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**INDIVIDUAL LOSS RESERVING
WITH THE MULTIVARIATE SKEW
NORMAL MODEL**

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Individual Loss Reserving with the Multivariate Skew Normal Model

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Abstract

In general insurance, the evaluation of future cash flows and solvency capital has become increasingly important. To assist in this process, the present paper proposes an individual discrete-time loss

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reserving model describing the occurrence, the reporting delay, the time to the first payment, and the cash flows associated with the settlement process of each individual claim. The approach uses development factors similar to those of the standard chain-ladder method. These are parametrically modeled by the Multivariate Skew Normal distribution. Empirical analyses using a realistic portfolio and out-of-sample prediction tests demonstrate the relevance of the model proposed.

Keywords: Stochastic Loss Reserving, General Insurance, Multivariate Skew Normal distribution, Chain-ladder

1 Introduction.

In a general insurance context, the typical evolution of a claim may be divided into three parts. From the occurrence of the claim to its notification to the insurance company, the insurer is liable for the claim amount but is unaware of the claim's existence. The claim is said to be Incurred But Not Reported (IBNR). After notification, the claim is known by the company and there may be few time units before the first payment is made. At this point, an initial case estimate is evaluated by a claim handler. In this paper, such a claim will be called Reported But Not Paid (RBNP). Then, the initial payment occurs and several partial payments and refunds can follow. The claim is finally closed at the closure date. From the first payment to the closure of the claim, the insurer is aware of the existence of the claim, but the final amount is still unknown: the claim is Reported But Not Settled (RBNS). This structure provides a flexible framework which can be simplified or extended if needed. The evolution of a general insurance claim is illustrated on a timeline in Figure 1.

At a certain point in time, usually the end of the last completed period or the first moment of the new one, an actuarial evaluation must be performed and technical provisions have to be estimated. This moment will be called the *evaluation date*. Loosely speaking, the insurer must predict, with maximum accuracy, the total amount needed to pay claims that he has legally committed to cover. One part of the total amount comes from the completion of Reported But Not Settled (RBNS) claims. Predictions for costs related to Reported But Not Paid (RBNP) claims and Incurred But Not Reported (IBNR) claims form the second part of the total amount.

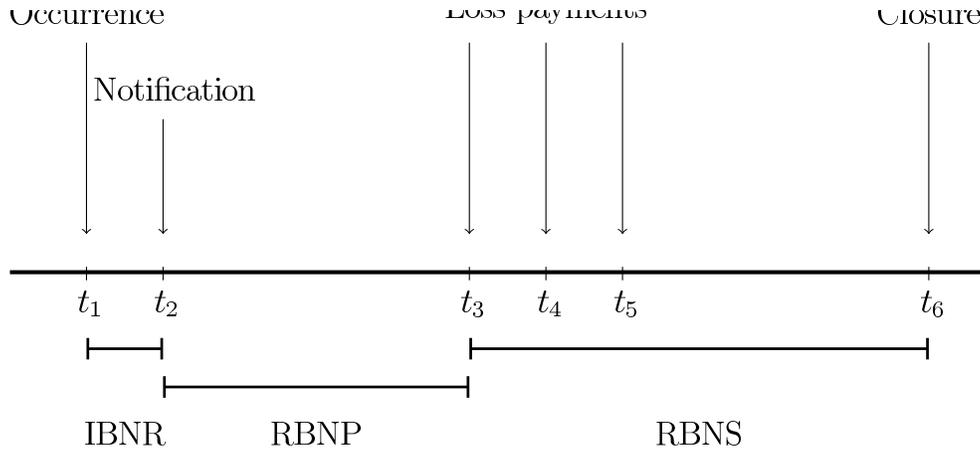


Figure 1: Evolution of a general insurance claim.

Both components are summed up and a *best estimate* of the total amount is obtained.

With the introduction of Solvency 2 (in 2012) and IFRS 4 Phase 2 (in 2013), the evaluation of future cash flows and regulatory required solvency capital becomes more important and current techniques for loss reserving may have to be improved, adjusted or extended. Generally, these claims reserving techniques are based on aggregated data, conveniently summarized in a run-off triangle per accident year and per development year. Tables 6 and 7 below are examples of that kind of structure. The chain-ladder approach (Mack's model in MACK (1993) and MACK (1999)) is the most popular member of this category. A rich literature exists about those techniques and an overview can be found in ENGLAND & VERRALL (2002) or WÜTHRICH & MERZ (2008). However, using aggregated data in combination with the chain-ladder approach gives rise to several issues which are enumerated in ANTONIO & PLAT (2011). Many practical solutions have been proposed, but they have not been applied simultaneously.

A mathematical framework for the development of individual claims was formulated in the last decade of the 20th century by ARJAS (1989), NORBERG (1993), HAASTRUP & ARJAS (1996) and NORBERG (1999). More recently, a semi-parametric model (ZHAO & ZHOU & WANG (2009)) and a so-called micro-model (ANTONIO & PLAT (2011)) have been introduced. The model developed in this paper is at the confluence of the latter and the

chain-ladder model. Instead of using the continuous timeline from ANTONIO & PLAT (2011), a micro-level discrete stochastic structure is used to model the occurrence times, the reporting delays, the first payment delays, the number of payments for each claim and the number of periods between two subsequent payments. The development pattern is modeled with a chain-ladder approach in the framework of a multivariate distribution. The model will conduct to an estimate of the total reserve and its distribution. Moreover, and unlike aggregated approaches often used, it will provide individual predictions.

Our paper is organized as follows. The statistical model is introduced in Section 2. In Section 3, the data are presented. These will be used in the example discussed in Sections 4 and 5. Finally, Section 6 concludes. Some technical developments have been gathered in two appendices, for the sake of completeness.

2 The Model.

2.1 General Structure of the Data.

The data set should contain detailed information about the development of individual claims. More specifically, the model will use:

- the occurrence date;
- the declaration date;
- the date(s) of payment(s) (and refund(s)) done for the claim;
- the amount(s) paid for the claim; and
- the closure date.

An extract of the data set used in Sections 4 and 5 is presented in Table 1. The data set is from a European insurance company and concerns a portfolio of general liability insurance policies for private individuals. More details on the data can be found in Section 3. Developments of claims occur in a continuous framework as is illustrated in Figures 2 and 3 for claims No 1567 and No 2680. In order to apply the actuarial model developed in this paper, a *time unit* and an evaluation date have to be chosen and the

data set must be transformed with respect to these choices. In the example, a time unit of one year is chosen and the transition from continuous micro-level data to discrete micro-level data for claims No 1567 and No 2680 is presented in Figure 4.

Event	No	Date MM/DD/YY	Amount (discounted)
Occurrence	1567	06/10/1997	-
	2680	01/03/1997	-
Declaration	1567	07/09/1997	-
	2680	01/06/1997	-
Payments	1567	08/20/1997	32.76
	1567	10/25/1997	16.32
	1567	03/18/1998	608.02
	2680	02/06/1997	135.84
	2680	04/19/1997	45.99
	2680	10/15/1997	313.60
Closure	1567	07/08/1998	-
	2680	04/04/1998	-

Table 1: Development of two claims from the data set.

To enable comparisons between results from the model presented in this paper and the classical Mack's model, individual information can be summarized on an annual basis and over groups of claims from the same accident year. A run-off triangle as illustrated in Tables 6 and 7 presents the results.

2.2 Structure of the Model.

2.2.1 Time Structure.

The evolution of the k^{th} claim from occurrence period i (the arrival period) is modeled in a discrete framework with a four-component structure:

- the random variable T_{ik} which is the *reporting delay* for claim (ik) , i.e. the difference between the occurrence period of the claim and the period of its notification to the insurance company (for claim No 1567 and claim No 2680, the observed t_{ik} is 0);

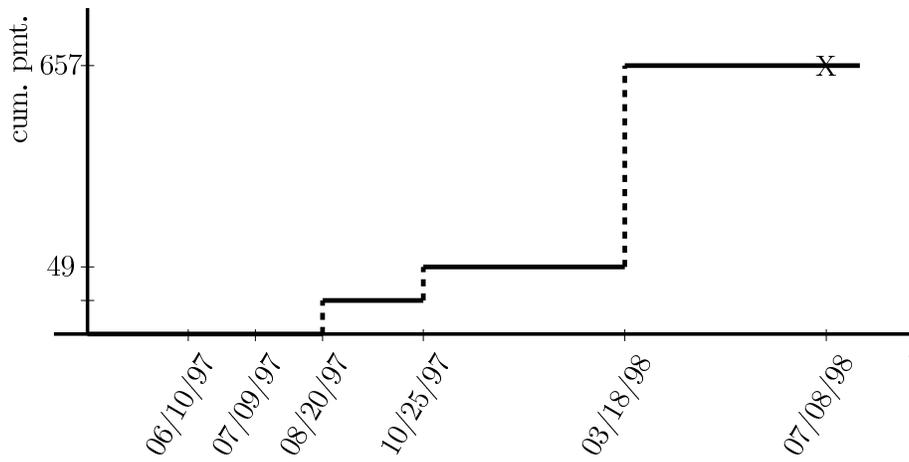


Figure 2: Development of claim No 1567 in a continuous framework. The x -axis represents the date of each event and the y -axis represents the cumulative payment (“cum. pmt.”) for the claim.

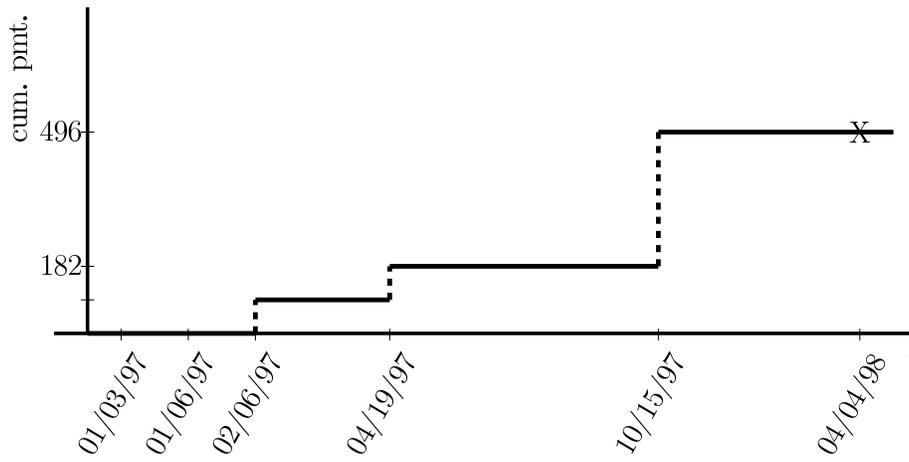


Figure 3: Development of the claim No 2680 in a continuous framework.

- the random variable Q_{ik} , the *first payment delay*, which represents the difference between the notification period and the first period with payment (i.e. the first time unit(s) during which at least one payment occurs) for claim (ik) (for claim No 1567 and claim No 2680, the

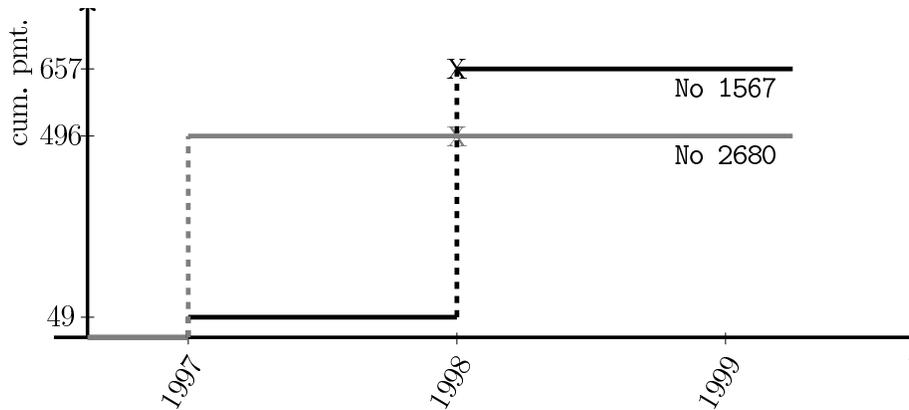


Figure 4: Development of both claims in a discrete framework (annual).

observed q_{ik} is 0 for both claims) ;

- the *number of period(s)* with partial payment after the first one is modeled with the random variable U_{ik} (for claim No 1567, the observed u_{ik} is 1 and for claim No 2680, it is 0); and
- the *delay between two periods with payment* is modeled with the random variable N_{ikj} which represents the number of periods between payments j and $j + 1$ and $N_{ik} = \sum_j N_{ikj}$, $j = 1, 2, \dots$ (for claim No 1567, the observed N_{ik} is 1 and for claim No 2680, there is no variable N_{ik}).

Each component is supposed to follow a discrete distribution $f : \mathbb{N} \rightarrow [0, 1]$, respectively $f_1(t; \nu)$, $f_2(q; \psi)$, $f_3(u; \delta)$ and $f_4(n; \phi)$. One should note that for the delay between two consecutive periods with payment, $\Pr(N_{ikj} = 0) = 0, \forall j$. The time line of claim (ik) is illustrated in Figure 5.

2.2.2 Exposure and Occurrence Measures.

The stochastic model presented in this paper requires a measure of the insurer's exposure to claims. The (inflation adjusted) earned premium is probably the most common measure used by both insurers and reinsurers when evaluating reserves, but written premium, number of policies in force, property values, etc. can also be used. Let $w(i)$ denote the exposure measure for the occurrence period $i, i = 1, \dots, I$.

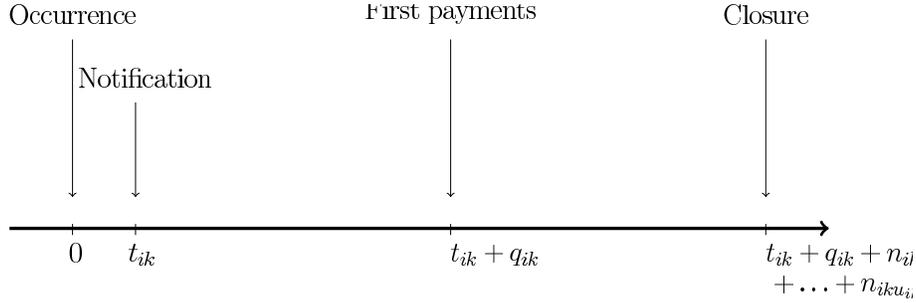


Figure 5: Time line for claim (ik) .

To distinguish explicitly between IBNR and RBNS/RBNP claims, an occurrence measure must be defined. The number of claims for the occurrence period i , K_i , is supposed to follow a Poisson process with occurrence measure $\theta w(i)$. However, since we are only able to observe reported claims, the Poisson process should be filtered in the following way:

$$\theta w(i) F_1(t_i^* - 1; \nu), \quad (1)$$

where t_i^* denotes the number of periods between the occurrence period i and the evaluation date.

2.2.3 Development Structure.

Let the random variable Y_{ikj} represent the j^{th} incremental partial amount for the k^{th} claim ($k = 1, \dots, K_i$) from the occurrence period i ($i = 1, \dots, I$). The cumulative amount paid for claim (ik) is obtained by multiplying the initial amount, Y_{ik1} , by one or more link ratios. The initial amount and the vector of link ratio(s) form the *development pattern* of the claim. This approach is similar to the one used in the chain-ladder model (Mack's model) where the index j is for development period instead of partial payment. Using a development-to-development model in an individual framework can be problematic because the length of the development pattern is supposed to be fixed and identical for all claims, and many development factors with a value of 1 would unnecessarily be included. In the payment-to-payment approach presented in the present paper, only development factors with non-one values are modeled.

For a claim (ik) with a positive value of the random variable U_{ik} , the development pattern is given by the vector $\Lambda^{(ik)} = [Y_{ik1} \ \lambda_1^{(ik)} \ \dots \ \lambda_{u_{ik}}^{(ik)}]'$, where

$$\lambda_j^{(ik)} = \frac{\sum_{r=1}^{j+1} Y_{ikr}}{\sum_{r=1}^j Y_{ikr}},$$

for $j = 1, \dots, u_{ik}$. The development pattern of a claim (ik) is supposed to follow a multivariate distribution $M : \mathbb{R}^{u_{ik}+1} \rightarrow [0, 1]$.

For claim No 1567 in the previous example, the initial amount is $y_{ik1} = 49.08$ and the link ratio is given by

$$\begin{aligned} \lambda_1^{(ik)} &= \frac{49.08 + 608.02}{49.08} \\ &= 13.3883. \end{aligned}$$

2.3 The Likelihood.

For the sake of clarity, the likelihood function will be divided into three parts: an expression for the likelihood of closed, RBNP and RBNS claims.

Completed claims. For completed claims (C), the likelihood function is given by

$$\begin{aligned} L^C &\propto \prod_{ik} M(\ln(\boldsymbol{\lambda}_{u_{ik}+1}); \boldsymbol{\mu}_{u_{ik}+1}, \boldsymbol{\Sigma}_{u_{ik}+1}^{1/2}, \boldsymbol{\beta}_{u_{ik}+1} | u_{ik}) \\ &\quad \times f_1(t_{ik}; \boldsymbol{\nu} | T_{ik} \leq t_{ik}^* - 1) f_2(q_{ik}; \boldsymbol{\psi} | Q_{ik} \leq t_{ik}^* - t_{ik} - 1) \\ &\quad \times f_3(u_{ik}; \boldsymbol{\delta} | U_{ik} \leq t_{ik}^* - q_{ik} - t_{ik} - 1) \\ &\quad \times [I(u_{ik} = 0)(1) \\ &\quad + I(u_{ik} \geq 1) f_4(n_{ik1}; \boldsymbol{\phi} | 0 < N_{ik1} \leq t_{ik}^* - t_{ik} - q_{ik} - u_{ik}) \\ &\quad + I(u_{ik} \geq 2) \prod_{j=2}^{u_{ik}} f_4 \left(n_{ikj}; \boldsymbol{\phi} | 0 < N_{ikj} \leq t_{ik}^* - t_{ik} - q_{ik} - (u_{ik} - j + 1) - \sum_{p=1}^{j-1} n_{ikp} \right) \Big]. \end{aligned}$$

The first component of the likelihood function is the multivariate distribution for the development pattern, i.e. the initial amount and the vector of link ratio(s), given the total number of payment(s). For the distributions of random variables involved in the time structure part of the function (T , Q ,

U and N), a condition must be added to take into account the censoring of the development at the evaluation date. More specifically, for completed claims the whole development must be observed before the evaluation date.

As an example, at evaluation date 01/01/2004, claim No 1567 previously presented contributes to the likelihood function in the following way:

$$\begin{aligned} L &\propto M(\ln(49.08, 13.3883); \boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2^{1/2}, \boldsymbol{\beta}_2 | U = 1) \\ &\quad \times f_1(0; \boldsymbol{\nu} | T \leq 6) f_2(0; \boldsymbol{\psi} | Q \leq 6) \\ &\quad \times f_3(1; \boldsymbol{\delta} | U \leq 6) \\ &\quad \times (1) f_4(1; \boldsymbol{\phi} | 0 < N_{ik1} \leq 6), \end{aligned}$$

with $t^* = 2004 - 1997 = 7$.

RBNS claims. For Reported But Not Settled claims (RBNS), the likelihood expression is

$$\begin{aligned} L^{RBNS} &\propto \prod_{ik} M(\ln(\lambda_{u_{ik}^*+1}); \boldsymbol{\mu}_{u_{ik}^*+1}, \boldsymbol{\Sigma}_{u_{ik}^*+1}^{1/2}, \boldsymbol{\beta}_{u_{ik}^*+1} | u_{ik}^*) \\ &\quad \times f_1(t_{ik}; \boldsymbol{\nu} | T_{ik} \leq t_{ik}^* - 1) f_2(q_{ik}; \boldsymbol{\psi} | Q_{ik} \leq t_{ik}^* - t_{ik} - 1) \\ &\quad \times (1 - F_3(u_{ik}^* - 1; \boldsymbol{\delta})) \\ &\quad \times [I(u_{ik}^* = 0)(1) \\ &\quad + I(u_{ik}^* \geq 1) f_4(n_{ik1}; \boldsymbol{\phi} | 0 < N_{ik1} \leq t_{ik}^* - t_{ik} - q_{ik} - u_{ik}^*) \\ &\quad + I(u_{ik}^* \geq 2) \prod_{j=2}^{u_{ik}^*} f_4 \left(n_{ikj}; \boldsymbol{\phi} | 0 < N_{ikj} \leq t_{ik}^* - t_{ik} - q_{ik} - (u_{ik}^* - j + 1) - \sum_{p=1}^{j-1} n_{ikp} \right)] , \end{aligned}$$

where u_{ik}^* represents the number of periods with payments after the first one for the claim (ik).

RBNP claims. Finally, for Reported But Not Paid claims (RBNP), the likelihood function is

$$L^{RBNP} \propto \prod_{ik} f_1(t_{ik}; \boldsymbol{\nu} | T_{ik} \leq t_{ik}^* - 1) (1 - F_2(t_{ik}^* - t_{ik} - 1; \boldsymbol{\psi})).$$

3 The Data.

3.1 Background.

The reserve model will be applied on a year by year basis to a data set from a European insurance company and concerns a portfolio of general liability insurance policies for private individuals. Available information is from January 1997 till December 2003 except for exposure which is missing from January 1997 to December 1999. Originally, information is available till August 2009, but to enable out-of-sample prediction we remove the observations from January 2004 to August 2009. The evaluation date is the first day of January 2004.

The exposure measure used in the data set is not the number of policies but the “earned” exposure which is the exposure units actually exposed to loss during the period. As said previously, some part of the information about exposure is missing for the period before January 2000, so the occurrence measure is given from 2000 till 2004.

The paper focuses on paid claims and not on incurred losses (paid losses plus case estimates) but incorporating information on incurred losses could be a topic for future research. No claim handling expenses are assumed in the data set since we had no information on expenses. The payments are discounted to year 1997 with the annual Dutch consumer price index. However, attention is paid in the model to the timing of the payments so the insurer can adjust the estimations and/or predictions by applying a suitable measure for inflation.

3.2 Descriptive Statistics.

A total of 279,094 claims are considered in the study and divided in two categories of risk: 273,977 claims are related to Material Damage (MD) and 5,117 claims are related to Bodily Injury (BI). 1,156 open MD claims and 694 open BI claims are left. Note that a claim could have a BI and a MD part.

Occurrence of claims. Figure 6 presents the exposure per year which appears to be fairly linear.

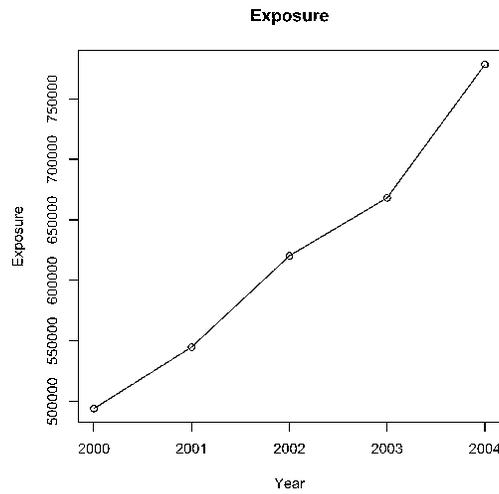


Figure 6: Exposure per year from 2000 to 2004.

Development pattern. For BI claims with exactly three payments, Figure 7 shows the dependence between the initial payment and the two development factors (on log scale). In Figure 8 the dependence is illustrated (on log scale) between the initial payment and the development factor for MD claims with exactly two payments. In both figures, the shape of each cloud illustrates the correlation between variables. Moreover, histograms on the main diagonal illustrate the shape of the marginal distribution of each variable. The presence of asymmetry, particularly in development factors, can be noted.

Distributions for number of periods. For BI and MD claims, Figure 9 presents the reporting delay, Figure 10 presents the first payment delay and Figure 11 presents the number of period(s) with payment after the first one.

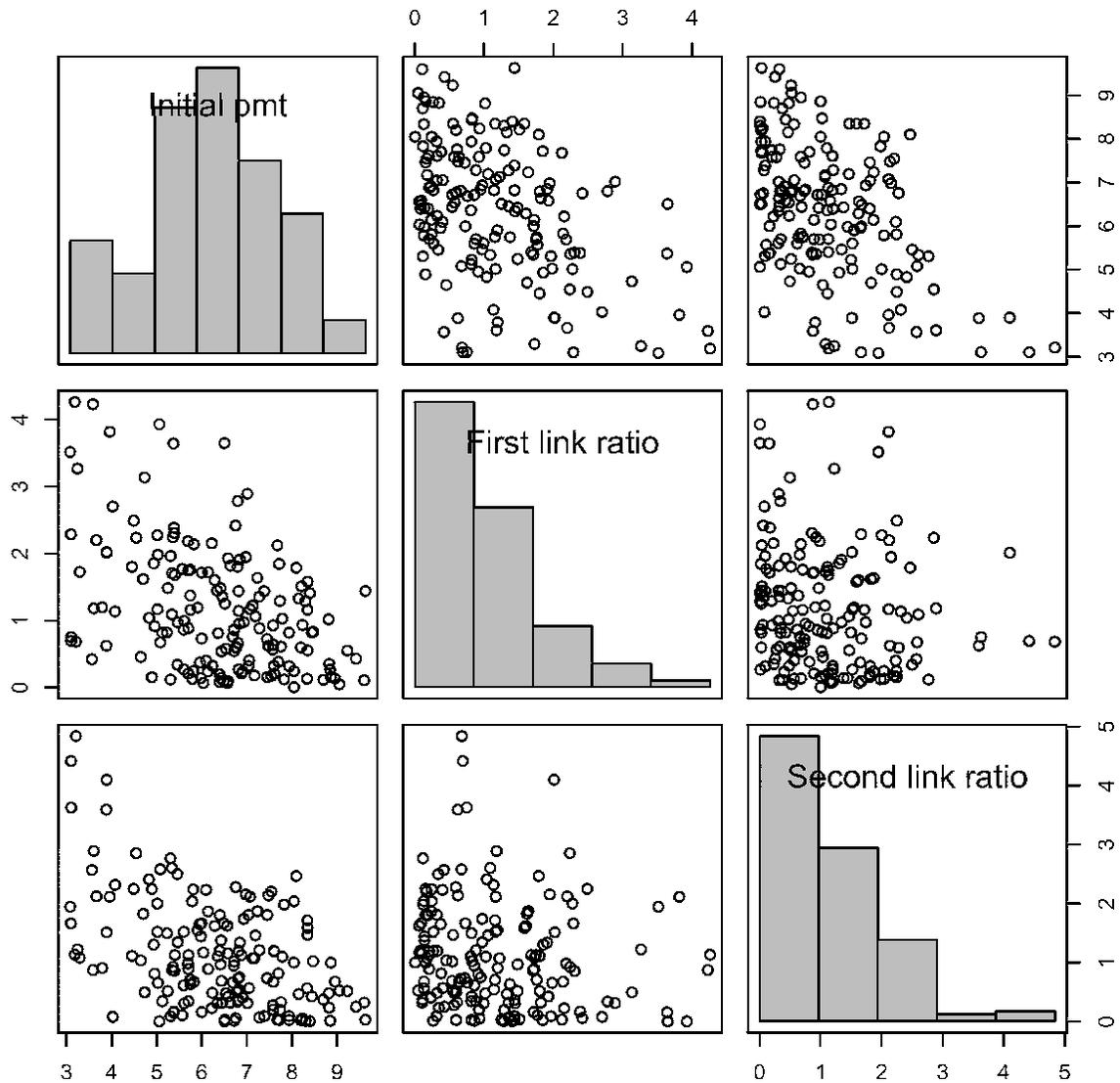


Figure 7: Observed values for Bodily Injury claims (on log scale) with exactly three payments (“pmt”).

4 Distributional Assumptions and Estimation Results.

4.1 Distributional Assumptions.

In this section, distributional assumptions for each component of the likelihood function ($\{T_{ik}\}, \{Q_{ik}\}, \{U_{ik}\}, \{N_{ik}\}$ and $\Lambda^{(ik)}$) are discussed.

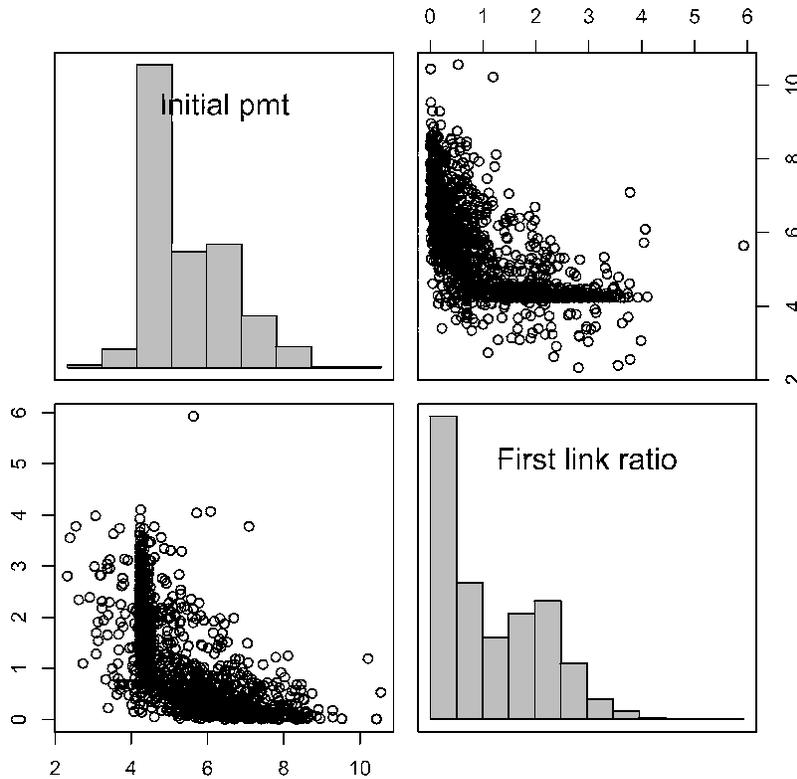


Figure 8: Observed values for Material Damage claims (on log scale) with exactly two payments (“pmt”).

Occurrence of claims. The random variable K_i representing the number of claims with actual payment(s) for the occurrence period i is supposed to follow a Poisson distribution with measure given by (1).

Development pattern. In the stochastic version of the chain-ladder model (Mack’s model), successive development factors are supposed to be non-correlated given the past information. Moreover, there is no link between the initial payment and the vector of development factors. If this hypothesis may seem justified from an aggregated point of view, it becomes more problematic in the individual framework presented in this paper.

In the model presented in the current paper, the development pattern, i.e. the logarithm of the initial payment, Y_{ik1} , and the logarithm of the vector of link ratios, is supposed to follow a *Multivariate Skew Normal*

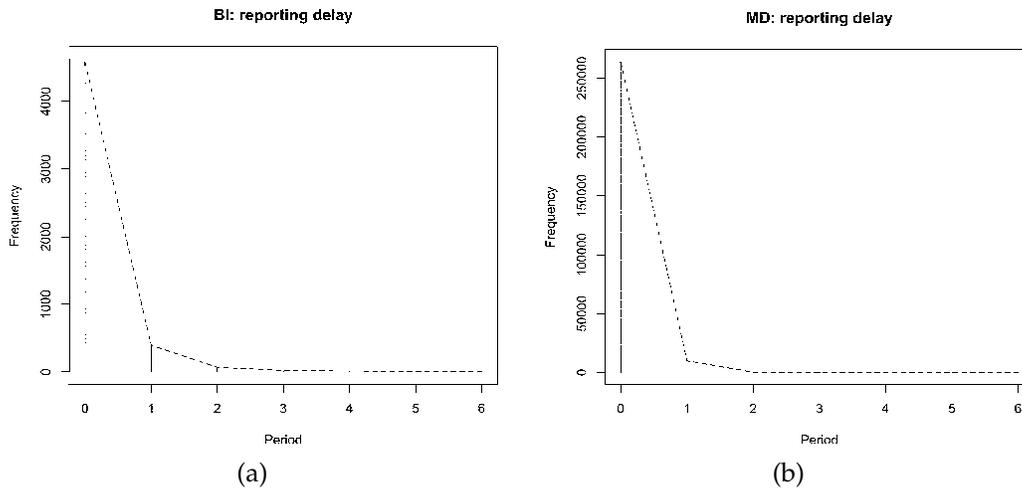


Figure 9: Reporting delay observed and estimated (broken line) for Bodily Injury (left) and Material Damage (right).

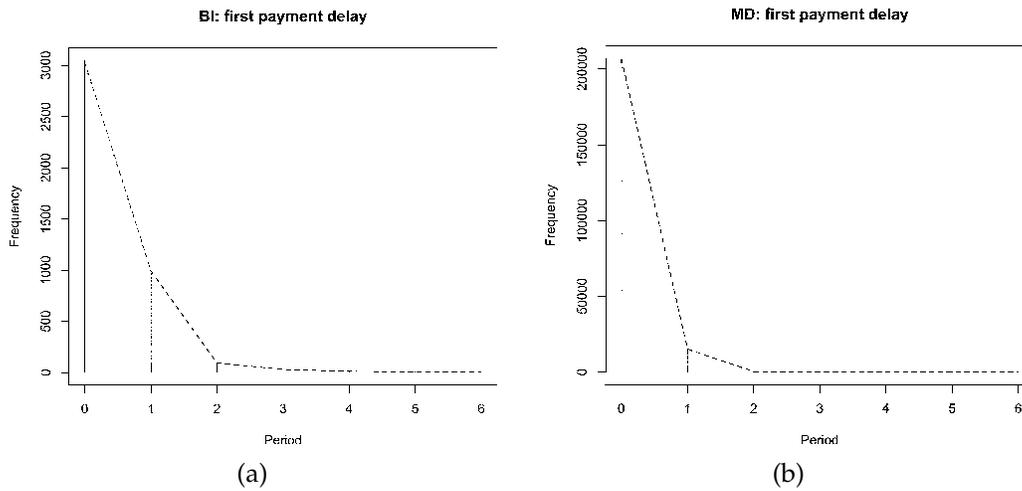


Figure 10: First payment delay observed and estimated (broken line) for Bodily Injury (left) and Material Damage (right).

(MSN) distribution which is introduced below. Some crucial properties and proofs are deferred to Appendix A.

The Univariate Skew Normal distribution has been fragmentarily introduced in ROBERTS & GEISSER (1966), but the first formal definition and

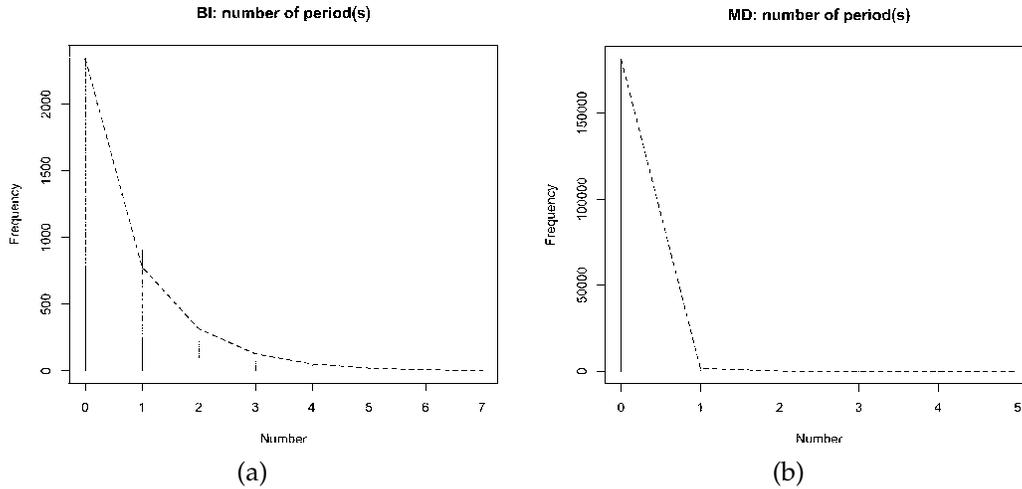


Figure 11: Number of period(s) with payment after the first one observed and estimated (broken line) for Bodily Injury (left) and Material Damage (right).

systematic study of its properties appeared in the seminal work of AZZALINI (1985). A random variable X with probability density function given by

$$f_X(x) = 2\phi(x)\Phi(\beta x), \quad -\infty < x < \infty,$$

where $\phi(\cdot)$, and $\Phi(\cdot)$ are the pdf and the cdf of the standard Normal distribution, respectively, is called a *Univariate Skew Normal* (USN) random variable with shape parameter β . If $\beta = 0$, $f_X(x) = \phi(x)$ and if $\beta \rightarrow \pm\infty$, the USN density approaches the distribution of \pm the absolute value of the standard Normal distribution. It is a natural extension of the family of Normal distributions to situations where the assumption of symmetry is quite unrealistic. Despite the absence of symmetry, many statistical properties of the Normal distribution can be verified for the Skew Normal distribution. The density of the USN random variable is exemplified in Figure 12.

Many extensions of the Univariate Skew Normal distribution to the multivariate case have been suggested by different authors (among others, see AZZALINI (1985) and AZZALINI & DALLA VALLE (1996)). The definition retained in this paper comes from GUPTA & CHEN (2004) and DENIZ (2009).

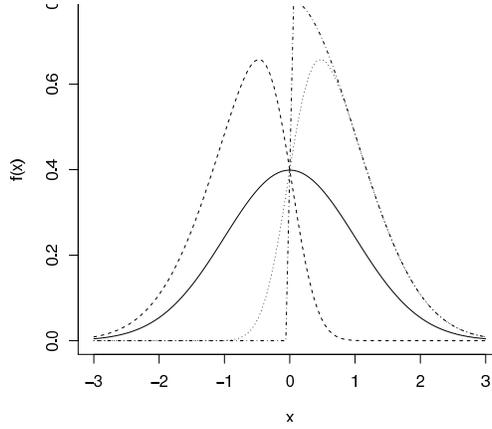


Figure 12: Univariate Skew Normal distribution with shape parameter $\beta = 0$ (solid line), $\beta = -3$ (dashed line), $\beta = 3$ (dotted line) and $\beta = 10,000,000$ (dot-dashed line).

Let $\boldsymbol{\mu} = [\mu_1 \ \dots \ \mu_k]'$ be a location parameter, $\boldsymbol{\Sigma}$ be a $(k \times k)$ positive definite symmetric scale matrix, $\boldsymbol{\beta} = [\beta_1 \ \dots \ \beta_k]'$ be a shape parameter. The $(k \times 1)$ random vector \mathbf{X} follows a *Multivariate Skew Normal* (MSN) distribution if its density function is of the form

$$MSN(\mathbf{X}; \boldsymbol{\mu}, \boldsymbol{\Sigma}^{1/2}, \boldsymbol{\beta}) = \frac{2^k}{\det(\boldsymbol{\Sigma})^{1/2}} \phi_k(\boldsymbol{\Sigma}^{-1/2}(\mathbf{X} - \boldsymbol{\mu})) \times \prod_{j=1}^k \Phi(\beta_j \mathbf{e}_j' \boldsymbol{\Sigma}^{-1/2}(\mathbf{X} - \boldsymbol{\mu})),$$

where $\phi_k(\cdot)$ denotes the pdf of the k -variate standard Normal distribution, $\Phi(\cdot)$ denotes the cdf of the Univariate standard Normal distribution and \mathbf{e}_j' are the elementary vectors of the coordinate system \mathbb{R}^k . One should note that $\boldsymbol{\Sigma}$ is not the usual variance-covariance matrix as in the Multivariate Normal distribution. A MSN random vector is defined by $\boldsymbol{\Sigma}^{1/2}$ in place of $\boldsymbol{\Sigma}$ because of the plurality of the square roots of $\boldsymbol{\Sigma}$. Without subscript, $\boldsymbol{\Sigma}^{1/2}$ designs the symmetric square root such that $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{1/2} (\boldsymbol{\Sigma}^{1/2})'$, where $\boldsymbol{\Sigma}^{1/2}$ is a symmetric matrix.

This model can properly represent the dependence in the data as illustrated in Figures 7 and 8. Moreover, unlike the Multivariate Normal

distribution, the MSN distribution can take into account the asymmetric structure of the data.

Distributions for number of periods. For the reporting delay random variable, a mixture of a Geometric distribution with few degenerate distributions for notification during the first periods is considered, as suggested in ANTONIO & PLAT (2011). Moreover, a Geometric distribution is often used to model tail factors in loss reserving models. The distribution is given by

$$f_1(t; \boldsymbol{\nu}) = \sum_{i=0}^p \nu_i I_i(t) + \left(1 - \sum_{i=0}^p \nu_i\right) f_{T|T>p}(t), \quad (2)$$

where $I_i = 1$ for the i^{th} period after occurrence time and $I_i = 0$ otherwise and $f(t)$ is the conditional probability mass function of the Geometric distribution with parameter ν_{p+1} . Similar distributions are assumed for the number of periods between the notification and the first payment, for the number of periods with payment after the first one and for the number of period(s) between two payments.

4.2 Estimation Results.

The model is fitted separately for both classes: Material Damage and Bodily Injury. Data manipulations and likelihood optimization are performed with R (using additional packages, among which package `ChainLadder` for Mack's model and package `sn` for Skew Normal distribution). Packages are available from the CRAN website.

Occurrence of claims. The parameter of the distribution of the random variable K (occurrence of claim) is estimated¹ for each class and results are $\hat{\theta}_{BI} = 0.7459$ (s.e. 0.016) and $\hat{\theta}_{MD} = 38.89$ (s.e. 0.112).

Development pattern. For the logarithm of the severity of the first and only payment, a univariate Skew Normal distribution is fitted by maximum likelihood and estimation results are presented in Table 2. A graphical comparison is presented in Figure 13. In this example, the shape pa-

¹Actually, the exposure measure used in the estimation process is $w(i)^* = w(i)/1000$.

parameter (α) is unnecessary and a Normal distribution could be used instead.

Parameter	Bodily Injury (s.e.)	Material Damage (s.e.)
μ	5.9168 (1.39)	4.9971 (0.18)
σ	1.3961 (0.02)	1.1636 (0.01)
α	-0.0098 (1.25)	0.0313 (0.19)

Table 2: Estimation results for the logarithm of the severity of the first payment in case there is only one payment.

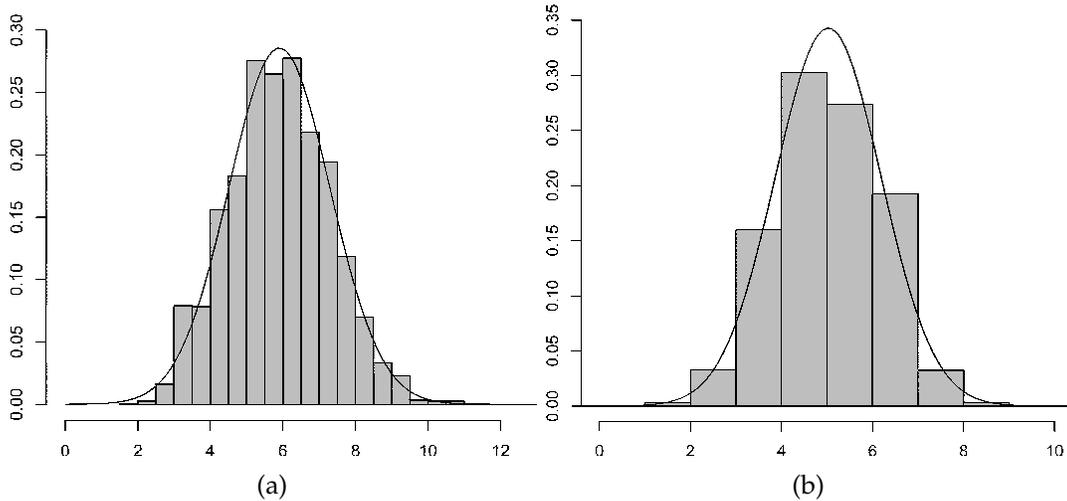


Figure 13: Histograms of severity of the first and only payment (on log scale) and the density fitted using the Skew Normal distribution for Bodily Injury (left) and Material Damage (right).

For the logarithm of the severity of the first payment when there is more than one payment, optimization has to be performed for each value of the random variable U in each class of risk. Results are presented in Tables 3 and 4. For the MSN distribution, several authors have reported problems with maximum likelihood estimation, e.g. PEWSEY (2000) and AZZALINI & CAPITANIO (1999). To deal with these problems, many methods have been proposed (see DENIZ (2009)). In this paper, the *maximum*

products of spacings estimation technique is used. More details are available in Appendix B.

Given the low frequency of Bodily Injury claims with more than 5 payments (23 claims) and the low frequency of Material Damage claims with more than 3 payments (6 claims), an annual tail factor of 20% is used to model the tail of the development. In the example, the impact of this tail factor is negligible because of the low probabilities for the number of payments to be larger than these values (respectively 0.004 and 0.0000002).

Distributions for number of periods. For discrete random variables $\{T\}$, $\{Q\}$, $\{U\}$ and $\{N\}$, estimation results are presented in Table 5. As defined in equation (2), a mixture of $p + 1$ degenerate distributions with a Geometric distribution is used in each case. $p + 2$ parameters have to be estimated for each variable. Observed and estimated results for the reporting delay, the first payment delay and for the number of partial payments are compared in Figures 9, 10 and 11.

5 Prediction Results.

Tables 6 and 7 summarize the database by arrival year and development year in run-off triangles. In the data set, information for years 2005 to 2009 (August) is available but not used in the analysis to enable out-of-sample prediction. Observed values for these years are presented in bold in run-off triangles and sums are given in Table 8. In order to make comparisons, the classical Mack model is fitted to both classes of risk and results are presented in Table 8. Note that the run-off triangles reported here are different from those used in ANTONIO & PLAT (2011). These differences arise from the definition of the risk class, i.e. ANTONIO & PLAT (2011) defines any claim with at least one BI payment, to be a Bodily Injury. The present paper allows a claim to have both a BI and a MD part.

Predictions for the total reserve are obtained by the sum of the components detailed in the following two Sections: IBNR and RBNP reserves on the one hand and RBNS reserves on the other hand. Note that the simulation approach presented here does not incorporate parameter uncertainty. In a future work the model will be adjusted to take this source of uncertainty into account. However, to the best of our knowledge no straightforward method is available to calculate standard errors corresponding with

parameters estimated in the MSN distribution. In line with the findings in ANTONIO & PLAT (2011) it can be anticipated that the inclusion of parameter uncertainty will have a minor impact on the variability of the reserves, given that the statistical model is estimated on a large data set.

5.1 Prediction of the IBNR and RBNP reserves.

For each occurrence period, the number of Incurred But Not Reported (IBNR) claims in each class of risk is simulated by using a Poisson distribution with occurrence measure given by

$$\hat{\theta}w(i)(1 - F_1(t_i^* - 1; \hat{\nu})).$$

Then, for each IBNR claim, say (ik) , the number of period(s) with partial payments $\{U_{ik}\}$ and the corresponding development pattern $\Lambda^{(ik)}$ are simulated. Finally, the IBNR reserve is calculated. To know the timing of the partial payments, each random variable, $\{T_{ik}\}$, $\{Q_{ik}\}$ and $\{N_{ikj}\}$, has to be simulated.

The prediction routine for the RBNP reserve is similar to the routine for the prediction of the IBNR reserve except for the first step. Actually, RBNP claims are reported claims and there is no need to simulate the number of reported but not paid claims for each occurrence period. Moreover, the variable $\{Q_{ik}\}$ should be simulated from a truncated distribution, using the condition $Q_{ik} > t_{ik}^* - t_{ik} - 1$.

10,000 simulations are performed. Numerical results are presented in Table 8 and graphical results are presented in Figure 14.

5.2 Prediction of the RBNS reserve.

The prediction routine for the RBNS reserve begins with the simulation of the number of period(s) with payment by using the conditional probability function $f_3(u|u \geq u^*)$ where u^* is the observed number of period(s) with payment after the first one. Then, the missing part of the development pattern is simulated using the conditional distribution of the MSN distribution (see result (ii) in the second Theorem presented in the Appendix A). Finally, the RBNS reserve is evaluated.

10,000 simulations are performed: numerical results are presented in Table 8 and graphical results are presented in Figure 15.

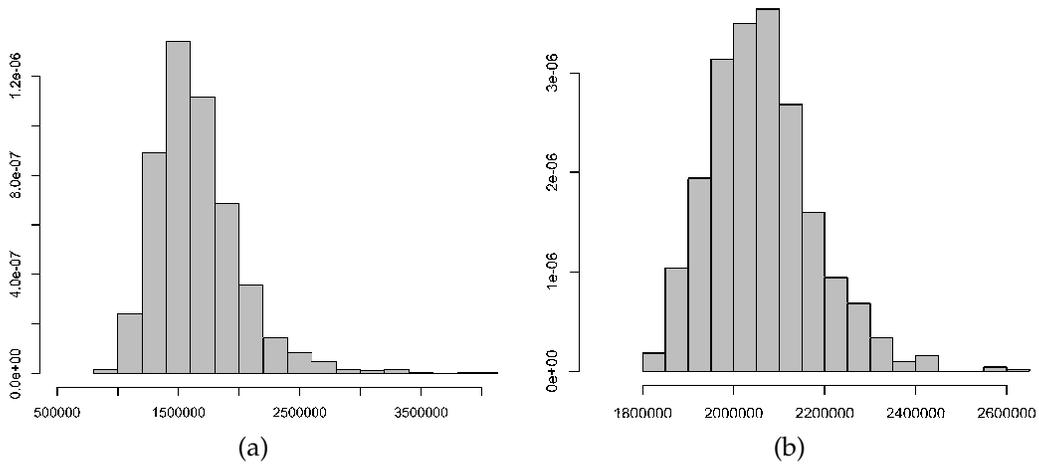


Figure 14: Histograms of the reserve obtained for IBNR and RBNP claims with the individual model for Bodily Injury (left) and Material Damage (right).

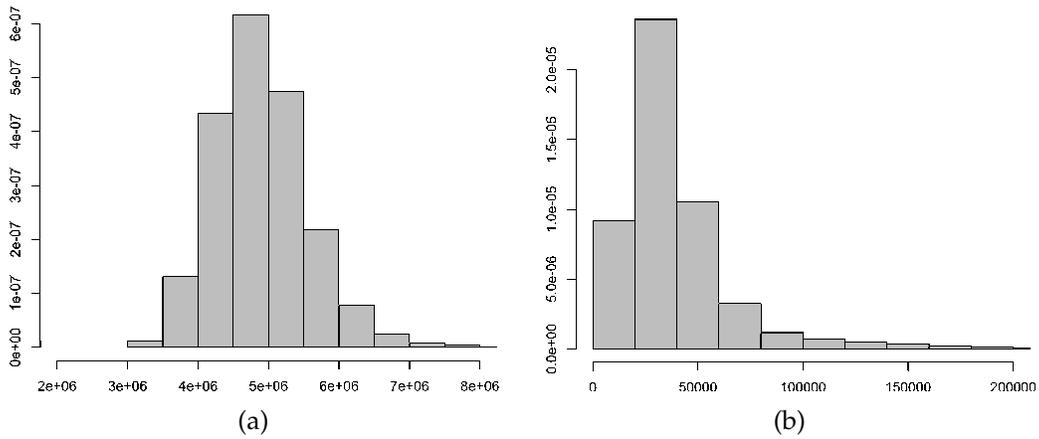


Figure 15: Histograms of the reserve obtained for RBNS claims with the individual model for Bodily Injury (left) and Material Damage (right).

5.3 Discussion.

Table 8 summarizes results from both chain-ladder model and individual MSN model. S.E. is the process standard error obtained from the simula-

tions. In addition, the Values-at-Risk² with 95% and 99.5% confidence levels have been calculated. Lognormal distributions with expected values and variances as obtained from Mack’s chain-ladder model have been fitted to calculate the VaRs for Mack’s model. A graphical comparison is performed in Figure 16. For short-term and long-term insurance (MD and BI

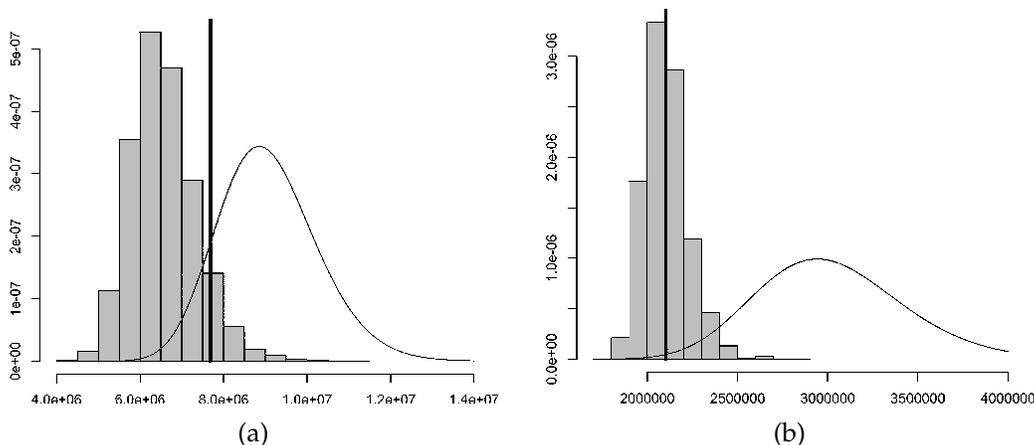


Figure 16: Histograms of the total reserve obtained with the individual model and the Lognormal density fitted on chain-ladder model for Bodily Injury (left) and Material Damage (right). Vertical lines represent the observed total payment for years 2005 to 2009 (August).

class respectively), best estimates (expected values) of the reserve appear to be over-evaluated by the chain-ladder model when compared to the MSN individual model. Moreover, the MSN individual model presents a lower variance than the chain-ladder model in both cases.

In each class, comparisons between the distribution of the total reserve and the observed total amount, i.e. the total reserve up to the three unknown years in the data set, are performed. Given that the lower triangle is almost complete, the observed total payment in the lower triangle is probably very close to the corresponding unobserved run-off. Note that

²The Value-at-Risk, or VaR is a well-known risk measure. In broad terms, the α -VaR represents the loss that, with probability α will not be exceeded. Since that may not define a unique value, for example if there is a probability mass around the value, the α -VaR can be defined more specifically, for $0 \leq \alpha \leq 1$, as $\min\{Q : \Pr = (L \leq Q) \geq \alpha\}$, where L is the loss random variable. For continuous distributions this simplifies to Q such that $\Pr(L \leq Q) = \alpha$.

the BI run-off triangle in Table 6 shows a very large total payment in occurrence year 2002, development period 8. This is caused by a single, extreme payment of 779,398 euro. The MSN individual model reflects this appropriately, i.e. the observed total amount is in the right tail of the predictive distribution obtained from the model. Thus, for the case-study under consideration, the individual model leads to a more realistic predictive distribution of the reserve than the one obtained from the chain-ladder model.

6 Conclusions.

The present paper proposed a discrete-time individual reserving model inspired from the well-known chain-ladder approach. The case study performed on a general liability insurance portfolio of an insurance company operating in the EU demonstrated the usefulness of the modeling of individual claim developments. Risk measures, including VaR, are easily calculated and can be used for solvency evaluations. Also, the impact of reinsurance treaties can be effectively taken into account using our individual model.

Several directions for future researches can be envisaged. The simulation approach used in Section 5 did not include parameter uncertainty, and could be adjusted for this source of uncertainty. The modeling of the first payment could be refined and adjusted to large losses, using the Lognormal-Pareto distribution (see PIGEON & DENUIT (2011)). More precise modeling of inflation effects and including the “time value of money” will be of importance in future researches. Studying the approach in light of the new solvency guidelines, is another path to be explored, as well as extending the model to the reinsurance industry.

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Appendix A.

Moments of the Multivariate Skew Normal distribution.

The moment-generating function of \mathbf{X} evaluated at \mathbf{t} ($k \times 1$) is

$$M(\mathbf{t}) = 2^k \exp\left(\mathbf{t}'\boldsymbol{\mu} + 0.5\mathbf{t}'\boldsymbol{\Sigma}^{1/2}(\boldsymbol{\Sigma}^{1/2})'\mathbf{t}\right) \prod_{j=1}^k \Phi\left(\frac{\beta_j \left((\boldsymbol{\Sigma}^{1/2})'\mathbf{t}\right)_j}{\sqrt{1 + \beta_j^2}}\right).$$

The expectation vector is given by

$$E(\mathbf{X}) = \sqrt{\frac{2}{\pi}}\boldsymbol{\Sigma}^{1/2} \begin{bmatrix} \frac{\beta_1}{\sqrt{1+\beta_1^2}} \\ \dots \\ \frac{\beta_k}{\sqrt{1+\beta_k^2}} \end{bmatrix} + \boldsymbol{\mu},$$

and the covariance matrix is given by

$$\text{Cov}(\mathbf{X}) = \boldsymbol{\Sigma}^{1/2} \left(I_k - \frac{2}{\pi} \text{diag} \left(\frac{\beta_1^2}{1 + \beta_1^2}, \dots, \frac{\beta_k^2}{1 + \beta_k^2} \right) \right) (\boldsymbol{\Sigma}^{1/2})'.$$

Properties of the Multivariate Skew Normal distribution.

Following proofs are based on DENIZ (2009).

Theorem 1: Closure under linear transformation.

Assume that the random vector \mathbf{X} follows a Multivariate Skew Normal distribution with location parameter $\boldsymbol{\mu}$ ($k \times 1$), scale parameter $\boldsymbol{\Sigma}^{1/2}$ ($k \times k$) and shape parameter $\boldsymbol{\beta}$ ($k \times 1$) and define $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$ with \mathbf{A} a ($k \times k$) matrix and \mathbf{b} a ($k \times 1$) real vector. Then \mathbf{Y} follows a MSN distribution with parameters $\mathbf{A}\boldsymbol{\mu} + \mathbf{b}$, $\mathbf{A}\boldsymbol{\Sigma}^{1/2}$ and $\boldsymbol{\beta}$.

Proof. The moment-generating function of \mathbf{Y} evaluated at $\mathbf{t} \in \mathbb{R}^k$ is

$$\begin{aligned} M_{\mathbf{Y}}(\mathbf{t}) &= \mathbb{E} \left(e^{\mathbf{t}'\mathbf{Y}} \right) \\ &= e^{\mathbf{t}'\mathbf{b}} M_{\mathbf{X}}(\mathbf{A}'\mathbf{t}) \\ &= 2^k e^{\mathbf{t}'(\mathbf{A}\boldsymbol{\mu}+\mathbf{b})+0.5\mathbf{t}'(\mathbf{A}\boldsymbol{\Sigma}^{1/2})(\mathbf{A}\boldsymbol{\Sigma}^{1/2})'\mathbf{t}} \\ &\quad \times \prod_{j=1}^k \Phi \left(\frac{\beta_j \left((\mathbf{A}\boldsymbol{\Sigma}^{1/2})'\mathbf{t} \right)_j}{\sqrt{1 + \beta_j^2}} \right), \end{aligned}$$

which is the moment-generating function of a MSN random vector with parameters $\mathbf{A}\boldsymbol{\mu} + \mathbf{b}$, $\mathbf{A}\boldsymbol{\Sigma}^{1/2}$ and $\boldsymbol{\beta}$. \square

Lemma 1

Let \mathbf{X} be a $(k \times 1)$ random vector following a Multivariate Skew Normal distribution with parameters $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}^{1/2}$ and $\boldsymbol{\beta}$ as defined previously. Let

$$\begin{aligned} \mathbf{X} &= \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \quad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix} \quad \boldsymbol{\beta} = \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix} \\ \boldsymbol{\Sigma}^{1/2} &= \begin{bmatrix} \boldsymbol{\Sigma}_{11}^{1/2} & \boldsymbol{\Sigma}_{12}^{1/2} \\ \boldsymbol{\Sigma}_{21}^{1/2} & \boldsymbol{\Sigma}_{22}^{1/2} \end{bmatrix}, \end{aligned}$$

where \mathbf{X}_1 , $\boldsymbol{\beta}_1$ and $\boldsymbol{\mu}_1$ are $l \times 1$ ($l < k$) and $\boldsymbol{\Sigma}_{11}^{1/2}$ is $l \times l$. If $\boldsymbol{\Sigma}_{12}^{1/2} = \mathbf{0}$ and $\boldsymbol{\Sigma}_{21}^{1/2} = \mathbf{0}$ then $\mathbf{X}_1 \sim \text{MSN}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}^{1/2}, \boldsymbol{\beta}_1)$ is independant of $\mathbf{X}_2 \sim \text{MSN}(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22}^{1/2}, \boldsymbol{\beta}_2)$.

Proof. If $\boldsymbol{\Sigma}_{12}^{1/2} = \mathbf{0}$ and $\boldsymbol{\Sigma}_{21}^{1/2} = \mathbf{0}$, then

$$\boldsymbol{\Sigma}^{-1/2} = \begin{bmatrix} \boldsymbol{\Sigma}_{11}^{-1/2} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{22}^{-1/2} \end{bmatrix}.$$

The quadratic form in the exponent of the MSN distribution can be expressed as

$$\begin{aligned} (\mathbf{X} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu}) &= (\mathbf{X}_1 - \boldsymbol{\mu}_1)'\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{X}_1 - \boldsymbol{\mu}_1) \\ &\quad + (\mathbf{X}_2 - \boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{X}_2 - \boldsymbol{\mu}_2) \end{aligned}$$

and

$$\begin{aligned} \prod_{j=1}^k \Phi \left(\beta_j \mathbf{e}'_j \Sigma^{-1/2} (\mathbf{X} - \boldsymbol{\mu}) \right) &= \prod_{j=1}^l \Phi \left(\beta_j \mathbf{e}'_j \Sigma_{11}^{-1/2} (\mathbf{X}_1 - \boldsymbol{\mu}_1) \right) \\ &\quad \times \prod_{j=l+1}^k \Phi \left(\beta_j \mathbf{e}'_j \Sigma_{22}^{-1/2} (\mathbf{X}_2 - \boldsymbol{\mu}_2) \right) \end{aligned}$$

Moreover,

$$\det(\Sigma) = \det(\Sigma_{11}) \det(\Sigma_{22}).$$

So, the probability density function of \mathbf{X} can be written as

$$\begin{aligned} \text{MSN} \left(\boldsymbol{\mu}, \Sigma^{1/2}, \boldsymbol{\beta} \right) &= \frac{2^l}{\det(\Sigma_{11})^{1/2}} \phi_l \left(\Sigma_{11}^{-1/2} (\mathbf{X}_1 - \boldsymbol{\mu}_1) \right) \\ &\quad \times \prod_{j=1}^l \Phi \left(\beta_j \mathbf{e}'_j \Sigma_{11}^{-1/2} (\mathbf{X}_1 - \boldsymbol{\mu}_1) \right) \\ &\quad \times \frac{2^{k-l}}{\det(\Sigma_{22})^{1/2}} \phi_{k-l} \left(\Sigma_{22}^{-1/2} (\mathbf{X}_2 - \boldsymbol{\mu}_2) \right) \\ &\quad \times \prod_{j=l+1}^k \Phi \left(\beta_j \mathbf{e}'_j \Sigma_{22}^{-1/2} (\mathbf{X}_2 - \boldsymbol{\mu}_2) \right) \\ &= \text{MSN} \left(\boldsymbol{\mu}_1, \Sigma_{11}^{1/2}, \boldsymbol{\beta}_1 \right) \text{MSN} \left(\boldsymbol{\mu}_2, \Sigma_{22}^{1/2}, \boldsymbol{\beta}_2 \right). \end{aligned}$$

□

Theorem 2: Conditional distribution.

Let \mathbf{X} be a $(k \times 1)$ random vector following a Multivariate Skew Normal distribution with parameters $\boldsymbol{\mu}$, $\Sigma_c^{1/2}$ and $\boldsymbol{\beta}$. $\Sigma_c^{1/2}$ is the square root of the matrix Σ by Cholesky decomposition³. Let

$$\begin{aligned} \mathbf{X} &= \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} & \boldsymbol{\mu} &= \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix} \\ \Sigma_c^{1/2} &= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} & \boldsymbol{\beta} &= \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix}, \end{aligned}$$

³If Σ is a positive definite symmetric matrix of size k , there exists a unique lower triangular matrix \mathbf{L} with positive diagonal elements such that $\Sigma = \mathbf{L}\mathbf{L}'$. This is called the Cholesky decomposition of the matrix Σ and \mathbf{L} is the Cholesky square root of the matrix.

where \mathbf{X}_1 , β_1 and $\boldsymbol{\mu}_1$ are $l \times 1$ ($l < k$), \mathbf{A}_{11} is $l \times l$ lower triangular matrix with positive diagonal elements and \mathbf{A}_{22} is $(k - l) \times (k - l)$ lower triangular matrix with positive diagonal elements. Then (i) \mathbf{X}_1 has a Multivariate Skew Normal distribution with parameters $\boldsymbol{\mu}_1$, \mathbf{A}_{11} and β_1 and (ii) $(\mathbf{X}_2|\mathbf{X}_1 = \mathbf{x}_1)$ has a Multivariate Skew Normal distribution with parameters $\boldsymbol{\mu}_2 + \mathbf{A}_{21}\mathbf{A}_{11}^{-1}(\mathbf{x}_1 - \boldsymbol{\mu}_1)$, \mathbf{A}_{22} and β_2 .

Proof. Let

$$\mathbf{C} = \begin{bmatrix} \mathbf{1}_l & \mathbf{0} \\ -\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{1}_{k-l} \end{bmatrix}$$

and consider the random vector

$$\mathbf{Y} = \mathbf{C}\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{Y}_2 \end{bmatrix}.$$

The random vector \mathbf{Y} follows a MSN distribution with parameters $\mathbf{C}\boldsymbol{\mu}$, $\mathbf{C}\boldsymbol{\Sigma}^{1/2}$ and β . From Lemma 1, \mathbf{X}_1 is independant of \mathbf{Y}_2 which both belong in the MSN family. The joint distribution of \mathbf{Y} is

$$\text{MSN}(\boldsymbol{\mu}_1, \mathbf{A}_{11}, \beta_1) \text{MSN}(\boldsymbol{\mu}_2 - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\boldsymbol{\mu}_1, \mathbf{A}_{22}, \beta_2).$$

Note that $\mathbf{X}_2 = \mathbf{Y}_2 + \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{X}_1$ and by Theorem 1, the distribution of $(\mathbf{X}_2|\mathbf{X}_1)$ is MSN with parameters $\boldsymbol{\mu}_2 + \mathbf{A}_{21}\mathbf{A}_{11}^{-1}(\mathbf{x}_1 - \boldsymbol{\mu}_1)$, \mathbf{A}_{22} and β_2 . \square

Appendix B.

Stochastic representation of the Univariate Skew Normal distribution.

The USN distribution has a convenient stochastic representation which will be useful for simulations. Let X_i and Y_i be two independent random variables with standard Normal distributions. For $\beta_i \in \mathbb{R}$, the random variable

$$Z_i = \left(\frac{\beta_i}{\sqrt{1 + \beta_i^2}} \right) |X_i| + \left(\frac{1}{\sqrt{1 + \beta_i^2}} \right) Y_i \quad (3)$$

follows a Univariate Skew Normal distribution with shape parameter β_i (a proof can be found in DENIZ (2009)). The random vector $\mathbf{Z} = [Z_1 \ \dots \ Z_k]'$

has a MSN distribution with $\mathbf{0}$ location vector, identity scale matrix and $\boldsymbol{\beta} = [\beta_1 \ \dots \ \beta_k]'$ shape parameter. By using Theorem 1, a MSN random vector with $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}^{1/2}$ and $\boldsymbol{\beta}$ can be obtained.

The maximum products of spacings estimation technique.

Let $X = \{x_{(1)}, \dots, x_{(n)}\}$ be an ordered random sample from the distribution $F_{\theta_0}(x)$ which belongs to a family $F = \{F_{\theta}(x) : \theta \in \Theta\}$ where the parameter θ may be vector-valued. $f_{\theta}(x)$ is the probability density function corresponding to $F_{\theta}(x)$. The maximum likelihood estimator arises from the maximization of the function

$$l(X; \theta) = \frac{1}{n} \sum_{i=1}^n \ln (f_{\theta}(x_{(i)})).$$

The maximum products of spacings estimator arises from the maximization of the function

$$sl(X; \theta) = \frac{1}{n+1} \sum_{i=1}^{n+1} \ln (F_{\theta}(x_{(i)}) - F_{\theta}(x_{(i-1)})),$$

with $F_{\theta}(x_0) = 0$ and $F_{\theta}(x_{(n+1)}) = 1$. Under very general conditions, both estimators are asymptotically equivalent (see CHENG & AMIN (1983)).

The following methodology is applied to estimate parameters $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$ and $\boldsymbol{\beta}$ of order k from a random sample $\mathbf{Y} = \{y_1, \dots, y_n\}$:

- a) Let $\boldsymbol{\beta}_0$ be a initial estimate for $\boldsymbol{\beta}$.
- b) Given an estimate of $\boldsymbol{\beta}$, estimate $\boldsymbol{\Sigma}_c^{1/2}$ and $\boldsymbol{\mu}$ by method of moments

$$\widehat{\boldsymbol{\Sigma}}_c^{1/2} = \mathbf{S}_c^{1/2} \left(\mathbf{I}_k - \left(\frac{2}{\pi} \right) \text{diag} \left(\frac{\beta_1^2}{1 + \beta_1^2}, \dots, \frac{\beta_k^2}{1 + \beta_k^2} \right) \right)^{-1/2}$$

$$\widehat{\boldsymbol{\mu}} = \bar{\mathbf{Y}} - \widehat{\boldsymbol{\Sigma}}_c^{1/2} \begin{bmatrix} \frac{\sqrt{2}\beta_1}{\sqrt{\pi}\sqrt{1+\beta_1^2}} \\ \dots \\ \frac{\sqrt{2}\beta_k}{\sqrt{\pi}\sqrt{1+\beta_k^2}} \end{bmatrix}$$

- c) Given an estimate of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, define $\mathbf{X}_i = \boldsymbol{\Sigma}_c^{-1/2}(\mathbf{Y}_i - \boldsymbol{\mu})$ with *univariate* i.i.d. distributions $\text{MSN}(0, 1, \beta_j)$, $j = 1, \dots, k$ and estimate components β_j of $\boldsymbol{\beta}$ by maximum products of spacings estimation.

d) Repeat b) and c) until convergence.

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Number of pmt after the first one (U)	Location par. (μ)	Scale par. (Σ)	Shape par. (β)
$U = 1$	$\mu_1 = 5.6019$	$\sigma_{11} = 2.1722$	$\beta_1 = 0.2134$
	$\mu_2 = 0.5664$	$\sigma_{22} = 2.4442$	$\beta_2 = 2.5263$
		$\sigma_{12} = -0.9495$	
$U = 2$	$\mu_1 = 6.3406$	$\sigma_{11} = 2.4008$	$\beta_1 = 0.0056$
	$\mu_2 = 0.1885$	$\sigma_{22} = 1.8621$	$\beta_2 = 3.1828$
	$\mu_3 = 0.5845$	$\sigma_{33} = 1.9804$	$\beta_3 = 86.6838$
		$\sigma_{12} = -0.6288$	
		$\sigma_{13} = -0.6865$	
$U = 3$	$\mu_1 = 4.4631$	$\sigma_{11} = 7.2241$	$\beta_1 = 38.7554$
	$\mu_2 = 0.9787$	$\sigma_{22} = 2.4592$	$\beta_2 = 2.2285$
	$\mu_3 = 0.4082$	$\sigma_{33} = 1.0268$	$\beta_3 = 4.0688$
	$\mu_4 = 1.1512$	$\sigma_{44} = 0.4742$	$\beta_4 = 0.0384$
		$\sigma_{12} = -2.7370$	
		$\sigma_{13} = -0.8189$	
		$\sigma_{14} = 0.0998$	
		$\sigma_{23} = -0.1371$	
		$\sigma_{24} = 0.0145$	
		$\sigma_{34} = -0.2451$	
$U = 4$	$\mu_1 = 6.9951$	$\sigma_{11} = 2.4487$	$\beta_1 = -0.0219$
	$\mu_2 = 1.1287$	$\sigma_{22} = 0.9936$	$\beta_2 = 0.0505$
	$\mu_3 = -0.1187$	$\sigma_{33} = 0.4695$	$\beta_3 = 5.7102$
	$\mu_4 = -0.0980$	$\sigma_{44} = 0.4383$	$\beta_4 = 7.3448$
	$\mu_5 = 1.3499$	$\sigma_{55} = 0.5898$	$\beta_5 = 0.0007$
		$\sigma_{12} = -0.7031$	
		$\sigma_{13} = -0.2317$	
		$\sigma_{14} = -0.1836$	
		$\sigma_{15} = -0.4310$	
		$\sigma_{23} = -0.0451$	
		$\sigma_{24} = 0.0173$	
		$\sigma_{25} = -0.0379$	
		$\sigma_{34} = -0.0024$	
	$\sigma_{35} = -0.1412$		
	$\sigma_{45} = -0.2273$		

Table 3: Estimation results for logarithms of development factors for Bodily Injury claims with more than one period with payment.

Number of pmt after the first one (U)	Location par. (μ)	Scale par. (Σ)	Shape par. (β)
$U = 1$	$\mu_1 = 5.7733$	$\sigma_{11} = 4.0848$	$\beta_1 = 4.8548$
	$\mu_2 = 1.2253$	$\sigma_{22} = 2.2893$	$\beta_2 = 2.5429$
		$\sigma_{12} = -2.1815$	
$U = 2$	$\mu_1 = 6.0698$	$\sigma_{11} = 2.6104$	$\beta_1 = 0.0187$
	$\mu_2 = 1.8193$	$\sigma_{22} = 3.0916$	$\beta_2 = 0.0590$
	$\mu_3 = 0.7243$	$\sigma_{33} = 0.5071$	$\beta_3 = -0.0004$
		$\sigma_{12} = -1.9334$	
		$\sigma_{13} = -0.8542$	
		$\sigma_{23} = 0.6764$	

Table 4: Estimation results for logarithms of development factors for Material Damage claims with more than one period with payment.

Class	i	Report delay ($T; \nu_i$) (s.e.)	First pmt delay ($Q; \psi_i$) (s.e.)	Number partial pmt ($U; \delta_i$) (s.e.)	Time between pmt ($N; \phi_i$) (s.e.)
	0	0.8961 (0.005)	0.7273 (0.007)	0.6409 (0.008)	0.8825 (0.009)
	1	0.0774 (0.004)	0.2384 (0.007)	0.5947 (0.012)	0.5604 (0.021)
BI	2	0.0139 (0.002)	0.0225 (0.002)	-	-
	3	0.3585 (0.010)	0.5972 (0.090)	-	-
		$p = 2$	$p = 2$	$p = 0$	$p = 0$
	0	0.9626 (< 0.001)	0.9298 (< 0.001)	0.9908 (< 0.001)	0.9857 (< 0.001)
	1	0.0362 (< 0.001)	0.0680 (< 0.001)	0.9754 (0.004)	0.268 (0.003)
MD	2	0.0008 (< 0.001)	0.0015 (< 0.001)	-	-
	3	0.7263 (0.030)	0.5785 (0.024)	-	-
		$p = 2$	$p = 2$	$p = 0$	$p = 0$

Table 5: Estimation results for a Geometric distribution, combined with degenerate components.

Arrival year	Development year							
	1	2	3	4	5	6	7	8
1997	261	614	359	526	546	137	130	339
1998	202	473	307	336	269	56	179	78
1999	238	569	393	270	249	286	132	97
2000	237	557	429	496	406	365	247	275
2001	389	628	529	559	446	375	147	239
2002	260	570	533	444	132	122	332	1,082
2003	236	743	558	237	217	205	171	
2004	248	794	401	236	254	98		

Table 6: Incremental run-off triangle for the Bodily Injury class (in thousands).

Arrival year	Development year							
	1	2	3	4	5	6	7	8
1997	4,427	992	89	13	39	27	37	11
1998	4,389	984	60	35	76	24	0.5	16
1999	5,280	1,239	76	110	113	12	0.4	0
2000	5,445	1,164	172	16	6	10	0	10
2001	5,612	1,838	156	127	13	3	0.4	3
2002	6,593	1,592	74	71	17	15	9	9
2003	6,603	1,660	150	52	37	18	3	
2004	7,195	1,417	109	86	39	15		

Table 7: Incremental run-off triangle for the Material Damage class (in thousands).

Model	Class	Expected value	S.E.	VaR _{0.95}	VaR _{0.995}
Chain-ladder model	BI	9,082,114	1,184,546	11,150,686	12,583,834
	MD	3,024,375	411,507	3,744,588	4,247,807
Ind. MSN (IBNR + RBNP)	BI	1,666,000	389,124	2,330,317	3,437,898
	MD	2,062,000	115,335	2,272,556	2,411,647
Ind. MSN (RBNS)	BI	4,895,000	668,333	6,055,026	7,110,197
	MD	40,430	34,065	93,585	234,049
Ind. MSN (TOTAL)	BI	6,560,000	779,120	7,918,944	9,152,913
	MD	2,102,000	120,109	2,318,484	2,480,923
Observed Payment	BI	7,684,000	-	-	-
	MD	2,102,800	-	-	-

Table 8: Estimated reserves. The reserves “Ind. MSN (TOTAL)” and “Chain-ladder” are both for the complete lower triangle including missing cells. The “Observed Payment” is only for years 2004 to 2009 (August).