Time-varying copulas: a survey

Olga Reznikova

Institute of Statistics
Université catholique de Louvain

joint with

Hans Manner
Department of Quantitative Economics
Maastricht University

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Motivation example 1

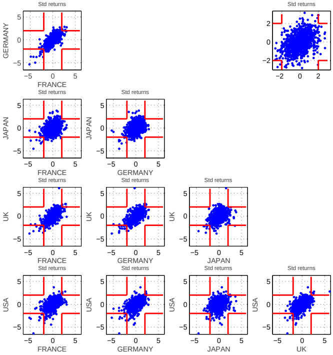

Figure: Scatter plots of standardized returns of G5 countries, weekly observations from 11 October 1989 till 31 May 2006

Motivation example 2

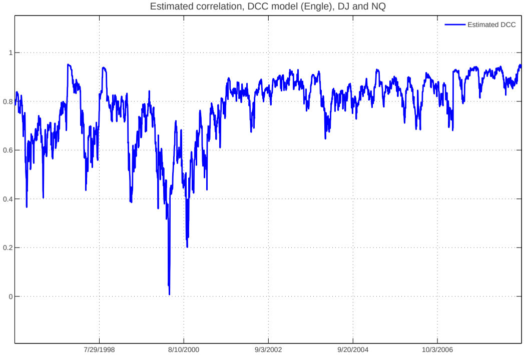

Figure: Correlation estimated with DCC model (Engle), DJ and NQ, daily observations 17 July 1996 till 21 October 2008

Estimating a copula model

The Copula model $F(X_{1t}, X_{2t}) = C\{F_1(X_{1t}), F_2(X_{2t})\}$
The joint *pdf*

$$f(X_{1t}, X_{2t}) = c(F_1(X_{1t}; \phi_1), F_2(X_{2t}; \phi_2); \theta) \prod_{i=1}^{2} f_i(X_{it}, ; \phi_i)$$

The joint log-likelihood

$$\begin{aligned}\mathbb{L}(\theta, \phi) &= \sum_{t=1}^{T} \ln c(F_1(X_{1t}; \phi_1), F_2(X_{2t}; \phi_2); \theta) \\ &\quad + \sum_{t=1}^{T} \ln f_1(X_{1t}; \phi_1) + \sum_{t=1}^{T} \ln f_2(z_{2t}; \phi_2)\end{aligned}$$

$$\mathbb{L}(\theta, \phi) = \mathbb{L}_C(\theta, \phi) + \mathbb{L}_V(\phi)$$

$(\phi, \theta) = [\phi_1', \phi_2', \theta]'$ is the parameter vector to be estimated

$c(u, v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$

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Estimating a copula model

Two-step Maximum likelihood

First step

$$\widetilde{\phi} = \arg\max_{\phi \in \Phi} \mathbb{L}_V(\phi)$$

Second step

$$\widetilde{\theta} = \arg\max \mathbb{L}_C(\theta, \widetilde{\phi})$$

Drawback *loss in efficiency*

Solution *apply Newton-Rhapson algorithm*

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Parametric models

Patton

Patton (2006): θ is a function of lagged past observations and autoregressive term

$$\begin{aligned}
\rho_t &= \Lambda_1\left(\omega + \beta\Lambda_1^{-1}(\rho_{t-1}) + \alpha\frac{1}{m}\sum_{i=1}^{m}\Phi^{-1}(U_{1,t-i})\Phi^{-1}(U_{2,t-i})\right)\\
\theta_t &= \Lambda_2\left(\omega + \beta\theta_{t-1} + \alpha\frac{1}{m}\sum_{j=0}^{m-1}|u_{t-j} - v_{t-j}|\right)
\end{aligned}$$

Dynamic conditional correlation (DCC)

Heinen, Valdesogo (2008): The correlation is driven by the crossproduct of lagged standardized residuals and autoregressive term

$$\begin{aligned}
R_t &= \mathrm{diag}\{Q\}^{-1/2} Q_t \,\mathrm{diag}\{Q\}^{-1/2}\\
Q_t &= \Omega(1-\alpha-\beta) + \alpha Y_{t-1} Y'_{t-1} + \beta Q_{t-1}\\
\tau_t &= \frac{2}{\pi} arcsin(\rho_t),\ \theta_t = \gamma(\tau_t)\\
\text{where} &\quad Y_{it} = \Phi^{-1}(U_{i,t}),\ Y_t = (Y_{1t}, Y_{2t})'
\end{aligned}$$

Stochastic and semiparametric models

Stochastic autoregressive copula (SCAR)

Hafner, Manner (2009): θ is driven by an independent stochastic process

$$\begin{aligned} \lambda_t &= \omega + \beta\lambda_{t-1} + \sigma_\eta \eta_t \\ \eta_t &\sim \textit{iid } N(0,1) \\ \theta &= \Lambda(\lambda_t) \end{aligned}$$

Semiparametric dynamic copula (SDC)

Hafner, Reznikova (2009): θ a smooth function of time

$$\begin{aligned} L_C(\theta; h, \tau) &= \sum_{t=1}^{T} \ell(U_{1t}, U_{2t}; \theta) K_h(t/T - \tau) \\ \widehat{\theta}(\tau) &= \textit{arg}\max_{\theta} L(\theta; h, \tau) \end{aligned}$$

where $K(\cdot)$ is a kernel and h is a bandwidth

Local parametric fitting

Local change point (LCP)

Giacomini et al. (2009): θ is approximated by a constant on a time invariant interval

$$I_t = [t - m_t, t[, t = 1, \ldots, T$$

Idea: test sequentially the nested intervals from I_t on the presence of the break point.

Regime switching copula (RSC)

Pelletier(2006), Garcia, Tsafack (2008), Chollete et al.(2008): allow for two regimes, characterized by different levels of dependence

$$\begin{aligned}
\mathcal{L}(\theta) &= \sum_{t=1}^{T} \log(1'(\widehat{\xi}_{t|t-1} \odot \eta_t)) \\
\eta_t &= \begin{pmatrix} c_1(U_{1t}, U_{2t}; \theta_1) \\ c_2(U_{1t}, U_{2t}; \theta_2) \end{pmatrix}
\end{aligned}$$

where

- $\widehat{\xi}_{t|t-1}$ is the vector of estimated transition probabilities using information until $(t-1)$
- $\odot$ is the Hadamard product

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Simulation study and model selection

Simulation design: Simulate 1000 observations from

Gaussian copula with ρ_t

Step: $\rho_t = 0.2 + 0.6 I_{t>500}$
Sine: $\rho_t = 0.5 + 0.4 \cos(2\pi t/400)$
AR(1):

$$\begin{aligned} \rho_t &= \frac{\exp(2\lambda_t) - 1}{exp(2\lambda_t) + 1} \\ \lambda_t &= 0.02 + 0.97\lambda_{t-1} + 0.1\epsilon_t \end{aligned}$$

Measures: MSE, Log-likelihood, Anderson-Darling test

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Simulation study: MSE

$$MSE = \frac{1}{K}\sum_{k=1}^{K}\frac{1}{T}\sum_{t=1}^{T}\left(\widehat{\rho}_t^k - \rho_t^{0k}\right)^2$$

MSE	Const	DCC	PATT	SDC	LCP	SCAR	RSC
Step	0.092	0.016	0.053	0.007	0.017	0.008	**0.004**
Sine	0.082	0.021	0.048	**0.006**	0.047	0.010	0.020
AR(1)	0.076	0.040	0.052	0.035	0.063	**0.025**	0.036

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Model selection by log-likelihood

The fraction of times each copula is selected as the best fitting.

	Sine						
	Const	**DCC**	**PATT**	**SDC**	**LCP**	**SCAR**	**RSC**
Gaussian	0.212	**0.981**	0.007	**1.000**	0.350	**1.000**	**0.999**
Clayton	0.008	0.002	0.002	0.000	0.010	0.000	0.000
Frank	**0.697**	0.006	0.488	0.000	0.260	0.000	0.000
Gumbel	0.083	0.011	**0.503**	0.000	**0.380**	0.000	0.001

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Model selection by Anderson-Darling test

Anderson-Darling test: Is the data generated by a C_i?

$$H_0 : C_i(u_t, v_t; \hat{\theta}_{it}) = C_0(u_t, v_t; \theta_t^0)$$

$$\hat{z}_t = C_i(u_t | v_t; \hat{\theta}_{it}) = \frac{\partial C_i(u_t, v_t; \hat{\theta}_{it})}{\partial v_t} \sim U(0,1)$$

The size and power for the AD test at 5% nominal level (the fraction of times the H_0 is rejected)

			Sine				
	Const	**DCC**	**PATT**	**SDC**	**LCP**	**SCAR**	**RSC**
Gaussian	0.352	**0.129**	0.324	**0.068**	0.260	**0.060**	**0.041**
Clayton	0.643	0.898	0.635	0.640	0.770	0.790	0.762
Frank	**0.051**	0.142	**0.134**	0.212	**0.110**	0.329	0.130
Gumbel	0.539	0.625	0.561	0.552	0.520	0.671	0.595

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Empirical example

Data set:

- exchange rates Euro-USD and Yen-USD
- from 31 December 1999 till 30 December 2005
- daily returns, T = 1564

Data is corrected for autocorrelation

$$
\begin{aligned}
X_t^E &= \underset{(1.7E-04)}{-9.7E-05} - \underset{(0.03)}{0.06}\, X_{t-1}^E + \epsilon_t^E \\
X_t^Y &= \underset{(1.5E-04)}{9.8E-05} - \underset{(0.03)}{0.04} + \epsilon_t^Y
\end{aligned}
$$

and conditional heteroscedasticity

$$
\begin{aligned}
h_t^E &= \underset{(1.3E-07)}{3.5E-07} + \underset{(0.01)}{0.02}\, \epsilon_{t-1}^E + \underset{(0.01)}{0.97}\, h_{t-1}^E, \ \nu^E = \underset{(12.03)}{28.83} \\
h_t^Y &= \underset{(1.5E-07)}{5.3E-07} + \underset{(0.01)}{0.02}\, \epsilon_{t-1}^Y + \underset{(0.01)}{0.96}\, h_{t-1}^E, \ \nu^Y = \underset{(1.15)}{7.11}
\end{aligned}
$$

Empirical example: Log-likelihood

(a) Log-likelihood							
	Const	**DCC**	**PATT**	**SDC**	**LCP**	**SCAR**	**RSC**
Gaussian	132.6	**194.3**	170.3	**228.9**	151.9	**202.2**	**207.63**
Gumbel	123.7	176.5	161.0	200.6	169.9	173.7	178.53
Clayton	113.4	145.2	142.9	161.9	135.3	149.5	151.86
Frank	**146.5**	**194.2**	**194.9**	**226.8**	**183.1**	**201.8**	**205.32**
rot Gumbel	134.4	182.9	169.5	198.3	169.3	177.6	169.04
rot Clayton	95.3	131.1	128.4	161.2	140.7	110.6	144.10

Empirical example: Anderson-Darling test

$$H_0 : C_i(u_t, v_t; \hat{\theta}_{it}) = C_0(u_t, v_t; \theta_t^0)$$

(b) AD test (Pvalues)

	Const	DCC	PATT	SDC	LCP	SCAR	RSC
Gaussian	0.00	0.00	0.00	0.00	0.00	0.03	0.03
Gumbel	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Clayton	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Frank	**0.14**	**0.16**	**0.51**	**0.48**	**0.17**	**0.32**	**0.25**
rot Gumbel	0.00	0.00	0.00	0.00	0.00	0.00	0.00
rot Clayton	0.00	0.00	0.00	0.00	0.00	0.00	0.00

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Empirical example: Dynamic Quantile (DQ) test

DQ test Engle and Manganelli (2004): is the model correctly specified?

- $VaR_t(\alpha) = F^{-1}_{t+1}(\alpha)$
- $hit^{\alpha}_t = \mathbb{I}(X_t \leq VaR_t(\alpha))$

$$hit^{\alpha}_t - \alpha = \delta_0 + \delta_1 hit^{\alpha}_{t-1} + \ldots + \delta_5 hit^{\alpha}_{t-5} + \delta_6 VaR_t(\alpha) + \nu_t$$

- $H_0 : \delta_0 = \ldots = \delta_6 = 0$

(c) DQ test (Pvalues)

	Const	**DCC**	**PATT**	**SDC**	**LCP**	**SCAR**	**RSC**
Gaussian	0.04	**0.07**	**0.17**	**0.07**	0.04	**0.64**	**0.07**
Gumbel	0.03	0.03	0.00	**0.29**	0.02	**0.10**	0.03
Clayton	**0.19**	**0.44**	0.00	**0.23**	0.00	0.02	**0.26**
Frank	**0.27**	0.04	**0.05**	**0.17**	0.03	**0.16**	0.03
rot Gumbel	**0.08**	0.00	0.02	0.00	0.03	0.00	0.02
rot Clayton	0.02	0.01	**0.05**	0.04	0.03	**0.11**	**0.07**

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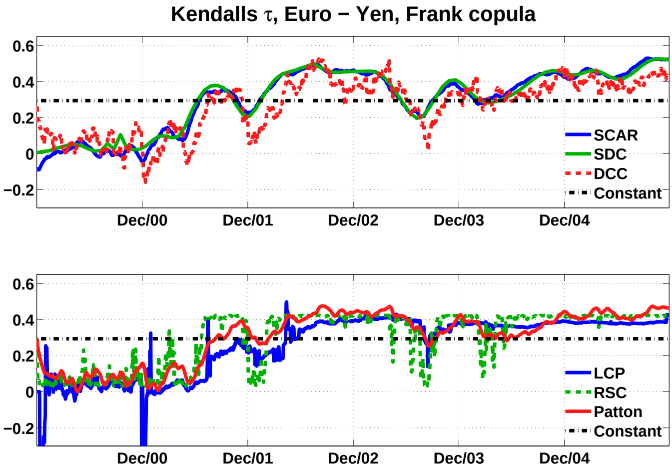
Empirical example: estimated dependence



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Conclusion

Results

- log-likelihood is a strong model selection criterion, when variation of the dependence parameter is taken into account
- Anderson-Darling test has acceptable size and power properties
- DQ test of Engle and Manganelli (2004) only shows if the model fits the data

Recommendations

- RSC model showed good performance in the simulation study, is easy to program and is not computationally tedious

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For Further Reading

- Chollete, Heinen, Valdesogo (2009) Modeling International Financial Returns with a Multivariate Regime-switching Copula. *Journal of Financial Econometrics*, 7(4):437–480
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- Patton (2006) Modelling asymmetric exchange rate dependence.*International Economic Review*, 47:527–556