The restriction of ℓ to the unit simplex

$$\Delta_d = \{(w_1, \dots, w_d) : \sum_{i=1}^{d} w_i = 1, w_j \geq 0, \forall j = 1, \dots, d\}$$

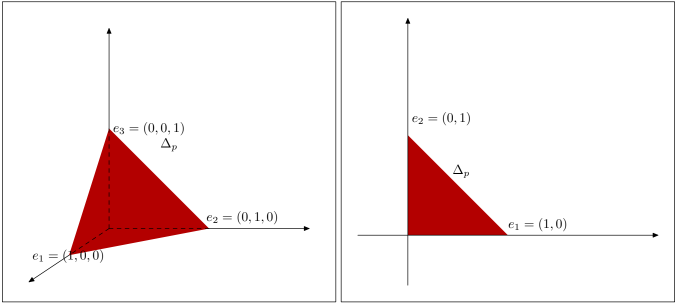
is called Pickands dependence function A.

$$\begin{aligned} C(\mathbf{u}) &= \exp\{-\ell(-\log u_1, \dots, -\log u_d)\} \\ &= \exp\left\{\left(\sum_{j=1}^{d} \log u_j\right) A\left(\frac{\log u_1}{\sum_{j=1}^{p} \log u_j}, \dots, \frac{\log u_{d-1}}{\sum_{j=1}^{d} \log u_j}\right)\right\} \end{aligned}$$

for $0 < u_j \leqslant 1$.

For dimension $d = 3$:

$$\Delta_3 = \{(w_1, w_2, w_3) : \sum_{i=1}^{3} w_i = 1, w_j \geq 0, \forall j = 1, \ldots, 3\}$$



A satisfies the following properties:

1. A is convex;
2. $\max(w_1, \ldots, w_d) \leqslant A(\mathbf{w}) \leqslant 1$.

$\Rightarrow A(\mathbf{e}_j) = 1$, for $\mathbf{e}_j = (0, \ldots, 1, \ldots, 0)$ being the d vertices.

Remark

Functions satisfying the previous conditions do not characterize the class of multivariate extreme value copulas except in dimension $d = 2$.

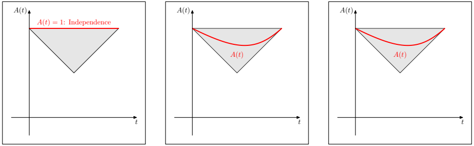
Theorem 6

A bivariate copula C is an extreme-value copula if and only if

$$C(u,v) = (uv)^{A(\log(v)/\log(uv))}, \qquad (u,v) \in (0,1]^2 \setminus \{(1,1)\}, \qquad (7)$$

where $A : [0,1] \to [1/2,1]$ is convex and satisfies $t \vee (1-t) \leqslant A(t) \leqslant 1$ for all $t \in [0,1]$.

Pickands dependence function A together with the region $t \vee (1-t) \leqslant A(t) \leqslant 1$.



Archimedean copula

- Archimedean copula with generator $\phi : [0,1] \to [0,\infty]$.

$$C_\phi(u_1, \ldots, u_d) = \phi^{\leftarrow}\big(\phi(u_1) + \cdots + \phi(u_d)\big), \quad (u_1, \ldots, u_d) \in [0,1]^d \tag{8}$$

- If the following limit exists,

$$\theta = -\lim_{s \downarrow 0} \frac{\phi(1-s)}{s\,\phi'(1-s)} \in [0,1] \tag{9}$$

then

$$C_\phi(u_1^{1/n}, \ldots, u_p^{1/n})^n \to C_\theta(u_1, \ldots, u_d)$$

with

$$C_\theta(u_1, \ldots, u_d) = \exp\big\{-\big((-\log u_1)^{1/\theta} + \cdots + (-\log u_d)^{1/\theta}\big)^\theta\big\},$$

Archimedean copula

- Gumbel–Hougaard or logistic copula

$$C_\theta(u_1, \dots, u_d) = \exp\{-((-\log u_1)^{1/\theta} + \cdots + (-\log u_d)^{1/\theta})^\theta\},$$

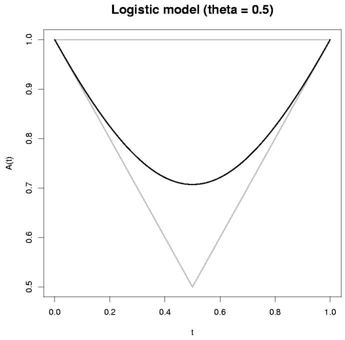
- θ: measures the degree of dependence

$$\theta = 1 \ : \ \text{independence}$$
$$\theta = 0 \ : \ \text{complete dependence.}$$

$$\ell(x_1, \dots, x_d) = \begin{cases} (x_1^{1/\theta} + \cdots + x_d^{1/\theta})^\theta & \text{if } 0 < \theta \leqslant 1, \\ x_1 \vee \cdots \vee x_d & \text{if } \theta = 0, \end{cases} \qquad (10)$$

Example: Logistic or Gumbel copula

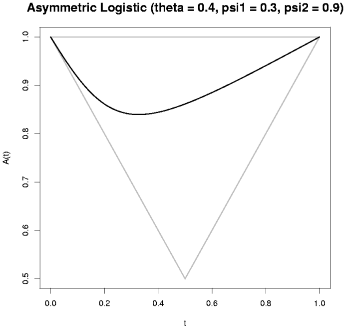
Gumbel (1961): $A(t) = [t^{1/\theta} + (1-t)^{1/\theta}]^{\theta}$



Example: Asymmetric logistic copula

Tawn (1988):

$$A(t) = (1-\psi_1)(1-t) + (1-\psi_2)t + [(\psi_1 t)^{1/\theta} + \{\psi_2(1-t)\}^{1/\theta}]^\theta$$



Example: Archimedean survival copulas

- Survival copula of an Archimedean copula:

$$C_\phi(u_1, u_2) = u_1 + u_2 - 1 + \phi^{\leftarrow}(\phi(1-u_1) + \phi(1-u_2))$$

- If

$$\theta = -\lim_{s \downarrow 0} \frac{\phi(s)}{s\phi'(s)} \in [0, \infty]$$

then $C_\phi(u_1^{1/n}, u_2^{1/n})^n \to C(u_1, u_2)$ where

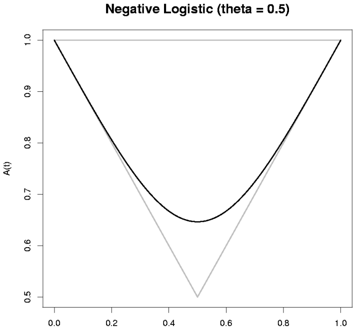
$$A(t) = 1 - \{t^{-1/\theta} + (1-t)^{-1/\theta}\}^{-\theta}$$

- C is the negative logistic or Galambos copula

Example: Negative logistic (Galambos) copula

GALAMBOS (1975):

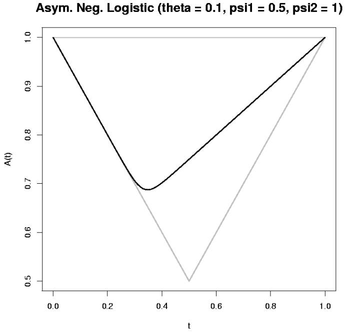
$$A(t) = 1 - \{t^{-1/\theta} + (1-t)^{-1/\theta}\}^{-\theta}$$



Example: Asym. negative logistic copula

Joe (1990)

$$A(t) = 1 - [\{\psi_1(1-t)\}^{-1/\theta} + (\psi_2 t)^{-1/\theta}]^{-\theta}$$



C admits the following properties:

- Positive quadrant dependent, $A \leqslant 1$ implies that $C(u, v) \geqslant uv$ for all $(u, v) \in [0, 1]^2$.
- Monotone regression dependent, that is, the conditional distribution of U given $V = v$ is stochastically increasing in v and *vice versa*; see GARRALDA-GUILLEM (2000).
- Kendall's τ and Spearman's ρ_S

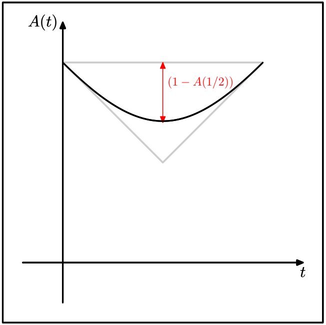
$$\tau = 4 \iint_{[0,1]^2} C(u, v) \, \mathrm{d}C(u, v) - 1 = \int_0^1 \frac{t(1-t)}{A(t)} \, \mathrm{d}A'(t),$$

$$\rho_S = 12 \iint_{[0,1]^2} uv \, \mathrm{d}C(u, v) - 3 = 12 \int_0^1 \frac{1}{(1 + A(t))^2} \, \mathrm{d}t - 3.$$

Coefficient of upper tail dependence: For a bivariate copula C_F in the domain of attraction of an extreme-value copula with tail dependence function ℓ and Pickands dependence function A, we find

$$\begin{aligned}\lambda_U &= \lim_{u \uparrow 1} \mathrm{P}(U > u \mid V > u) \\ &= \lim_{t \downarrow 0} t^{-1}\big(2t - 1 + C_F(1-t, 1-t)\big) \\ &= 2 - \ell(1,1) = 2\,\big(1 - A(1/2)\big) \in [0,1].\end{aligned}$$

$$\begin{aligned}\lambda_L &= \lim_{u \downarrow 0} \mathrm{P}(U \leqslant u \mid V \leqslant u) \\ &= \lim_{u \downarrow 0} u^{(2A(1/2)-1)}\end{aligned}$$



Estimation

- Parametric estimation: not treated here
- Nonparametric estimation

Known margins: apply *probability integral transform* (i.i.d. sample from C)

$$(U_{i,1},\ldots,U_{i,d}) := (F_1(X_{i,1}),\ldots,F_d(X_{i,d}))$$

Unknown margins: use *ranks* (empirical copula)

$$(\hat{U}_i,\ldots,\hat{U}_{i,d}) := \left(\frac{R_{i,1}}{n+1},\ldots,\frac{R_{i,d}}{n+1}\right)$$

, where $R_{i,1} = \sum_{k=1}^{n} I(X_{k,1} \leqslant X_{i,1})$.

Pickands estimator

For $t \in [0, 1]$, define

$$\xi_i(t) = \min\left(\frac{\log U_{i,1}}{1-t}, \frac{\log U_{i,2}}{t}\right),$$

with the obvious conventions for division by zero.

$$\begin{aligned} \mathrm{P}[\xi_i(t) > x] &= \mathrm{P}[U_i < e^{-(1-t)x}, V_i < e^{-tx}] \\ &= C(e^{-(1-t)x}, e^{-tx}) = e^{-x\,A(t)}. \end{aligned} \tag{11}$$

Pickands (1981) proposed the estimator:

$$\frac{1}{\hat{A}^P(t)} = \frac{1}{n}\sum_{i=1}^{n} \xi_i(t). \tag{12}$$

Deheuvels estimator:

Deheuvels (1991):

- $$\frac{1}{\hat{A}^D(t)} = \frac{1}{n}\sum_{i=1}^{n} \xi_i(t) - t\,\frac{1}{n}\sum_{i=1}^{n} \xi_i(1) - (1-t)\frac{1}{n}\sum_{i=1}^{n} \xi_i(0) + 1. \tag{13}$$
- Endpoint constraints verified: $\hat{A}^D(0) = \hat{A}^D(1) = 1$
- Weights $(1-t)$ and t rather pragmatic choices:

Deheuvels estimator:

Segers (2007):

$$\frac{1}{\hat{A}^D(t)} = \frac{1}{n}\sum_{i=1}^{n}\xi_i(t) - \beta_1(t)\,\frac{1}{n}\sum_{i=1}^{n}\xi_i(1) - \beta_2(t)\,\frac{1}{n}\sum_{i=1}^{n}\xi_i(0) + 1. \tag{14}$$

Variance-minimizing weight functions via a linear regression:

$$\xi_i(t) = \beta_0(t) + \beta_1(t)\,\{\xi_i(0) - 1\} + \beta_2(t)\,\{\xi_i(1) - 1\} + \epsilon_i(t).$$

$\hat{\beta}_0(t)$: minimum-variance estimator for $1/A(t)$ in the class of Deheuvels estimators.

CFG-estimator

Capéraà, Fougères and Genest (1997):

$$\log \hat{A}^{CFG}(t) = -\frac{1}{n} \sum_{i=1} \log \xi_i(t) - (1-t)\, \frac{1}{n} \sum_{i=1}^{n} \log \xi_i(0) - t\, \frac{1}{n} \sum_{i=1}^{n} \log \xi_i(1) \quad (15)$$

for $t \in [0,1]$, using

$$\mathrm{E}[-\log \xi_i(t)] = \log A(t) + \gamma, \qquad t \in [0,1],$$

with the Euler–Mascheroni constant $\gamma = 0.5772\ldots$

$$\log \hat{A}^{CFG}(t) = -\frac{1}{n} \sum_{i=1} \log \xi_i(t) + \beta_1(t)\, \frac{1}{n} \sum_{i=1}^{n} \log \xi_i(0) + \beta_2(t)\, \frac{1}{n} \sum_{i=1}^{n} \log \xi_i(1).$$

Estimation

- HALL-TAJVIDI (2000)
- Optimize weight functions $\beta_1(\mathbf{w}), \ldots, \beta_d(\mathbf{w})$; see SEGERS (2007) and GUDENDORF & SEGERS (2009).
- Estimation using unknown margins; see GENEST & SEGERS (2009).
- Other estimators ...

Shape constraints

- Even after endpoint correction, the estimators still do not satisfy the shape constraints
  - A is convex
  - $\max(t, 1-t) \leq A(t) \leq 1$
- Possible solutions:
  - Convex minorant to $\{A(t) \vee t \vee (1-t)\} \wedge 1$ (Deheuvels 1991)
  - Spline smoothing (Hall & Tajvidi 2000)
  - L_2 projection (Fils-Villetard, Guillou & Segers 2008)

Reading

References can be found in:

- G. Gudendorf & J. Segers (2009)
  "Extreme-Value Copulas"
  IS - DP 2009-26