

# Generic Insecurity of Cliques-Type Authenticated Group Key Agreement Protocols

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## Abstract

The A-GDH.2 and SA-GDH.2 authenticated group key agreement protocols showed to be flawed in 2001. Even though the corresponding attacks (or some variants of them) have been rediscovered in several different frameworks, no fixed version of these protocols has been proposed until now.

In this paper, we prove that it is in fact impossible to design a scalable authenticated group key agreement protocol based on the same design assumptions as the A-GDH ones. We proceed by providing a systematic way to derive an attack against any A-GDH-type protocol with at least four participants and exhibit protocols with two and three participants which we cannot break using our technique. As far as we know, this is the first generic insecurity result reported in the literature concerning authentication protocols.

## 1 Introduction

The A-GDH.2 and SA-GDH.2 authenticated group key agreement protocols [1, 2], which are part of the Cliques protocol suites, have been shown to be flawed in 2001 [17, 18]. Even though the corresponding attacks (or some variants of them) have been rediscovered in several different frameworks (using the Casper tool [5], rank functions [6] or the constraint solving approach [13] for instance), no fixed version of these protocols has been proposed until now.

As we tried to design such fixes, i.e. authenticated group key agreement protocols built from the same ingredients as the A-GDH protocols, we found

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that the method we proposed in [18] could always be used to find attacks against our candidates.

Actually, we prove in this paper that it is impossible to build a scalable authenticated group key agreement protocol using the technique adopted for the A-GDH protocols, that is, by constructing a group Diffie-Hellman key  $\alpha^{r_1 \dots r_n}$  through the exchange of partial group Diffie-Hellman values of form  $\alpha^{\prod r_i}$ , possibly exponentiated with long-term symmetric keys shared between the different group members. Our proof proceeds by providing a systematic procedure allowing the building of an attack against the implicit key authentication property for any protocol of the family we consider (provided that the protocol is executed by at least four principals). As far as we know, this is the first such impossibility result reported in the literature concerning authentication protocols.

In the next section, we will define the protocol family we want to analyze. The next step of our analysis, exposed in Section 3, will consist in the definition of several properties all protocols of our family must exhibit, mainly due to the fact that the different group members must be able to compute the same group key. The main result of this section will be the proof that the (secret) computation that each group member performs at the end of a protocol execution in order to obtain the group key can be written as the composition of computations executed by honest users during different protocol sessions, computations whose inputs and outputs can be eavesdropped.

This result does not however guarantee that the routing of the messages as given in the protocol definition will allow an active attacker to compose these computations as he would like to do: we will exhibit a three-party protocol for which we cannot use our Section 3 result to derive an attack. However, in Section 4, we will prove that it is possible to exploit that result in order to undermine the implicit key authentication property for at least one member of any protocol of our family provided that it is executed by at least four users.

## 2 The GDH-Protocols

### 2.1 Authenticated Group Key Agreement Protocols

The protocols of the family we consider, which we will call the *GDH-Protocols*, are group key agreement protocols.

**Definition 2.1** *A Group Key Agreement Protocol is a protocol enabling a group of  $n$  users  $M = \{M_1, \dots, M_n\}$  to contributively generate a key that should be known by all group members at the end of a protocol execution.*

It can be observed that the protocols we consider are different from the key exchange protocols usually analyzed through Dolev-Yao-type methods

(such as the Needham-Schroeder symmetric key protocol [14], the Otway-Rees protocol [15], the Woo and Lam protocol [22], for instance), because they are required to be contributive: no group member can choose the key in advance.

In the presence of an active attacker, i.e. an attacker who can intercept, reroute and send messages of his own, authentication properties are often required. The most usual authentication property, which is the one we will consider in this paper, is the implicit key authentication (IKA) [12].

**Definition 2.2** *A protocol expected to be executed by a group of users  $\mathbf{M}$  is said to achieve Implicit Key Authentication (IKA) if, when he completed his role in a session of the protocol, each  $M_i \in \mathbf{M}$  is assured that no party  $M_I \notin \mathbf{M}$  can learn the key  $S(M_i)$  (i.e.  $M_i$ 's view of the session key).*

The attacker is assumed to be a regular network user (and possibly to have several identities), and to be a regular member of some groups. His goal is then to obtain the key computed by honest users in groups from which he is expected to be excluded. We may observe that this property does not mean that all group members have any knowledge of a group key at the end of the protocol execution, nor that they agree on its value. These last properties could be achieved through key confirmation extensions of the protocols and will not be considered here.

Besides the IKA property, two other types of security properties are usually desirable: forward secrecy which guarantees that the compromise of long-term keys cannot result in the compromise of past session keys; and resistance to known session-secret attacks which guarantees that the compromise of old session-secrets cannot result in the compromise of future session keys [18]. We do not discuss these properties anymore and will only consider the IKA property in the rest of this paper.

## 2.2 The A-GDH.2 Protocol

A well-known example of authenticated group key agreement protocol is the A-GDH.2 protocol [1, 2] which we will use in order to provide intuitions about our attack methodology. The A-GDH.2 protocol is executed by a pool of users  $\mathbf{M}$  who agreed on performing all computations in an algebraic group  $\mathcal{G}$  of prime order  $q$ , group in which the Decisional Diffie-Hellman problem is believed to be hard (the subgroup of order  $q$  of  $\mathbb{Z}_p^*$  where  $p$  and  $q$  are large prime numbers can be chosen to this effect). All users also agree on the use of a specific generator  $\alpha$  of  $\mathcal{G}$ , and these two choices are public.

The authentication mechanism adopted in the A-GDH protocols relies on the assumption that each pair of users  $(M_i, M_j)$  share a long-term secret key  $K_{ij} \in \mathbb{Z}_q^*$ .

These assumptions and notations having been introduced, we now define the way the A-GDH.2 protocol is executed.

When starting a protocol execution, each group member  $M_i \in \mathbf{M}$  selects a random key contribution  $r_i \in \mathbb{Z}_q^*$ . The first group member,  $M_1$ , computes  $\alpha^{r_1}$  and sends  $\langle \alpha, \alpha^{r_1} \rangle$  to  $M_2$ . Then, all group members from  $M_2$  to  $M_{n-1}$  perform the following sequence of actions:  $M_i$  exponentiate the  $i$  elements of  $\mathcal{G}$  he received with  $r_i$ , inserts the last received element in the next to last position and sends the result to the next group member. Finally, for each but the last element of  $\mathcal{G}$  that  $M_n$  received, he exponentiates the  $i$ -th of them with  $r_n K_{in}$  and finally broadcasts the result. The messages transcript is then as follows.

**Protocol 1: A-GDH.2 Protocol**

**Round  $i$**  ( $1 \leq i < n$ ):

$$M_i \rightarrow M_{i+1} : \quad \left\{ \alpha^{\frac{r_1 \cdots r_i}{r_j}} \mid j \in [1, i] \right\}, \alpha^{r_1 \cdots r_i}$$

**Round  $n$ :**

$$M_n \rightarrow \text{All } M_i : \quad \left\{ \alpha^{\frac{r_1 \cdots r_n}{r_i} K_{in}} \mid i \in [1, n] \right\}$$

Upon receipt of the above, every  $M_i \in \mathbf{M}$  computes the group key as:

$$S_n(M_i) = \alpha^{\frac{r_1 \cdots r_n}{r_i} \cdot r_i \cdot K_{in}^{-1}} = \alpha^{r_1 \cdots r_n},$$

except  $M_n$  who computes:

$$S_n(M_n) = \alpha^{(r_1 \cdots r_{n-1}) \cdot r_n}$$

from the last element of  $\mathcal{G}$  he received during the  $(n - 1)$ -th round.

A typical run of this protocol with 3 participants is represented using the strand space notations in Fig. 1. In this figure,  $\rightarrow$ -arrows denote message transmission and  $\Rightarrow$ -arrows distinguish successive actions performed by a given protocol participant. Note that both horizontal arrows on the last line of this figure correspond to the broadcast sent by  $M_3$ .

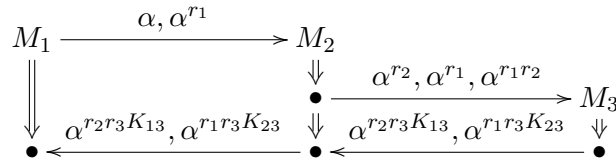


Figure 1: A-GDH.2 Protocol Run with 3 Participants

An important feature of this protocol is that all group members do not check anything about the sequences of elements of  $\mathcal{G}$  they receive (except that these sequences have the expected length) and, as a result, that the attacker can always generate messages with the structure expected by the different users. This is not the case for the classical authentication protocols

usually analyzed in Dolev-Yao-type models where messages are typically expected to be encrypted through keys the attacker does not know or to contain secret values such as nonces that the attacker should not be able to guess. In this protocol, and in all protocols we will consider in this paper, the key authentication relies on the fact that the values that the different group members are using for computing their view of the group key are not known by the attacker.

### 2.3 An Attack Against the A-GDH.2 Protocol

In order to provide intuitions regarding the systematic attack construction process we will describe further, we now describe an attack against the A-GDH.2 protocol when it is executed by three participants.

Let us consider an attacker whose identifier is  $M_I$  and who wants to undermine the IKA property by fooling  $M_2$  into accepting a key he knows in a session  $M_2$  thinks he is executing with  $M_1$  and  $M_3$ . Since  $M_2$  computes his view of the group key by exponentiating the second term of  $M_3$ 's final broadcast with  $r_2K_{23}^{-1}$ , the goal of the intruder consists in obtaining a pair of elements of the form  $(\alpha^x, \alpha^{xr_2K_{23}^{-1}})$  and in replacing the second term of that broadcast with  $\alpha^x$  so that  $M_2$  will compute  $\alpha^{xr_2K_{23}^{-1}}$  as group key.

The attacker can obtain such a pair by using the protocol participants as oracles, that is, by exploiting the services they provide. We call *service* a computation carried out by a honest user during a protocol execution; computation of which the input and result can be eavesdropped by the intruder. In most cases, the intruder will furthermore be able to exploit these services in a more efficient way: he will be able to replace services' input with a value of his own choice and then to obtain the result of these services provided for this chosen input. All services provided during an A-GDH.2 protocol execution are (modular) exponentiations. As it does not cause any ambiguity, we will therefore call a service consisting in exponentiating an element  $\alpha^x$  with a value  $s$  as providing the  $s$ -service. If we look at the protocol execution described in Fig. 1, we may observe that  $M_1$  provides the  $r_1$ -service, that  $M_2$  provides the  $r_2$ -service, and that  $M_3$  provides the  $r_3K_{13}$ - and  $r_3K_{23}$ -services.

Let us now consider a second protocol session executed by  $M_I$ ,  $M_2$  and  $M_3$ . If we use  $r'_i$  to denote  $M_i$ 's contribution in that session, the provided services are  $r'_2$ ,  $r'_3K_{I3}$  and  $r'_3K_{23}$  (we do not consider the actions of  $M_I$  since they would only involve values that the intruder knows).

It can now be observed that a pair of form  $(\alpha^x, \alpha^{xr_2K_{23}^{-1}})$  can be built by exploiting the services  $r_2$ ,  $r'_3K_{I3}$  and  $r'_3K_{23}$ . Actually, if the intruder replaces the input values of these last two services with a random value he knows, say  $\alpha^y$ ,  $M_3$  will send the values  $\alpha^{yr'_3K_{I3}}$  and  $\alpha^{yr'_3K_{23}}$ . Then, if  $M_I$  replaces the input of the  $r_2$ -service with  $\alpha^{yr'_3K_{I3}}$ ,  $M_2$  will send the value  $\alpha^{yr'_3K_{I3}r_2}$ . Finally, if the intruder exponentiates this last value with  $K_{I3}^{-1}$ ,

he will be in possession of the pair  $(\alpha^{yr'_3K_{23}}, \alpha^{yr'_3r_2})$  which has the desired form. The final step of this attack consists in replacing the second term of the broadcast sent by  $M_3$  with  $\alpha^{yr'_3K_{23}}$ , in such a way that  $M_2$  will compute  $\alpha^{yr'_3r_2}$  as group key. So, at this point,  $M_2$  expects that only  $M_1$  and  $M_3$  could know  $\alpha^{yr'_3r_2}$ , but it is known by  $M_I$  and this is in opposition with the implicit key authentication property.

We would like to distinguish two phases in this attack. The first one consists in finding which services can be used in order to obtain a pair of the desired form. This comes down to trying to write the value that  $M_2$  will use in order to compute his view of the group key as a product of services and values that the intruder knows: in the attack above, we found that  $r_2K_{23}^{-1} = r_2 \cdot r'_3K_{I3} \cdot (r'_3K_{23})^{-1} \cdot (K_{I3})^{-1}$ . In Section 3, we will show that, for the family of protocols we consider, such equations can always be found, provided that the protocol is executed by at least 3 users.

The second phase consists in finding a way to exploit the equation found during the first step in order to obtain an attack scenario. In our example above, it consisted in starting with a pair of form  $(\alpha^y, \alpha^y)$  and replacing the input of services inverted in the previous equation with the first term of the pair while the input of non-inverted services was replaced by the second term. So, we successively constructed the pairs  $(\alpha^y, \alpha^y)$ ,  $(\alpha^{yr'_3K_{23}}, \alpha^{yr'_3K_{I3}})$  and  $(\alpha^{yr'_3K_{23}}, \alpha^{yr'_3K_{I3}r_2})$ . This was however an easy case: if we had to use the  $r_1$ -service for instance, we would not have been able to replace the input of this service with a value of our choice since  $M_1$  always uses  $\alpha$  as input value for this service. Our goal in Section 4 will be to prove that at least one of the equations obtained during our first attack phase uses services which can be collected in order to build an attack against the protocols we consider, provided that they are executed by at least four users.

## 2.4 A Fix for the A-GDH.2 Protocol?

As we found an attack against the A-GDH.2 protocol, we now naturally wonder how this protocol could be fixed, that is, how we could write a protocol based on the same design assumptions as the A-GDH.2 protocol and guaranteeing the expected implicit key authentication property.

A first design assumption we keep is that we only consider protocols executed by exchanging elements of the public group  $\mathcal{G}$ , those elements being built by exponentiating a public generator  $\alpha$  with a product of:

- random values generated during the protocol execution and only known by the user who generated them and
- long-term shared keys of form  $K_{ij}$ , where  $K_{ij}$  is only known by  $M_i$  and  $M_j$ .

So, we only consider the exchange of elements of  $\mathcal{G}$  of form  $\alpha^{\prod r_i \prod K_{jk}}$  (but not elements obtained by multiplying two other elements of  $\mathcal{G}$  for instance).

A second design assumption is that we consider protocols aiming at building a shared group key of form  $\alpha^{r_1 \cdots r_n}$  where  $r_i$  has been generated by the  $i$ -th group member  $M_i$ . This guarantees that the protocol is contributive, which is required for a key agreement protocol.

A third design assumption is that these protocols are constant under member substitution: substituting member  $M_i$  with a user  $M_j$  in the group constitution will only change the protocol execution by substituting keys of the form  $K_{ik}$  with  $K_{jk}$ . This assumption excludes protocols the definition of which would contain rules such as: “User  $M_i$  exponentiates the term intended to  $M_j$  with  $K_{ij}^x$  where  $x$  is the last bit of  $M_j$ 's identifier” for instance.

## 2.5 Modelling the GDH-Protocols

The protocols based on the design assumptions we just described will be called *GDH-Protocols*. Before giving a precise definition of this protocol family and starting our analysis, we propose a second example of GDH-Protocol, the Ex-GDH protocol, which we will use to illustrate our further discussion.

**Example 2.3** We describe the Ex-GDH protocol in a similar form as the one commonly used in the literature and in [2] for instance. This protocol allows a group of three users  $M_1, M_2$  and  $M_3$  to contributively generate a key  $\alpha^{r_1 r_2 r_3}$ .

### Protocol 2: Ex-GDH Protocol

Let  $r_i, \hat{r}_i \in \mathbb{Z}_q^*$  be random values generated by  $M_i$  at the beginning of each protocol session. The three group members  $M_1, M_2$  and  $M_3$  then generate the group key by exchanging the following messages:

$$\begin{aligned} M_1 \rightarrow M_2 & : \alpha^{\hat{r}_1}, \alpha^{r_1} \\ M_2 \rightarrow M_3 & : \alpha^{\hat{r}_1 r_2 K_{23}}, \alpha^{r_1 K_{23}}, \alpha^{r_1 r_2} \\ M_3 \rightarrow M_1, M_2 & : \alpha^{\hat{r}_1 r_2 r_3 K_{13}}, \alpha^{r_1 r_3 K_{23}^2} \end{aligned}$$

When executing this protocol,  $M_1$  computes  $\alpha^{\hat{r}_1}, \alpha^{r_1}$  and sends these values to  $M_2$ , then  $M_2$  exponentiates the first term he received with  $r_2 K_{23}$ , the second with  $K_{23}$  and also with  $r_2$ , sends the three resulting elements of  $\mathcal{G}$  to  $M_3$ ; and finally,  $M_3$  exponentiates the first term he received with  $r_3 K_{13} K_{23}^{-1}$  and the second with  $r_3 K_{23}$ .

Upon receipt of the above,  $M_1$  computes the group key  $\alpha^{r_1 r_2 r_3}$  from  $\alpha^{\hat{r}_1 r_2 r_3 K_{13}}$ ,  $M_2$  from  $\alpha^{r_1 r_3 K_{23}^2}$  and  $M_3$  from  $\alpha^{r_1 r_2}$ .

We now present our modelling of the GDH-Protocol families, and start by defining the set of messages which can be exchanged.

**Definition 2.4** *Let:*

1.  $R$  be the set of symbols representing random values generated during the protocol execution.
2.  $K$  be the set of symbols representing the long-term symmetric keys shared by pairs of users. We assume that  $R \cap K = \emptyset$  and call elements of  $R \cup K$  atoms. Furthermore, we denote  $K_i$  the subset of keys of  $K$  known by  $M_i$  and  $K_{ij}$  a key of  $K$  known by  $M_i$  and  $M_j$ .
3.  $(P, \cdot)$  be the commutative group freely generated from  $R \cup K$ .
4.  $G$  be defined from  $P$  through a bijection  $\mathbf{alphaexp} : P \rightarrow G$  which represents the exponentiation of the public group generator  $\alpha$  with a product of random values and keys. This set will be used to model the finite group  $\mathcal{G}$ .

As it can be seen, we do not take any arithmetic relation that could exist between elements of  $R$  and  $K$  into account. It can also be observed that, according to our definitions, the set  $G$  is infinite, while  $\mathcal{G}$  is a finite group of prime order. It would be interesting to relate this abstraction of  $\mathcal{G}$  with the pseudo-freeness computational assumption introduced by S. Hohenberger and R. Rivest [9, 19].

In order to make the use of these sets more convenient, we introduce the following notations:

**Definition 2.5** *Suppose  $p \in P$  and  $g \in G$ .*

1.  $p_a$  denotes the projection of  $p$  on  $a$ , that is,  $a^e$  where  $p = a^e a_1^{e_1} \cdots a_n^{e_n}$ ,  $a \notin \{a_1, \dots, a_n\}$ , and  $a, a_1, \dots, a_n$  are atoms.
2.  $p_R$  and  $p_K$  denote the projection of  $p$  on  $R$  and  $K$  respectively, that is, are elements of  $P$  such that  $p = p_R \cdot p_K$  where  $p_R$  is a product of elements of  $R$  and  $p_K$  a product of elements of  $K$ .
3.  $\alpha^p \in G$  denotes  $\mathbf{alphaexp}(p)$  ( $\alpha^1$  will usually be abbreviated as  $\alpha$ ).
4.  $g^p$  denotes  $\mathbf{alphaexp}(\mathbf{alphaexp}^{-1}(g) \cdot p)$

We illustrate these definitions through the following example.

**Example 2.6** If we consider our Ex-GDH protocol,  $\{r_1, \hat{r}_1, r_2, r_3\} \subset R$  and  $\{K_{13}, K_{23}\} \subseteq K$ ;  $p = r_1 \cdot r_3 \cdot K_{23}^2$  is an element of  $P$ ,  $p_R = r_1 \cdot r_3$ ,  $p_K = K_{23}^2$ ,  $p_{K_{23}} = K_{23}^2$ ,  $p_{r_2} = 1$  and  $\alpha^{r_2 K_{23}}$  is an element of  $G$ .

The messages of the protocols we consider are all constituted of sequences of elements of  $\mathcal{G}$  (modelled as elements of  $G$ ). In order to simplify our notations, and since an active attacker has complete control over concatenation,

we can model (without loss of generality) the sending (resp. the reception) of the concatenation of  $n$  elements of  $\mathcal{G}$  as  $n$  sending (resp. receptions) of elements of  $G$ . So, in our model, all messages are single elements of  $G$ . We will call these messages *GDH-Terms*.

In order to describe our protocols, we now exploit the strand-space and bundle definitions, which are given in Appendix A. A strand is a sequence of nodes representing some party's view of a protocol run. Associated with each node is a GDH-Term with a sign,  $+$  or  $-$ , indicating that the GDH-Term is sent or received, respectively, on that node. The function  $term(n)$  (resp.  $uns.term(n)$ ) provides the signed (resp. unsigned) GDH-Term associated with the node  $n$ , while  $\langle s, i \rangle$  denotes the  $i$ -th node of the strand  $s$  and  $strand(n)$  the strand to which  $n$  belongs. A bundle is a directed graph whose edges express the causal dependencies of the nodes: a " $\rightarrow$ "-edge connects two nodes whose associated GDH-Terms are of form  $+t$  and  $-t$ , while a " $\Rightarrow$ "-edge connects two consecutive nodes of a strand. We also use the notation  $n \Rightarrow^+ n'$  to express that  $n$  and  $n'$  are connected through a sequence of " $\Rightarrow$ "-edges and " $\rightarrow_C$ " and " $\Rightarrow_C$ " to denote the " $\rightarrow$ " and " $\Rightarrow$ "-edges of the bundle  $\mathcal{C}$ . In a bundle, " $\rightarrow$ " and " $\Rightarrow$ "-edges allow defining a partial order relation between nodes: the node  $n$  precedes the node  $n'$ , written  $n \prec n'$  if there is a path made of " $\rightarrow$ " and " $\Rightarrow$ "-edges from  $n$  to  $n'$ . The following example shows a bundle representing a session of our Ex-GDH protocol.

**Example 2.7** Let  $s_1, s_2$  and  $s_3$  be three strands representing the roles of  $M_1, M_2$  and  $M_3$  in the Ex-GDH protocol. A bundle containing these three strands is represented in Fig. 2 (all four arrows of the last two rows of this figure originate on nodes of the  $s_3$  strand).

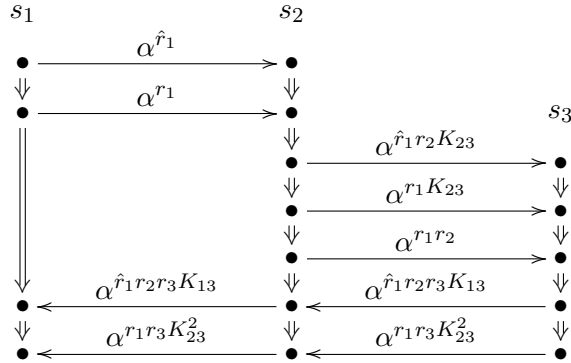


Figure 2: A run of the Ex-GDH protocol

Considering a bundle allows us to understand the way messages are exchanged during a protocol run. However, it does not express how these messages are built, which is an important property for the class of protocols

we are analyzing. As explained in the literature concerning the A-GDH protocols [2], the protocols we consider are executed in a very regular way: the group members receive elements of  $\mathcal{G}$  and exponentiate these elements with products of known random values and keys in order to construct the messages they send. So, for any element used by a group member to compute his view of the group key, it is possible to write a history describing how this element has been built from the group generator  $\alpha$ . This history can be described as a simple path since the combination of two elements of  $\mathcal{G}$  into a third one never occurs.

**Definition 2.8** *Given a bundle  $\mathcal{C}$ , a path  $\pi$  of length  $m$  in  $\mathcal{C}$  is a sequence of nodes  $\langle n_1, \dots, n_m \rangle$  of  $\mathcal{N}_{\mathcal{C}}$  such that:*

- $term(n_1) = +t$  and  $term(n_m) = -t'$
- $(n_{2i+1}, n_{2i+2}) \in \rightarrow_{\mathcal{C}}$  ( $0 \leq i < m/2$ )
- $(n_{2i}, n_{2i+1}) \in \Rightarrow_{\mathcal{C}}^+$  ( $0 < j < m/2$ )

*We furthermore consider that each path has at its extremities two “virtual” nodes  $n_0$  and  $n_{m+1}$  which are assumed to belong to the same strands as  $n_1$  and  $n_m$  respectively, and the associated GDH-Terms of which are  $uns\_term(n_0) = \alpha$  and  $uns\_term(n_{m+1}) = \alpha^{r_1 \dots r_n}$  (for a protocol executed by  $n$  parties).*

These two virtual nodes (which do not correspond to the transmission of any GDH-Term) are added in order to make the further notations more convenient. They correspond to the fact that the first element of an history is always computed from  $\alpha$  and that the last element of an history is used in order to compute a group key  $\alpha^{r_1 \dots r_n}$ .

We now introduce a few more definitions about paths:

**Definition 2.9** *Consider a path  $\pi = \langle n_1, \dots, n_m \rangle$  in a bundle  $\mathcal{C}$*

1.  $\pi(i) = n_i$ . As we will often be interested in the end of these paths, we also define  $\pi(\bar{i}) = n_{m+1-i}$
2.  $P(\pi(i)) = \frac{\mathbf{alphaexp}^{-1}(uns\_term(\pi(i)))}{\mathbf{alphaexp}^{-1}(uns\_term(\pi(i-1)))}$  is the element of  $\mathbf{P}$  which must be used to compute  $term(\pi(i))$  from  $term(\pi(i-1))$
3.  $Id(\pi(i)) = M_j$  where  $M_j$  is the user executing strand( $\pi(i)$ )
4.  $length(\pi) = m$

From this definition,  $\pi(i)$  is the  $i$ -th node of  $\pi$  (starting from the end of  $\pi$  if  $i$  is complemented). We may also note that  $term(\pi(\bar{1}))$  is the last GDH-Term of  $\pi$ , which will be used for computing the group key, and that  $P(\pi(\bar{0}))$ , the element of  $\mathbf{P}$  required for computing  $term(\pi(m+1))$  from

$term(\pi(m))$ , is the product of random values and long-term keys which will be used to compute the group key from  $term(\pi(\bar{1}))$ . The length of a path  $\pi$ , denoted  $length(\pi)$ , is defined as the number of nodes belonging to this path, without including the two virtual nodes.

These notions are exemplified below.

**Example 2.10** If we consider the bundle of Example 2.7, a path  $\pi$  describing the history of  $\langle s_2, 7 \rangle$  is

$$\pi = \langle \langle s_1, 2 \rangle, \langle s_2, 2 \rangle, \langle s_2, 4 \rangle, \langle s_3, 2 \rangle, \langle s_3, 5 \rangle, \langle s_2, 7 \rangle \rangle$$

This path is represented on Fig. 3, on which we added the virtual nodes at the beginning of  $s_1$  and at the end of  $s_2$ .

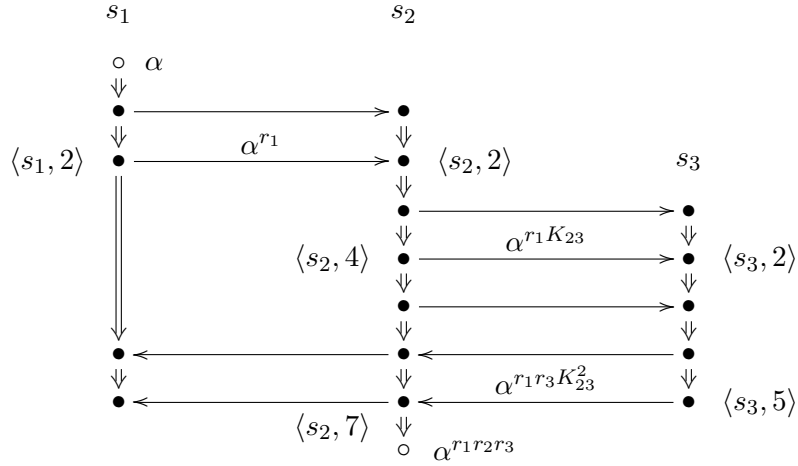


Figure 3: Example of Path

$$term(\pi(1)) = +\alpha^{r_1}, term(\pi(\bar{4})) = +\alpha^{r_1 K_{23}}$$

$$P(\pi(1)) = r_1, P(\pi(2)) = 1, P(\pi(\bar{0})) = r_2 K_{23}^{-2}$$

$$strand(\pi(2)) = s_2, Id(\pi(6)) = M_2, length(\pi) = 6$$

As paths will be used in order to describe the way messages are transformed along strands, they can also be used to define a notion of knowledge expressing that a party must know specific values in order to be able to perform the transformation required at some node. This notion will make use of the subterm relation  $\sqsubset$  defined as follows:

**Definition 2.11** Suppose  $a$  is an atom,  $p \in \mathbb{P}$  and  $g \in \mathbb{G}$ .

- $a \sqsubset p$  if  $p_a \neq 1$

- $a \sqsubset g$  iff  $a \sqsubset \mathbf{alphaexp}^{-1}(g)$

If  $a \sqsubset x$ , we say that  $a$  is a subterm of  $x$  or that  $x$  contains  $a$ .

The following example illustrates this definition.

**Example 2.12** Let  $g = \alpha^{r_1 K_{23}}$  be an element of  $G$ . Then  $r_1 \sqsubset g$  and  $K_{13} \not\sqsubset g$ .

We can now define our notions of knowledge.

**Definition 2.13** Consider a set  $\pi = \{\pi_1, \dots, \pi_n\}$  of paths in  $\mathcal{C}$ . We say that:

1.  $p \in \mathbf{P}$  is known on  $\pi_i(j)$  iff for any atom  $a \sqsubset p$ , we have that  $a \sqsubset P(\pi_i(j))$ ,
2.  $p \in \mathbf{P}$  is known on the strand  $s$  iff for any atom  $a \sqsubset p$ , there are values for  $i$  and  $j$  such that  $a \sqsubset P(\pi_i(j))$  and  $\text{strand}(\pi_i(j)) = s$ ,
3.  $p \in \mathbf{P}$  is locally known on the strand  $s$  of  $\mathcal{C}$  if  $s$  is the only strand of  $\mathcal{C}$  on which  $p$  is known.

**Example 2.14** If we consider our Ex-GDH protocol as represented in Example 2.7 and the three paths  $\pi_1, \pi_2$  and  $\pi_3$  defined as

$$\begin{aligned} \pi_1 &= \langle \langle s_1, 1 \rangle, \langle s_2, 1 \rangle, \langle s_2, 3 \rangle, \langle s_3, 1 \rangle, \langle s_3, 4 \rangle, \langle s_1, 3 \rangle \rangle \\ \pi_2 &= \langle \langle s_1, 2 \rangle, \langle s_2, 2 \rangle, \langle s_2, 4 \rangle, \langle s_3, 2 \rangle, \langle s_3, 5 \rangle, \langle s_2, 7 \rangle \rangle \\ \pi_3 &= \langle \langle s_1, 2 \rangle, \langle s_2, 2 \rangle, \langle s_2, 5 \rangle, \langle s_3, 3 \rangle \rangle \end{aligned}$$

then  $r_3 K_{13}$  is known on  $\pi_1(5)$ . We also have that  $K_{23}$  is known on  $s_3$  but not locally known on that strand as it is also known on  $s_2$ . Finally,  $r_2$  is locally known on  $s_2$  since it is only known on  $\pi_1(3)$ , on  $\pi_2(7)$  and on  $\pi_3(3)$  which all belong to  $s_2$  (note that  $\pi_2(7)$  is the virtual node at the end of  $\pi_2$ ).

Equipped with these definitions, we can now define the GDH-Protocols. The first part of this definition expresses how GDH-Terms are exchanged, which is described through a bundle containing  $n$  strands; while the second part expresses how the exchanged GDH-Terms are computed by the different group members, which is expressed through  $n$  paths whose final element is the element of  $G$  used by the different group members to compute their view of the group key. The last part of this definition expresses constraints on the use of the different protocol variables (random key contributions and long-term keys). The first constraint expresses that the random values generated by the protocol participants can be used on only one strand: they are never transmitted to other users, and the probability of two users generating the same random value can be considered negligible. The second constraint expresses that each contribution  $r_i$  included in the group key  $\alpha^{r_1 \dots r_n}$  must

be generated by the user  $M_i$ . Finally, the last constraint expresses that the long-term symmetric keys can only be used by the two users who are assumed to know them.

**Definition 2.15** A GDH-Protocol on a group of  $n$  principals  $\mathbb{M} = \{M_1, \dots, M_n\}$  is a protocol aiming at enabling a key  $\alpha^{r_1 \dots r_n}$  to be shared by the principals in  $\mathbb{M}$  and the regular execution of which can be described through two elements:

1. a bundle  $\mathcal{C}_{GDH}$  containing  $n$  strands  $s_1, \dots, s_n$ ,  $M_i$  being the active principal for  $s_i$ .
2. a set  $\pi = \{\pi_1, \dots, \pi_n\}$  of  $n$  paths in  $\mathcal{C}_{GDH}$ . These specific paths are called histories and express how the GDH-Terms exchanged in the  $\mathcal{C}_{GDH}$  bundle are built:  $Id(\pi_i(2j))$  computes  $term(\pi_i(2j+1))$  from  $term(\pi_i(2j))$  and  $M_i$  computes the group key  $\alpha^{r_1 \dots r_n}$  from  $term(\pi_i(\bar{1}))$  (so,  $strand(\pi_i(\bar{1})) = s_i$ ).

Furthermore, in a GDH-Protocol,

- a. If  $a \in \mathbb{R}$  is known on  $\pi_i(j)$  then it is locally known
- b. The random contribution  $r_i \in \mathbb{R}$  is locally known on  $s_i$
- c. If  $a \in \mathbb{K}$  is known on  $\pi_i(j)$  then  $a \in \mathbb{K}_{Id(\pi_i(j))}$

We may verify that the Ex-GDH protocol, defined by the bundle represented in Fig. 2 and by the three histories given by the paths in Example 2.14, is a GDH-Protocol.

From now on, except when specified otherwise, we always refer to GDH-Protocols executed by a group  $\mathbb{M} = \{M_1, \dots, M_n\}$  and described through a GDH-Bundle  $\mathcal{C}_{GDH}$  and through histories  $\pi_1, \dots, \pi_n$ .

We introduce one last (and central) definition before turning to properties of GDH-Protocols: the definition of the notion of *contribution*. Given a session of a GDH-Protocol, the product of the elements of  $\mathbb{P}$  used by  $M_i$  in order to exponentiate elements of  $\mathbb{G}$  belonging to  $\pi_j$  will be called the contribution of  $M_i$  to  $M_j$ , denoted  $C(M_i \rightarrow M_j)$ . In other words,  $C(M_i \rightarrow M_j)$  is the product of the services offered by  $M_i$  on the path  $\pi_j$ .

**Definition 2.16** Given a GDH-Protocol with paths  $\pi_1, \dots, \pi_n$ , the contribution of  $M_i$  to  $M_j$ , denoted  $C(M_i \rightarrow M_j)$ , is defined as

$$\prod_{0 < k \leq \text{length}(\pi_j)} P(\pi_j(k))^{e_k}$$

where  $e_k = 1$  if  $Id(\pi_j(k)) = M_i$  and  $e_k = 0$  otherwise.

**Example 2.17** The table below indicates the value of  $C(M_i \rightarrow M_j)$  for the Ex-GDH protocol in the line  $M_i$  of column  $M_j$ .

	$M_1$	$M_2$	$M_3$
$M_1$	$\hat{r}_1$	$r_1$	$r_1$
$M_2$	$r_2 K_{23}$	$K_{23}$	$r_2$
$M_3$	$r_3 K_{13} K_{23}^{-1}$	$r_3 K_{23}$	1

### 3 Properties of GDH-Protocols

#### 3.1 Introduction

Starting from the definitions of the previous section, we now define a few constitutive properties of GDH-Protocols. These properties express characteristics that GDH-Protocols must respect if they conform to their definition. The goal of this section will be to achieve our first attack phase described in Section 2.3, that is, to obtain a relation expressing the value that each user  $M_i$  executing a session of a GDH-Protocol uses for computing his view of the group key (i.e.  $P(\pi_i(\bar{0}))$ ) as a product of contributions (which are themselves a product of services) and of long-term keys the attacker knows.

In the following paragraphs, we will never precisely specify to which session of a protocol we refer: we simply state the corresponding group constitution when it is different from  $M = \{M_1, \dots, M_n\}$ . This is because we will always consider a single protocol execution for each specified group constitution.

#### 3.2 Properties of GDH-Protocols

We start this subsection with two simple observations which will be useful further.

**Observation 3.1** *Let  $p_1, p_2$  and  $p_3$  be elements of  $\mathbb{P}$  and  $a$  be an atom. If  $p_1 = p_2 \cdot p_3$  and  $a \sqsubset p_1$ , then  $a \sqsubset p_2$  or  $a \sqsubset p_3$ . Similarly, If  $p_1 = p_2 \cdot p_3$  and  $(p_1)_a = (p_2)_a$  then  $a \not\sqsubset p_3$ .*

These observations result from the definition of  $\mathbb{P}$ .

**Observation 3.2** *From the definition of the notion of contribution,*

$$\text{term}(\pi_j(\bar{1})) = -\alpha^{\prod_{i=1 \dots n} C(M_i \rightarrow M_j)}.$$

This observation can be verified in Example 2.17 if we keep in mind that  $\text{term}(\pi_1(\bar{1})) = -\alpha^{\hat{r}_1 r_2 r_3 K_{13}}$ ,  $\text{term}(\pi_2(\bar{1})) = -\alpha^{r_1 r_3 K_{23}^2}$  and  $\text{term}(\pi_3(\bar{1})) = -\alpha^{r_1 r_2}$ .

We can now write a first proposition about the value of  $C_{\mathbb{R}}(M_i \rightarrow M_j)$  when  $i \neq j$ , where  $C_{\mathbb{R}}(M_i \rightarrow M_j)$  denotes the projection of  $C(M_i \rightarrow M_j)$  on the free abelian group generated from  $\mathbb{R}$  (as given in Def. 2.5).

**Proposition 3.3** *For any GDH-Protocol, if  $1 \leq i, j \leq n$  and  $i \neq j$ , then  $C_{\mathbb{R}}(M_i \rightarrow M_j) = r_i$ .*

*Proof.* From Observation 3.2, we can write

$$\prod_{i=1 \dots n} C_{\mathbb{R}}(M_i \rightarrow M_j) \cdot P_{\mathbb{R}}(\pi_j(\bar{0})) = r_1 \cdots r_n. \quad (1)$$

We observe that  $r_i \not\sqsubseteq C_{\mathbb{R}}(M_k \rightarrow M_j)$  when  $k \neq i$  or else  $r_i$  would be known on  $s_k$  which is in contradiction with Point (b) of Def. 2.15 of GDH-Protocols. Furthermore,  $r_i \not\sqsubseteq P_{\mathbb{R}}(\pi_j(\bar{0}))$  for the same reason. We can deduce from these remarks and from Observation 3.1 that  $r_i \sqsubset C_{\mathbb{R}}(M_i \rightarrow M_j)$  and that  $C_{r_i}(M_i \rightarrow M_j) = (r_1 \cdots r_n)_{r_i} = r_i$ .

Let us now imagine that  $C_{\mathbb{R}}(M_i \rightarrow M_j) = r_i \cdot p$ . Then  $r_i \not\sqsubseteq p$ . Suppose  $r_a \sqsubset p$ . From Observation 3.1,  $r_a \sqsubset C_{\mathbb{R}}(M_i \rightarrow M_j)$ , so  $r_a \in \mathbb{R}$  is locally known on  $s_i$  (from Point (a) of Def. 2.15). Therefore, it is not known on  $s_k$  when  $k \neq i$ ,  $r_a \notin \{r_1, \dots, r_n\}$ ,  $r_a \not\sqsubseteq C_{\mathbb{R}}(M_k \rightarrow M_j)$  ( $k \neq i$ ) and  $r_a \not\sqsubseteq P_{\mathbb{R}}(\pi_j(\bar{0}))$ . But this is in contradiction with Observation 3.1 and Equation (1) and we must have that  $p = 1$ . ■

Concerning the value of  $C_{\mathbb{R}}(M_i \rightarrow M_i)$ , the following relation must be valid:

**Proposition 3.4** *For any GDH-Protocol,  $C_{\mathbb{R}}(M_i \rightarrow M_i) = r_i \cdot P_{\mathbb{R}}(\pi_i(\bar{0}))^{-1}$ .*

*Proof.* Definition 2.9 gives us that

$$P(\pi_i(\bar{0})) = \prod_{j=1 \dots n} r_j \cdot (\mathbf{alphaexp}^{-1}(\mathit{uns\_term}(\pi_i(\bar{1}))))^{-1}$$

So, by successively exploiting Observation 3.2 and Proposition 3.3, we can write:

$$\begin{aligned} P_{\mathbb{R}}(\pi_i(\bar{0})) &= \prod_{j=1 \dots n} r_j \cdot \left( \prod_{j=1 \dots n} C_{\mathbb{R}}(M_j \rightarrow M_i) \right)^{-1} \\ &= \prod_{j=1 \dots n} r_j \cdot \left( \prod_{j=1 \dots n, j \neq i} r_j \right)^{-1} \cdot C_{\mathbb{R}}(M_i \rightarrow M_i)^{-1} \\ &= r_i \cdot C_{\mathbb{R}}(M_i \rightarrow M_i)^{-1} \end{aligned}$$

■

These two propositions can be checked for the Ex-GDH protocol in the tables of Example 2.17.

Having characterized the value of  $C_{\mathbb{R}}(M_i \rightarrow M_j)$ , we will now write two propositions concerning the value of  $C_{\mathbb{K}}(M_i \rightarrow M_j)$ .

**Proposition 3.5** *For any GDH-Protocol, if  $C_{K_{jk}}(M_j \rightarrow M_i) = K_{jk}^a$  ( $K_{jk} \notin \mathcal{K}_i$ ) then  $C_{K_{jk}}(M_k \rightarrow M_i) = K_{jk}^{-a}$ .*

*Proof.* Observation 3.2 gives us that  $\prod_{l=1 \dots n} C_{\mathcal{K}}(M_l \rightarrow M_i) \cdot P_{\mathcal{K}}(\pi_i(\bar{0})) = 1$ ; so the sum of the powers of  $K_{jk}$  in the components of the left part of this equation must be null. But  $K_{jk} \not\in P_{\mathcal{K}}(\pi_i(\bar{0}))$  since  $K_{jk} \notin \mathcal{K}_i$ . Just as  $K_{jk} \not\in C_{\mathcal{K}}(M_l \rightarrow M_i)$  ( $l \neq j, k$ ) since  $K_{jk} \notin \mathcal{K}_l$ . Therefore,  $K_{jk}$  can only be a subterm of  $C_{\mathcal{K}}(M_j \rightarrow M_i)$  and of  $C_{\mathcal{K}}(M_k \rightarrow M_i)$ , and the powers of  $K_{jk}$  in these two contributions must be of the form  $a$  and  $-a$  since their sum is null. ■

Until now, we considered the relations between values inside one session of a protocol. Now, we would like to write a proposition concerning the use of long-term keys in sessions executed by different groups of users. To this purpose, we introduce a substitution operator: if  $p \in \mathcal{P}$  is such that  $p_{\mathcal{R}} = 1$  and is a function of the keys of a bundle corresponding to a session of a GDH-Protocol executed by the group  $\mathcal{M}$ , then  $[M_i \setminus M_I : p]$  (where  $M_i \in \mathcal{M}$  and  $M_I \notin \mathcal{M}$ ) refers to the value that  $p$  would have in a session with the same participants except that  $M_i$  is substituted with  $M_I$ . More precisely:

**Definition 3.6** *If  $p = \prod_j K_{ij}^{e_{ij}} \cdot K_x$  where  $K_{ij} \not\in K_x$  ( $\forall j$ ) then  $[M_i \setminus M_I : p] = \prod_j K_{Ij}^{e_{ij}} \cdot K_x$ . More generally, if  $\mathcal{S} = \{M_{i_1}, \dots, M_{i_s}\}$ ,  $[\mathcal{S} \setminus M_I : p] = [M_{i_1} \setminus M_I : [(\mathcal{S} - \{M_{i_1}\}) \setminus M_I : p]]$ .*

**Example 3.7** In the Ex-GDH protocol,  $[M_1 \setminus M_I : C_{\mathcal{K}}(M_3 \rightarrow M_1)] = K_{I3} K_{23}^{-1}$  and  $[\{M_1, M_2\} \setminus M_I : C_{\mathcal{K}}(M_3 \rightarrow M_1)] = K_{I3} K_{I3}^{-1} = 1$

As above,  $M_I$  denotes a user that is not a member of the group  $\mathcal{M}$  and plays the role of the intruder. This user is however considered as a legitimate member of some other groups,  $K_{Ij} \in (\mathcal{K}_I \cap \mathcal{K}_j)$  denoting a long-term key shared by  $M_I$  and  $M_j$ .

We can now write a proposition relating the key part of the contribution of a honest member  $M_j$  to  $M_i$ , i.e.  $C_{\mathcal{K}}(M_j \rightarrow M_i)$ , with his contribution  $[\mathcal{S} \setminus M_I : C_{\mathcal{K}}(M_j \rightarrow M_i)]$  in a session where a set of honest members  $\mathcal{S} \subset \mathcal{M}$  has been replaced with the intruder. These two values are in fact equal, excepted that all occurrences of keys shared between  $M_j$  and users in  $\mathcal{S}$  will be replaced by keys shared between  $M_j$  and  $M_I$ .

**Proposition 3.8** *Suppose  $\mathcal{S} \subset \mathcal{M}$  and  $M_j \notin \mathcal{S}$ . Then,*

$$\begin{aligned} C_{\mathcal{K}}(M_j \rightarrow M_i) &= [\mathcal{S} \setminus M_I : C_{\mathcal{K}}(M_j \rightarrow M_i)] \\ &\cdot \prod_{M_k \in \mathcal{S}} C_{K_{jk}}(M_j \rightarrow M_i) \\ &\cdot \prod_{M_k \in \mathcal{S}} [\mathcal{S} \setminus M_I : C_{K_{jk}}^{-1}(M_j \rightarrow M_i)]. \end{aligned}$$

*Proof.*  $C_K(M_j \rightarrow M_i)$  is known on  $s_j$ , so it can be written as a product of keys of the form  $K_{jx}$ . A possible way to write  $C_K(M_j \rightarrow M_i)$  is therefore  $\prod_{M_k \in \mathcal{S}} K_{jk}^{e_k} \cdot K_x$  where  $K_{jk} \not\subseteq K_x$  for all  $M_k \in \mathcal{S}$  and  $K_x$  is a product of keys in  $\mathcal{K}_j$ . Definition 3.6 now implies that  $[\mathcal{S} \setminus M_I : C_K(M_j \rightarrow M_i)] = \prod_{M_k \in \mathcal{S}} K_{jI}^{e_k} \cdot K_x$ .

This proposition now results from the fact that  $C_{K_{jk}}(M_j \rightarrow M_i) = K_{jk}^{e_k}$  and  $[M_k \setminus M_I : C_{K_{jk}}(M_j \rightarrow M_i)] = K_{jI}^{e_k}$ . ■

**Example 3.9** Consider the Ex-GDH protocol and  $\mathcal{S} = \{M_1\}$ . In this case,  $C_K(M_3 \rightarrow M_1) = K_{13}K_{23}^{-1}$ ,  $[M_1 \setminus M_I : C_K(M_3 \rightarrow M_1)] = K_{I3}K_{23}^{-1}$ ,  $C_{K_{13}}(M_3 \rightarrow M_1) = K_{13}$  and  $[M_1 \setminus M_I : C_{K_{13}}(M_3 \rightarrow M_1)] = K_{I3}$ . Then we can check that  $K_{13}K_{23}^{-1} = K_{I3}K_{23}^{-1} \cdot K_{13} \cdot K_{I3}^{-1}$  as expected from our previous proposition.

All these propositions can be used to prove our main property concerning contributions: the product  $P(\pi_i(\bar{0}))$  that user  $M_i$  uses when computing his view of the group key can be written as a product of contributions and keys that the intruder knows.

**Theorem 3.10** *For any GDH-Protocol executed by a group of users  $\mathcal{M} = \{M_1, \dots, M_n\}$  where  $n \geq 3$ , it is possible to write any secret  $P(\pi_i(\bar{0}))$  as a product of contributions  $C(M_j \rightarrow M_k)$  ( $M_j, M_k \in \mathcal{M} \cup \{M_I\}$ ) and of keys known by  $M_I$ .*

*Proof.* (See Appendix B for details)

Let  $\mathcal{S}_j$  and  $\mathcal{S}_k$  be two disjoint sets of users such that  $M_k \in \mathcal{S}_j$ ,  $M_j \in \mathcal{S}_k$ ,  $M_i \notin \mathcal{S}_j \cup \mathcal{S}_k$  and  $\mathcal{S}_j \cup \mathcal{S}_k \cup \{M_i\} = \mathcal{M}$ . Then, it can be checked that:

$$\begin{aligned} P(\pi_i(\bar{0})) &= C^{-1}(M_i \rightarrow M_i) \cdot C(M_i \rightarrow M_j) \\ &\cdot [\mathcal{S}_j \setminus M_I : C^{-1}(M_i \rightarrow M_j) \cdot C(M_i \rightarrow M_k)] \\ &\cdot \prod_{M_l \in \mathcal{S}_k} [\mathcal{S}_j \setminus M_I : C^{-1}(M_l \rightarrow M_i) \cdot C(M_l \rightarrow M_k)] \\ &\cdot \prod_{M_l \in \mathcal{S}_j} [\mathcal{S}_k \setminus M_I : C^{-1}(M_l \rightarrow M_i) \cdot C(M_l \rightarrow M_j)] \\ &\cdot \prod_{M_l \in \mathcal{M}} K_{Il}^{e_l} \end{aligned}$$

This relation was obtained from the observation that  $P_R(\pi_i(\bar{0})) = C_R^{-1}(M_i \rightarrow M_i) \cdot C_R(M_i \rightarrow M_j)$ , that  $P_K(\pi_i(\bar{0})) = \prod_{M_l \in \mathcal{M}} C_K^{-1}(M_l \rightarrow M_i)$  and from the use of the previous propositions. ■

**Example 3.11** If we consider the Ex-GDH Protocol with  $i = 1$ ,  $j = 2$  and  $k = 3$ , we must choose  $\mathcal{S}_j = \{M_k\}$  and  $\mathcal{S}_k = \{M_j\}$  and, applying

Theorem 3.10, we have that:

$$\begin{aligned}
r_1 \hat{r}_1^{-1} K_{13}^{-1} &= \hat{r}_1^{-1} \cdot r_1 \\
&\cdot (r'_1)^{-1} \cdot r'_1 \\
&\cdot (r'_2 K_{2I})^{-1} \cdot r'_2 \\
&\cdot (r''_3 K_{13} K_{I3}^{-1})^{-1} \cdot r''_3 K_{I3} \\
&\cdot K_{I3}^{-2} \cdot K_{2I}.
\end{aligned}$$

where we considered  $r_i$  to be contributions sent in the session executed by the group  $\{M_1, M_2, M_3\}$ ,  $r'_i$  to be contributions in the session executed by the group  $\{M_1, M_2, M_I\}$ , and  $r''_i$  to be contributions in the session executed by the group  $\{M_1, M_I, M_3\}$ .

### 3.3 Conclusion

We have shown that, for any GDH-Protocol executed by at least three users, it is possible to write the secret value that each group member will use when computing the group key (i.e.  $P(\pi_i(\bar{0}))$ ) as a product of contributions of different group members during different sessions of the protocol.

In other words, we have shown that the first phase of Section 2.3's attack process can always succeed, provided that we consider at least three group members and some well chosen protocol sessions.

A natural question comes about the security of two-party GDH-Protocols. In fact, it is easy to exhibit a GDH-Protocol for which this first attack phase cannot succeed: the A-GDH.2 protocol with two participants. With our usual notations, this protocol executes as described in Fig. 4.

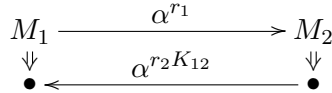


Figure 4: A-GDH.2 Protocol with two parties

Considering the role of  $M_1$ , we try to write  $P(\pi_1(\bar{0})) = r_1 K_{12}^{-1}$  as a product of services and keys the adversary knows. The only service containing  $r_1$  is the service provided by  $M_1$  during the session we try to attack, so we have no choice but using that service. The key part of  $P(\pi_1(\bar{0}))$ , i.e.  $K_{12}^{-1}$ , appears only in services provided by users in sessions executed by the group  $\{M_1, M_2\}$  and in sessions executed by the group  $\{M_2, M_1\}$ . However, this key always comes together with a random value ( $r_2$  in Fig. 4) which only appears in that service, so we will never find another service allowing us to cancel it. This shows that it is impossible to write  $P(\pi_1(\bar{0}))$  as a product of services and keys known by the adversary when we only consider the services provided during sessions of the A-GDH.2 protocol with two participants.

It can be easily verified that the same problem occurs with the role of  $M_2$ .

## 4 Collecting Contributions

### 4.1 Introduction

At this point, we know that the first phase of our attack process described in Section 2.3 always succeed for protocols executed by at least three parties.

We come now to the second phase and will now see how (and if) the required services can be exploited by an intruder who wants to undermine the IKA property. More precisely, we will examine how an intruder who wants to attack  $M_i$  can obtain a pair  $(g_1, g_2)$  of elements of  $\mathbf{G}$  such that  $g_2 = g_1^{P(\pi_i(\bar{0}))}$  and send  $g_1$  to the node  $\pi_i(\bar{1})$ .

### 4.2 Collecting Pairs of Contributions

If we look at Theorem 3.10, we can observe that we are interested in collecting pairs  $(g_1, g_2)$  such that  $g_2 = g_1^p$  where  $p$  is a product of terms of the form  $C^{-1}(M_i \rightarrow M_j) \cdot C(M_i \rightarrow M_k)$ . The following proposition is a first step in the obtention of such pairs.

**Proposition 4.1** *For any session of a GDH-Protocol executed by a group of users  $\mathbf{M}$  of cardinality  $n$ , an active attacker can obtain a pair  $(g_1, g_2)$  of elements of  $\mathbf{G}$  such that  $g_2 = g_1^{C^{-1}(M_i \rightarrow M_j) \cdot C(M_i \rightarrow M_k)}$ .*

*Proof.* Consider a session of a GDH-Protocol executed by the members of the group  $\mathbf{M}$ . If we initialize  $g_1$  and  $g_2$  to  $\alpha$ , Algorithm 1 gives the intruder a pair  $(g_1, g_2)$  of the desired form (actually, in order to prevent the message recipient to observe that the message he receives is simply the group generator  $\alpha$ , we also can initialize  $g_1$  and  $g_2$  to any random elements of  $\mathbf{G}$ , say  $\alpha^x$  and  $\alpha^y$ , and, at the end of the algorithm, exponentiate  $g_1$  with  $x^{-1}$  and  $g_2$  with  $y^{-1}$ ).

This algorithm can be explained as follows. Let  $s_i$  be a strand that corresponds to  $M_i$ 's role in an execution by the group  $\mathbf{M}$  of the considered protocol. We proceed by constructing a strand  $s_I$  matching  $s_i$  (i.e. a strand such that  $term(\langle s_i, x \rangle) = -term(\langle s_I, x \rangle)$ ), while collecting the services on  $\pi_j$  belonging to  $s_i$  into  $g_1$  and the services on  $\pi_k$  belonging to  $s_i$  into  $g_2$  (excepted for the common parts of  $\pi_j$  and  $\pi_k$ ). So, by executing this strand, the intruder will have a conversation with  $M_i$  at the end of which  $M_i$  will have completed his role in the considered session of the protocol without interacting with any other member of  $\mathbf{M}$ .

The  $s_I$  strand is constructed by receiving the messages  $M_i$  sends and by sending a random element of  $\mathbf{G}$  when  $M_i$  is waiting for a message, except

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**Algorithm 1** Defines a strand  $s_I$  which, when executed together with  $s_i$ , provides a pair  $(g_1, g_2)$  such that  $g_2 = g_1^{C^{-1}(M_i \rightarrow M_j) \cdot C(M_i \rightarrow M_k)}$  ( $M_j \neq M_k$ ) if the precondition  $g_1 = g_2$  is verified.

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for  $z := 1$  to  $length(s_i)$  do
  if  $\exists t : term(\langle s_i, z \rangle) = +t$  then
     $term(\langle s_I, z \rangle) := -t$ 
    if  $\exists x : \langle s_i, z \rangle = \pi_j(x)$  and  $\pi_j(x) \neq \pi_k(x)$  then
       $g_1 := uns\_term(\pi_j(x))$ 
    end if
    if  $\exists y : \langle s_i, z \rangle = \pi_k(y)$  and  $\pi_j(y) \neq \pi_k(y)$  then
       $g_2 := uns\_term(\pi_k(y))$ 
    end if
  else
     $t :=$  a random element of  $G$ 
    if  $\exists x : \langle s_i, z \rangle = \pi_j(x)$  and  $\pi_j(x+1) \neq \pi_k(x+1)$  then
       $t := g_1$ 
    end if
    if  $\exists y : \langle s_i, z \rangle = \pi_k(y)$  and  $\pi_j(y+1) \neq \pi_k(y+1)$  then
       $t := g_2$ 
    end if
     $term(\langle s_I, z \rangle) = +t$ 
  end if
end for

```

---

when the considered nodes of  $s_i$  are nodes of the histories  $\pi_j$  or  $\pi_k$ . In this last case, different actions are performed according to the sign of the term on the considered node of  $s_i$  (to which we will refer as  $n$ ) and the histories we consider:

- **if**
  - $term(n)$  is negative and
  - $n$  belongs to  $\pi_j$  (resp.  $\pi_k$ ), that is,  $term(n)$  is the input of a service that is part of  $C(M_i \rightarrow M_j)$  (resp.  $C(M_i \rightarrow M_k)$ ) and
  - the next node on  $\pi_j$  (resp. on  $\pi_k$ ) is not part of both  $\pi_j$  and  $\pi_k$  (i.e. the output of the corresponding service(s) is not part of both  $\pi_j$  and  $\pi_k$ )

then the term in the corresponding node of  $s_I$  is set to  $g_1$  (resp.  $g_2$ ), that is, the intruder provides  $g_1$  (resp.  $g_2$ ) as input for this service

- **if**
  - $term(n)$  is positive and
  - $n$  belongs to  $\pi_j$  (resp.  $\pi_k$ ) (that is,  $term(n)$  is the output of a service that is part of  $C(M_i \rightarrow M_j)$  (resp.  $C(M_i \rightarrow M_k)$ )) and

- the output of the considered service is not part of both  $\pi_j$  and  $\pi_k$ ,

then the intruder assigns the output of this service to  $g_1$  (resp.  $g_2$ ).

In fact, when  $n = \pi_j(x)$  (resp.  $n = \pi_k(x)$ ), this process allows to perform the operation  $g_1 := g_1^{P(\pi_j(x+1))}$  (resp.  $g_2 := g_2^{P(\pi_k(x+1))}$ ), which eventually provides the expected values, as it can be checked from the definition of contributions (Def. 2.16). ■

**Example 4.2** We apply Algorithm 1 in order to obtain a pair  $(g_1, g_2)$  such that  $g_2 = g_1^{C^{-1}(M_2 \rightarrow M_2) \cdot C(M_2 \rightarrow M_3)}$  in our Ex-GDH protocol. For that protocol,

$$\begin{aligned}\pi_2 &= \langle \langle s_1, 2 \rangle, \langle s_2, 2 \rangle, \langle s_2, 4 \rangle, \langle s_3, 2 \rangle, \langle s_3, 5 \rangle, \langle s_2, 7 \rangle \rangle \\ \pi_3 &= \langle \langle s_1, 2 \rangle, \langle s_2, 2 \rangle, \langle s_2, 5 \rangle, \langle s_3, 3 \rangle \rangle\end{aligned}$$

where  $s_1$ ,  $s_2$  and  $s_3$  are executed by  $M_1$ ,  $M_2$  and  $M_3$  respectively.

Our algorithm assumes  $g_1$  and  $g_2$  have been initialized to  $\alpha$  and successively considers all the nodes of  $s_2$  in order to build  $s_I$ , the variable  $z$  indicating the index of the node of  $s_i$  which is examined.

$z = 1$   $term(\langle s_2, 1 \rangle)$  is negative, so we define  $t := \langle \alpha^r \rangle$  (where  $\alpha^r$  is a random element of  $\mathbf{G}$ ). The next two tests are false, so  $term(\langle s_I, 1 \rangle) := +t$ .

$z = 2$   $term(\langle s_2, 2 \rangle)$  is also negative but  $\langle s_2, 2 \rangle$  is part of both  $\pi_2$  and  $\pi_3$ , so  $term(\langle s_I, 2 \rangle)$  is defined as  $+\alpha$ , that is, the value to which  $g_2$  has been initialized.

$z = 3$   $term(\langle s_2, 3 \rangle)$  is positive, so we define  $term(\langle s_I, 3 \rangle) := -t$ . The next two tests are false.

$z = 4$   $term(\langle s_2, 4 \rangle)$  is positive, so we define  $term(\langle s_I, 4 \rangle) := -t$ , where  $t = \langle \alpha^{K_{23}} \rangle$ . Since the choice  $x = 3$  matches the first **if** clause, we update the value of  $g_1$  to  $\alpha^{K_{23}}$ .

$z = 5$   $term(\langle s_2, 5 \rangle)$  is positive, so we define  $term(\langle s_I, 5 \rangle) := -t$ , where  $t = \langle \alpha^{r^2} \rangle$ . Since the choice  $y = 3$  matches the first **if** clause, we update the value of  $g_2$  to  $\alpha^{r^2}$ .

$z = 6$   $term(\langle s_2, 6 \rangle)$  is negative, and  $\langle s_2, 6 \rangle$  does not belong to  $\pi_2$  nor  $\pi_3$ , so we define  $term(\langle s_I, 6 \rangle) := +\alpha^r$ .

$z = 7$   $term(\langle s_2, 7 \rangle)$  is also negative, but  $\langle s_2, 7 \rangle$  is part of  $\pi_2$ , so we define  $term(\langle s_I, 7 \rangle) := +\alpha^{K_{23}}$ .

We can easily verify that  $g_2 = g_1^{r_2 K_{23}^{-1}} = g_1^{C^{-1}(M_2 \rightarrow M_2) \cdot C(M_2 \rightarrow M_3)}$  as expected. The strands  $s_2$  and  $s_I$  are represented in Fig. 5.

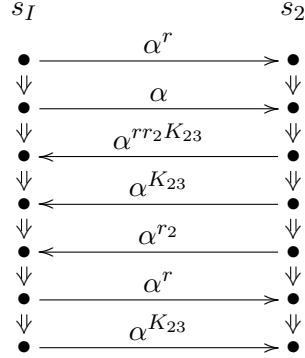


Figure 5: Representation of  $s_I$  and  $s_2$

### 4.3 Composing Contributions

In the previous subsection, we have shown that we can obtain pairs  $(g_1, g_2)$  of elements of  $\mathbf{G}$  such that  $g_2 = g_1^p$  where  $p$  is a product of terms of the form  $C^{-1}(M_i \rightarrow M_j) \cdot C(M_i \rightarrow M_k)$ . Now, we would like to be able to reuse Algorithm 1 with the obtained values of  $g_1$  and  $g_2$  as starting values in order to obtain a pair of the more complex form described in Theorem 3.10.

Unfortunately, this is not always possible, as we will show through the following example.

**Example 4.3** We introduce a new protocol, which we call the *Tri-GDH* protocol. This protocol can be defined through three strands and three histories:

#### Protocol 3: Tri-GDH Protocol

$$\begin{aligned}
s_1 &= \langle +\alpha^{r_1}, -\alpha^{r_3}, +\alpha^{r_1 r_3 K_{12}}, -\alpha^{r_2 r_3 K_{13}} \rangle \\
s_2 &= \langle +\alpha^{r_2}, -\alpha^{r_1}, +\alpha^{r_1 r_2 K_{23}}, -\alpha^{r_1 r_3 K_{12}} \rangle \\
s_3 &= \langle +\alpha^{r_3}, -\alpha^{r_2}, +\alpha^{r_2 r_3 K_{13}}, -\alpha^{r_1 r_2 K_{23}} \rangle \\
\pi_1 &= \langle \langle s_2, 1 \rangle, \langle s_3, 2 \rangle, \langle s_3, 3 \rangle, \langle s_1, 4 \rangle \rangle \\
\pi_2 &= \langle \langle s_3, 1 \rangle, \langle s_1, 2 \rangle, \langle s_1, 3 \rangle, \langle s_2, 4 \rangle \rangle \\
\pi_3 &= \langle \langle s_1, 1 \rangle, \langle s_2, 2 \rangle, \langle s_2, 3 \rangle, \langle s_3, 4 \rangle \rangle
\end{aligned}$$

A run of this protocol is represented in Fig. 6 (we do not use the classical strand notation in order to avoid crossing arrows). During the first round of the protocol, the three central messages are exchanged, while the three external ones are computed from those just received and sent during the second round.

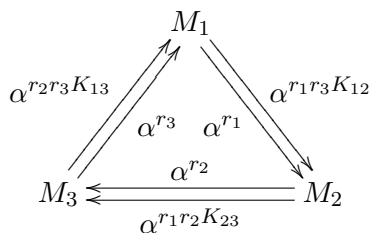


Figure 6: A run of the Tri-GDH protocol

An application of Theorem 3.10 for this protocol with  $i = 1$ ,  $j = 2$  and  $k = 3$  gives:

$$r_1 \cdot K_{13}^{-1} = 1 \cdot r_1 K_{12} \cdot (r'_1 K_{12})^{-1} \cdot r'_1 \cdot r_2^{-1} \cdot r'_2 K_{2I} \cdot (r''_3 K_{13})^{-1} \cdot r''_3 \cdot K_{2I}^{-1}$$

where  $r_i$ ,  $r'_i$ ,  $r''_i$  represent random values generated during three sessions of the protocol, the participants of these sessions being  $\{M_1, M_2, M_3\}$ ,  $\{M_1, M_2, M_I\}$  and  $\{M_1, M_I, M_3\}$  respectively.

Among these contributions we first consider  $r'_1$ ,  $r_2^{-1}$  and  $r''_3$ . These three services are provided as first elements of histories: the values  $\alpha^{r'_1}$ ,  $\alpha^{r_2^{-1}}$  and  $\alpha^{r''_3}$  are provided independently of any input value that the intruder could choose. Unfortunately, in order to build a pair  $(g_1, g_2)$  such that  $g_2 = g_1^p$  where  $p = r'_1 r_2^{-1} r''_3$ , we would need to submit  $\alpha^{r'_1}$  as input of the  $r''_3$ -service or, conversely, to submit  $\alpha^{r''_3}$  as input of the  $r'_1$ -service, which is unfortunately impossible.

Guided by this example, we can more generally observe that we are usually unable to compose two contributions containing initial parts of the corresponding histories if we have to collect these contributions in the same variable, which occurs when their powers have the same sign in the expression of  $P(\pi_i(\bar{0}))$ . This problem could not occur in the case we considered in the previous section since we had to collect contributions with exponents of different signs.

Another kind of services can be problematic: if two services have the same input and two distinct outputs, we may observe that  $\pi_j(x) = \pi_k(y)$  for these service's input and that the corresponding element of the GDH-Term  $t$  will be affected twice in Algorithm 1. This was not a problem when the precondition  $g_1 = g_2$  was verified, but becomes awkward when we try to reuse this algorithm in order to build more complex pairs. Indeed we will lose any non trivial relation that could exist between  $g_1$  and  $g_2$  before starting Algorithm 1. We illustrate this problem in the following example.

**Example 4.4** Suppose we applied Algorithm 1 and obtained two values  $g_1 = \alpha$  and  $g_2 = \alpha^p$ . Now, we would like to reuse the same algorithm with

the product of contributions  $C^{-1}(M_1 \rightarrow M_2) \cdot C(M_1 \rightarrow M_3)$  (in order to obtain a pair  $(g_1, g_2)$  where  $g_2 = g_1^{p \cdot C^{-1}(M_1 \rightarrow M_2) \cdot C(M_1 \rightarrow M_3)}$ ), the strand  $s_1$  being defined as

$$\begin{array}{ccc} & \xrightarrow{\alpha^x} & s_1 \\ & & \downarrow \\ & & \bullet \xrightarrow{\alpha^{xr_1}} \\ & & \downarrow \\ & & \bullet \xrightarrow{\alpha^{x\hat{r}_1}} \end{array}$$

and given that  $\pi_2(2) = \pi_3(2) = \langle s_1, 1 \rangle$ ,  $\pi_2(3) = \langle s_1, 2 \rangle$  and  $\pi_3(3) = \langle s_1, 3 \rangle$ .

Applying Algorithm 1 anew provides the following conversation:

$$\begin{array}{ccc} s_I & \xrightarrow{\alpha^p} & s_1 \\ \downarrow & \alpha^{pr_1} & \downarrow \\ \bullet & \longleftarrow & \bullet \\ \downarrow & \alpha^{p\hat{r}_1} & \downarrow \\ \bullet & \longleftarrow & \bullet \end{array}$$

The resulting pair will be  $(g_1, g_2) = (\alpha^{pr_1}, \alpha^{p\hat{r}_1})$ , so we will have  $g_2 = g_1^{r_1^{-1}\hat{r}_1}$  instead of the relation  $g_2 = g_1^{pr_1^{-1}\hat{r}_1}$  we need to obtain.

We now define the notions of *starting* and *splitting* points of histories, which will be useful to describe the problems we just described. We will use these notions in the next proposition, which gives sufficient conditions for products of contributions to be collectible by the intruder.

**Definition 4.5** Consider a GDH-Protocol executed by  $n$  participants and let  $s_1, \dots, s_n$  be the  $n$  strands and  $\pi_1, \dots, \pi_n$  be the  $n$  histories given in this protocol's definition. We define  $\text{start}(\pi_i)$  as the first node of  $\pi_i$ , that is,  $\pi_i(1)$ .

We then say that the product of contributions  $\prod_{i \in \mathcal{I}} C^{e_i}(M_{j_i} \rightarrow M_{k_i})$  (with  $\mathcal{I}$  a set of indices,  $e_i \in \{-1, 1\}$ ,  $1 \leq j_i, k_i \leq n$ ) contains  $x$   $\text{start}^+$  (resp.  $\text{start}^-$ ) if there exist  $x$  indices in  $\mathcal{I}$  such that  $e_i = 1$  (resp.  $e_i = -1$ ) and  $\text{start}(\pi_{k_i})$  belongs to  $s_{j_i}$ .

By extension, we say that  $\prod_{i \in \mathcal{I}} C^{e_i}(M_{j_i} \rightarrow M_{k_i})$  contains  $x$  starts (or starting points) if it contains  $x_1$   $\text{start}^+$ ,  $x_2$   $\text{start}^-$  and  $x_1 + x_2 = x$ .

**Definition 4.6** Consider a GDH-Protocol executed by  $n$  participants and let  $s_1, \dots, s_n$  be the  $n$  strands and  $\pi_1, \dots, \pi_n$  be the  $n$  histories given in this protocol's definition. We define  $\text{split}(\pi_i, \pi_j)$  as the last node which is part of both  $\pi_j$  and  $\pi_k$ , that is,  $\pi_i(k)$  where  $k = \max_l (\pi_i(l) = \pi_j(l))$  ( $\text{split}(\pi_i, \pi_j)$  is undefined if  $\pi_i(k) \neq \pi_j(k) \forall k$ ).

We say that the product of contributions  $\prod_{i \in \mathcal{I}} C^{-1}(M_{j_i} \rightarrow M_{k_i}) \cdot C(M_{j_i} \rightarrow M_{l_i})$  (with  $\mathcal{I}$  a set of indices,  $1 \leq j_i, k_i, l_i \leq n$ ) contains  $x$  splits (or splitting points) if there exist  $x$  indices in  $\mathcal{I}$  such that  $\text{split}(\pi_{k_i}, \pi_{l_i})$  belongs to  $s_{j_i}$ .

One last notion will be useful for our next proposition: the notion of precedence of contributions. We say that contribution  $C(M_i \rightarrow M_j)$  precedes contribution  $C(M_i \rightarrow M_k)$  if every node of  $\pi_k$  belonging to  $s_i$  is preceded by a node of  $\pi_j$  belonging to the same strand.

**Definition 4.7** Consider a GDH-Protocol executed by  $n$  participants and let  $s_1, \dots, s_n$  be the  $n$  strands and  $\pi_1, \dots, \pi_n$  be the  $n$  histories given in this protocol's definition. We say that  $C(M_i \rightarrow M_j)$  precedes (written  $\preceq$ )  $C(M_i \rightarrow M_k)$  iff

$$\begin{aligned} & \forall y \text{ such that } \pi_k(y) \text{ belongs to } s_i, \\ & \exists x : \pi_j(x) \text{ belongs to } s_i \text{ and } \pi_j(x) \preceq \pi_k(y). \end{aligned}$$

Given a node  $n$  on  $s_i$ , we also write that  $C(M_i \rightarrow M_j) \preceq n$  if  $\exists x : \pi_j(x)$  belongs to  $s_i$  and  $\pi_j(x) \preceq n$ , and that  $n \preceq C(M_i \rightarrow M_j)$  when  $\forall x : \pi_j(x)$  belongs to  $s_i$ ,  $n \preceq \pi_j(x)$ .

The strict precedence relation  $\prec$  corresponds to the precedence relation except that we replace " $\preceq$ " with " $\prec$ " in its definition.

These definitions are used in the following proposition in which we state sufficient conditions for the possibility of building more complex pairs of elements of  $\mathsf{G}$  than those described in Proposition 4.1.

**Proposition 4.8** Consider a GDH-Protocol with  $n$  participants and let  $p = \prod_{i \in \mathcal{I}} C^{-1}(M_{j_i} \rightarrow M_{k_i}) \cdot C(M_{j_i} \rightarrow M_{l_i})$  (with  $1 \leq j_i, k_i, l_i \leq n$  and  $\mathcal{I}$  being a set of indices) be a product of contributions such that all pairs of contributions are provided on different strands. Then an active attacker can obtain a pair  $(g_1, g_2)$  of elements of  $\mathsf{G}$  such that  $g_2 = g_1^p$  if one of the following conditions is verified:

1.  $p$  contains at most one splitting point and no starting point,
2.  $p$  contains no splitting point, one  $start^+$  and no  $start^-$ ,
3.  $p$  contains no splitting point, no  $start^+$  and one  $start^-$ ,
4.  $p$  contains no splitting point, one  $start^+$  (for the index  $i_+ \in \mathcal{I}$ ), one  $start^-$  (for the index  $i_- \in \mathcal{I}$ ),  $k_{i_+} = k_{i_-}$  and  $l_{i_+} = l_{i_-}$ .

*Proof.* Our proof of this proposition proceeds by using Algorithm 1 (or slight variants of it) and by verifying that, when any condition stated above is respected, the resulting pair  $(g_1, g_2)$  has the expected form.

1.  $p$  contains at most one splitting point and no starting point  
Let  $m \in \mathcal{I}$  be the index such that  $split(\pi_{k_m}, \pi_{l_m})$  belongs to  $s_{j_m}$ , or  $m$  be a random element of  $\mathcal{I}$  if  $p$  does not contain any splitting point. If we initialize  $g_1$  and  $g_2$  to  $\alpha$  (or to any random value) and execute Algorithm 1 for the product  $C^{-1}(M_{j_m} \rightarrow M_{k_m}) \cdot C(M_{j_m} \rightarrow M_{l_m})$ , we obtain a first pair  $(g_1, g_2)$ .

Then, we may successively apply Algorithm 1 for all products of contributions corresponding to indexes in  $i \in \mathcal{I} - \{m\}$ , using the values obtained for  $g_1$  and  $g_2$  at the end of each execution as input for the next one.

This procedure provides the expected values because Algorithm 1 will transform pairs  $(g_1, g_2)$  such that  $g_2 = g_1^{p_x}$  into pairs  $(g_1, g_2)$  such that  $g_2 = g_1^{p_x \cdot C^{-1}(M_{j_i} \rightarrow M_{k_i}) \cdot C(M_{j_i} \rightarrow M_{l_i})}$  for each value of  $i \in \mathcal{I} - \{m\}$  as long as the considered product of contributions does not contain any splitting or starting point.

2. *p contains no splitting point, one  $start^+$  and no  $start^-$*

The process is nearly identical to the one above. Let  $m \in \mathcal{I}$  be the index such that  $start(\pi_{l_m})$  belongs to  $s_{j_m}$ . If we initialize  $g_1$  and  $g_2$  to  $\alpha$  and execute Algorithm 1 for the product  $C^{-1}(M_{j_m} \rightarrow M_{k_m}) \cdot C(M_{j_m} \rightarrow M_{l_m})$ , we obtain a first pair  $(g_1, g_2)$ .

Then, we may successively execute Algorithm 1 for all products of contributions corresponding to indexes in  $i \in \mathcal{I} - \{m\}$ , providing the values obtained for  $g_1$  and  $g_2$  at the end of each execution as input for the next one.

The correctness of this procedure relies on the same observations as above.

3. *p contains no splitting point, no  $start^+$  and one  $start^-$*

The procedure is the same as the previous one.

4. *p contains no splitting point, one  $start^+$  (for the index  $i_+ \in \mathcal{I}$ ), one  $start^-$  (for the index  $i_- \in \mathcal{I}$ ),  $k_{i_+} = k_{i_-}$  and  $l_{i_+} = l_{i_-}$ .*

Suppose first  $i_+ = i_-$ . In that case, the process described for the first condition applies.

Suppose now  $i_+ \neq i_-$ ,  $k = k_{i_+} = k_{i_-}$  and  $l = l_{i_+} = l_{i_-}$ . Suppose also  $C(M_{j_{i_-}} \rightarrow M_k) \prec C(M_{j_{i_-}} \rightarrow M_l)$ . If this condition holds and if we define  $\pi_k(1) = \langle s_{j_{i_-}}, \hat{z} \rangle$ , we can proceed as follows. First, initialize  $g_1$  and  $g_2$  to  $\alpha$  and apply Algorithm 1 for the product  $C^{-1}(M_{j_{i_-}} \rightarrow M_k) \cdot C(M_{j_{i_-}} \rightarrow M_l)$  until  $z = \hat{z}$  (we also execute this algorithm for this value of  $z$ ). At this point, our precedence assumption guarantees us that  $g_1 = uns\_term(\pi_k(1))$  and  $g_2 = \alpha$ . Then, with the current values of  $g_1$  and  $g_2$ , execute the same algorithm for the product  $C^{-1}(M_{j_{i_+}} \rightarrow M_k) \cdot C(M_{j_{i_+}} \rightarrow M_l)$ , obtaining a pair  $(g_1, g_2) = (uns\_term(\pi_k(1))^{C(M_{j_{i_+}} \rightarrow M_k)}, \alpha^{C(M_{j_{i_+}} \rightarrow M_l)})$ . Now complete the first execution of the algorithm for the values of  $z$  going from  $\hat{z} + 1$  to  $length(s_j)$  with the updated values of  $g_1$  and  $g_2$ . Finally, execute Algorithm 1 for the indexes  $i \in \mathcal{I} - \{i_+, i_-\}$ , always updating  $g_1$  and  $g_2$ . This provides the desired pair.

Suppose now  $C(M_{j_{i_-}} \rightarrow M_k) \not\prec C(M_{j_{i_-}} \rightarrow M_l)$ . This means that there is an index  $y$  such that  $\pi_l(y)$  belongs to  $s_{j_{i_-}}$  and such that  $\pi_l(y) \preceq \pi_k(x)$  for every index  $x$  such that  $\pi_k(x)$  belongs to  $s_{j_{i_-}}$ . So, in particular, we have

that  $\pi_l(y) \preceq \pi_k(1)$  since  $\pi_k(1)$  belongs to  $s_{j_{i_-}}$ . But this last observation guarantees us that  $\pi_l(1)$ , which belongs to  $s_{j_{i_+}}$ , strictly precedes all nodes of  $\pi_k$  belonging to that strand, and therefore that  $C(M_{j_{i_+}} \rightarrow M_k) \prec C(M_{j_{i_+}} \rightarrow M_l)$ . This precedence relation is symmetric to the one above, and allows us to collect  $g_1$  and  $g_2$  by using a symmetric treatment. ■

All these sufficient conditions can be verified by checking on which strands histories split and start and are independent of the other aspects of the routing of the messages. This will be very convenient to check whether a pair of the form given in Theorem 3.10 can be obtained by the attacker, as we will see.

Unfortunately, in some cases, it will not be possible to be sure that one of these conditions is verified. So, we will define one more sufficient condition. Its verification will be slightly more demanding as it will require to check precedence relations on contributions.

**Proposition 4.9** *The following condition is sufficient to make the wording of Proposition 4.8 correct:*

5.  $p$  contains no splitting point, one  $start^+$  (for the index  $i_+ \in \mathcal{I}$ ), one  $start^-$  (for the index  $i_- \in \mathcal{I}$ ,  $i_+ \neq i_-$ ) and  $C(M_{j_{i_-}} \rightarrow M_{k_{i_-}}) \prec C(M_{j_{i_-}} \rightarrow M_{l_{i_-}})$  or  $C(M_{j_{i_+}} \rightarrow M_{l_{i_+}}) \prec C(M_{j_{i_+}} \rightarrow M_{k_{i_+}})$

*Proof.* This condition explicitly states that one of the precedence conditions we proved when examining the fourth condition of Prop. 4.8 is valid. The desired pair  $(g_1, g_2)$  can therefore be built by using exactly the same technique. ■

An example of the treatment described in these proofs is provided in the full attack process example we propose in Appendix C.

#### 4.4 Attacking GDH-Protocols with four and five participants

We will now prove that, when considering GDH-Protocols with four or five participants, the product of contributions given in the proof of Theorem 3.10 respects one of the conditions of Proposition 4.8 and Proposition 4.9 for at least one choice of the users  $M_i, M_j, M_k$  and of the sets  $S_j$  and  $S_k$ . In order to make it easier to extend this result to an unbounded number of protocol participants, we will in fact consider only two possible choices for the set  $S_j$  and  $S_k$ :

- $S_j = \{M_k\}$  and  $S_k = M - \{M_i, M_k\}$  and
- $S_j = M - \{M_i, M_j\}$  and  $S_k = \{M_j\}$ .

It is easy to check that these two choices of  $S_j$  and  $S_k$  respect the conditions given in the proof of Theorem 3.10.

This will prove that the attacker is always able to obtain a pair of values  $(g_1, g_2)$  such that  $M_i$  would compute  $g_2$  as his view of the group key if he uses

$g_1$  as input value for this computation. We are not sure however whether the attacker can complete his attack by submitting  $g_1$  to  $M_i$ :

1. building  $g_1$  could require the use of services that  $M_i$  provides after having computed his view of the group key or
2.  $M_I$  could need to use the value that  $M_i$  will use to compute the group key in order to build the pair  $(g_1, g_2)$ .

We can check that the first problem cannot occur: when building  $g_1$ , the only contribution that uses the strand from which  $M_i$  is computing his view of the group key is  $C(M_i \rightarrow M_i)$  and we know that all nodes which have to be exploited when collecting  $C(M_i \rightarrow M_i)$  by using Algorithm 1 strictly precede  $\pi_i(\bar{1})$ , that is, the node on which  $g_1$  has to be sent to  $M_i$ .

Let us now consider the second problem. We already know (from our treatment of the first problem) that we will never need to submit a specific value instead of the last element of  $\pi_i$  when computing  $g_1$ . It is however possible that this element has to be used when computing  $g_2$ . The only contribution that uses the strand from which  $M_i$  is computing his view of the group key in order to build  $g_2$  is  $C(M_i \rightarrow M_j)$ . We can also observe that if the last element of  $\pi_i$  has to be affected to some specific value when collecting  $C(M_i \rightarrow M_j)$ , then the last element of  $\pi_i$  is also part of  $\pi_j$  and, therefore,  $split(\pi_i, \pi_j)$  belongs to  $s_i$ .

For that reason, instead of simply checking if there is a choice of users  $M_i, M_j, M_k$  and of sets  $S_j$  and  $S_k$  such that the product of contributions given in the proof of Theorem 3.10 respects one of the conditions of Proposition 4.8 and 4.9, we will also require this choice of indices and sets to be such that  $split(\pi_i, \pi_j)$  does not belong to  $s_i$ .

Obtaining such a result will allow us to conclude that it is impossible to build a secure GDH-Protocol with four or five participants. In Section 4.5, we will show how to extend this result to GDH-Protocols with more than five participants.

**Theorem 4.10** *For any GDH-Protocol with four or five participants, it is possible for an active attacker to obtain a pair  $(g_1, g_2)$  of elements of  $\mathbf{G}$  such that  $g_2 = g_1^p$  where*

$$\begin{aligned}
p &= C^{-1}(M_i \rightarrow M_i) \cdot C(M_i \rightarrow M_j) \\
&\quad \cdot [S_j \setminus M_I : C^{-1}(M_i \rightarrow M_j) \cdot C(M_i \rightarrow M_k)] \\
&\quad \cdot \prod_{M_l \in S_k} [S_j \setminus M_I : C^{-1}(M_l \rightarrow M_i) \cdot C(M_l \rightarrow M_k)] \\
&\quad \cdot \prod_{M_l \in S_j} [S_k \setminus M_I : C^{-1}(M_l \rightarrow M_i) \cdot C(M_l \rightarrow M_j)] \\
&\quad \cdot \prod_{l \in 1 \dots n} K_{I l}^{e_l}
\end{aligned}$$

for some choice of  $M_i, M_j, M_k, S_j, S_k$  and  $e_l$ ; where  $M_i, M_j$  and  $M_k$  are

three different members of the group  $M$  while  $S_j$  and  $S_k$  are two disjoint sets of users defined either as

- $S_j = \{M_k\}$  and  $S_k = M - \{M_i, M_k\}$  or
- $S_j = M - \{M_i, M_j\}$  and  $S_k = \{M_j\}$ .

Furthermore, it is possible to select  $M_i$  and  $M_j$  in such a way that  $split(\pi_i, \pi_j)$  does not belong to  $s_i$ .

*Proof.* If we suppress the factor  $\prod_{l \in 1 \dots n} K_{ll}^{e_l}$  which is known by  $M_I$  from the product  $p$ , we can check that  $p$  has the form considered in Proposition 4.8. We will therefore try to verify that all GDH-Protocols with at least four participants respect at least one of the four sufficient condition of Proposition 4.8 for an adequate choice of  $M_i, M_j, M_k, S_j$  and  $S_k$ . We will also require this choice to be such that  $split(\pi_i, \pi_j)$  does not belong to  $s_i$ , which guarantees us that the attacker will also be able to replace  $\pi_i(\bar{1})$  with  $g_1$  in the session from which he is excluded. There will be cases for which it will not be possible to verify one condition of Proposition 4.8. For those cases, we will verify the condition stated in Proposition 4.9.

The verification of the conditions of Proposition 4.8 will be carried out by considering all possible ways for four histories (say  $\pi_1, \pi_2, \pi_3$  and  $\pi_4$ ) to split and start in four and five-parties GDH-Protocols. In fact, it can be verified that there are only six ways of making these four histories split and start: we represented them on Fig. 7.<sup>1</sup>

On this figure, we represented the nodes on which these histories start (i.e.  $\pi_i(1)$ ) on the left of each subfigure, then, when it occurs, the nodes on which they split and, finally, the nodes on which they finish (i.e.  $\pi_i(\bar{1})$ ). As an example, the representation of the three histories of the Ex-GDH protocol according to this convention is given in Fig. 8.

Then, for the four histories we consider, we select  $M_i, M_j$  and  $M_k$  among the four corresponding group members (that is,  $M_1, M_2, M_3$  and  $M_4$ ) and consider the two possible choices for  $S_j$  and  $S_k$  described in this theorem's statement.

We then perform an exhaustive search, considering all possible strands for the splitting and starting points and verifying for each of them if one of the sufficient conditions stated in Prop. 4.8 could be verified for at least one specific choice of  $M_i, M_j, M_k, S_j$  and  $S_k$ . This exhaustive search was performed automatically with the assistance of a small program described in Algorithm 2.

---

<sup>1</sup>This comes down to generating all binary forests with four leaves. A way to proceed consists in generating all partitions of these four leaves into trees (there are five possible such partitions as  $4 = 1 + 1 + 1 + 1 = 1 + 1 + 2 = 2 + 2 = 1 + 3$ ; they respectively correspond to one tree with 4 leaves, four trees with 1 leaf, two trees with 1 leaf and one tree with 2 leaves, ...) and, for each of these partitions into trees, generating all trees with the desired number of leaves (the number of such trees is given by the Wedderburn-Etherington numbers).

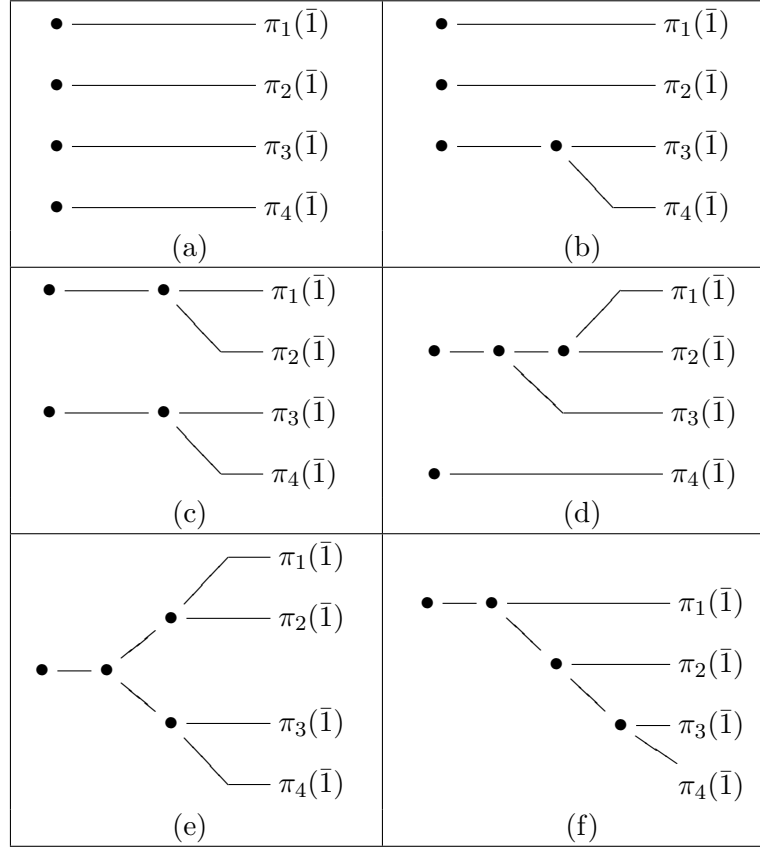


Figure 7: Six varieties of binary forests with four leafs.

This program provided us with an adequate choice in all cases, except nine.

As an example, if we look at the Ex-GDH protocol, such an adequate choice is  $M_i = M_3$ ,  $M_j = M_2$ ,  $M_k = M_1$ ,  $S_j = \{M_1\}$  and  $S_k = \{M_2\}$ : in that case, the product

$$\begin{aligned}
 P(\pi_3(\bar{0})) &= C^{-1}(M_3 \rightarrow M_3) \cdot C(M_3 \rightarrow M_2) \\
 &\quad \cdot [M_1 \setminus M_I : C^{-1}(M_3 \rightarrow M_2) \cdot C(M_3 \rightarrow M_1)] \\
 &\quad \cdot [M_1 \setminus M_I : C^{-1}(M_2 \rightarrow M_3) \cdot C(M_2 \rightarrow M_1)] \\
 &\quad \cdot [M_2 \setminus M_I : C^{-1}(M_1 \rightarrow M_3) \cdot C(M_1 \rightarrow M_2)]
 \end{aligned}$$

contains no splitting point and two starting points in the term  $[M_2 \setminus M_I : C^{-1}(M_1 \rightarrow M_3) \cdot C(M_1 \rightarrow M_2)]$ . Actually, a more detailed analysis shows that these two starting points do not raise any difficulty as  $\pi_2$  and  $\pi_3$  have a splitting point on  $s_2$ , which implies that we do not have to use the two corresponding starting services when applying Algorithm 1.

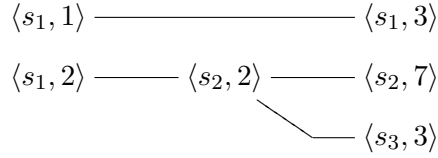


Figure 8: Spitting and starting points in the Ex-GDH Protocol.

As we said, our automatic treatment allowed us to obtain adequate choices for all configurations of histories, except in nine cases, all of them in the forests represented in Fig. 7(a). The strands on which  $\pi_1$ ,  $\pi_2$ ,  $\pi_3$  and  $\pi_4$  start in these nine cases are given in Table 1.

Table 1: Problematic strands for the starting points of  $\pi_1$ ,  $\pi_2$ ,  $\pi_3$  and  $\pi_4$ .

	$\pi_1$	$\pi_2$	$\pi_3$	$\pi_4$
1)	$s_2$	$s_1$	$s_4$	$s_3$
2)	$s_2$	$s_3$	$s_4$	$s_1$
3)	$s_2$	$s_4$	$s_1$	$s_3$
4)	$s_3$	$s_1$	$s_4$	$s_2$
5)	$s_3$	$s_4$	$s_1$	$s_2$
6)	$s_3$	$s_4$	$s_2$	$s_1$
7)	$s_4$	$s_1$	$s_2$	$s_3$
8)	$s_4$	$s_3$	$s_1$	$s_2$
9)	$s_4$	$s_3$	$s_2$	$s_1$

Unfortunately, in these nine cases, it is not possible to exhibit a choice of  $M_i$ ,  $M_j$ ,  $M_k$ ,  $S_j$  and  $S_k$  respecting one of the conditions of Proposition 4.8. So, we treated them by hand, each time exhibiting two possible choices of  $M_i$ ,  $M_j$ ,  $M_k$ ,  $S_j$  and  $S_k$  and verifying that at least one of them verifies the sufficient condition described in Proposition 4.9. As an example, we explain our treatment of the ninth case, where  $start(\pi_1)$  belongs to  $s_4$ ,  $start(\pi_2)$  belongs to  $\pi_3$ ,  $start(\pi_3)$  belongs to  $s_2$  and  $start(\pi_4)$  belongs to  $s_1$ . The treatment of the other cases is similar, and appropriate choices of  $M_i$ ,  $M_j$ ,  $M_k$ ,  $S_j$  and  $S_k$  are given in Appendix D.

Since the four histories represented on Fig. 7(a) have no splitting point,  $split(\pi_i, \pi_j)$  does obviously not belong to  $s_i$ . If we look at the possible choices for  $M_i$ ,  $M_j$ ,  $M_k$ ,  $S_j$  and  $S_k$  in the case we are examining, we observe that we always have to choose one  $start^+$  and one  $start^-$ . Unfortunately, there is no possible choice verifying the fourth condition of Proposition 4.8.

---

**Algorithm 2** Returns a list of the forests and values of splitting and starting points for which we cannot find a choice of  $M_i, M_j, M_k, S_j$  and  $S_k$  such that  $split(\pi_i, \pi_j)$  does not belong to  $s_i$  and the product  $p$  defined in the wording of Thm. 4.10 respects one of the conditions of Proposition 4.8.

---

```

for all Forest in Fig. 7 do
  for all strands  $(s_a, s_b, s_c, s_d)$  in  $\{s_1, s_2, s_3, s_4, s_5\}$  on which the histories
  of the current forest can split and start do
    SolutionFound := False
    for all distinct  $M_i, M_j, M_k$  selected in  $\{M_1, M_2, M_3, M_4\}$  do
       $S_j := \{M_1, M_2, M_3, M_4, M_5\} \setminus \{M_i, M_j\}$     $S_k := \{M_j\}$ 
      if one of the conditions of Prop. 4.8 is verified for this choice of  $M_i,$ 
       $M_j, M_k, S_j$  and  $S_k$  and  $split(\pi_i, \pi_j)$  does not belong to  $s_i$  then
        SolutionFound := True
      end if
       $S_j := \{M_k\}$     $S_k := \{M_1, M_2, M_3, M_4, M_5\} \setminus \{M_i, M_k\}$ 
      if one of the conditions of Prop. 4.8 is verified for this choice of  $M_i,$ 
       $M_j, M_k, S_j$  and  $S_k$  and  $split(\pi_i, \pi_j)$  does not belong to  $s_i$  then
        SolutionFound := True
      end if
    end for
  if SolutionFound = False then
    Write “The forest Forest with splitting and starting points on  $s_a,$ 
     $s_b, s_c, s_d$  is problematic.”
  end if
end for
end for

```

---

We now show that it is however possible to verify the condition given in Proposition 4.9.

Suppose we choose  $M_i = M_1, M_j = M_2, M_k = M_4, S_j = \{M_4\}$  and  $S_k = M - \{M_1, M_4\}$ . This choice implies that the product  $p$  of this theorem’s statement contains one  $start^+$  (in the term  $C^{-1}(M_1 \rightarrow M_2) \cdot C(M_1 \rightarrow M_4)$ ), one  $start^-$  (in the term  $C^{-1}(M_4 \rightarrow M_1) \cdot C(M_4 \rightarrow M_2)$ ), and no splitting point.

Assume first that  $C(M_1 \rightarrow M_4) \prec C(M_1 \rightarrow M_2)$ . This relation is one of those described in Prop. 4.9, so it is possible to obtain a pair  $(g_1, g_2)$  of the form desired. The same technique can be adopted if  $C(M_4 \rightarrow M_1) \prec C(M_4 \rightarrow M_2)$ .

Suppose now that  $C(M_1 \rightarrow M_4) \not\prec C(M_1 \rightarrow M_2)$  and  $C(M_4 \rightarrow M_1) \not\prec C(M_4 \rightarrow M_2)$ . From these assumptions, the definition of paths, and the fact that  $C(M_4 \rightarrow M_1)$  contains a  $start^+$ , we can write:

$$\pi_2(1) \prec C(M_4 \rightarrow M_2) \preceq \pi_1(1).$$

We suggest now a second choice of the parameters for  $p$ :  $M_i = M_2$ ,  $M_j = M_1$ ,  $M_k = M_3$ ,  $S_j = \{M_3\}$  and  $S_k = \mathbb{M} - \{M_2, M_3\}$ . This choice implies that the product  $p$  contains one  $\text{start}^+$  (in the term  $C^{-1}(M_2 \rightarrow M_1) \cdot C(M_2 \rightarrow M_3)$ ), one  $\text{start}^-$  (in the term  $C^{-1}(M_3 \rightarrow M_2) \cdot C(M_3 \rightarrow M_1)$ ) and no splitting point.

If  $C(M_2 \rightarrow M_3) \prec C(M_2 \rightarrow M_1)$  or if  $C(M_3 \rightarrow M_2) \prec C(M_3 \rightarrow M_1)$ , the condition described in Prop. 4.9 is verified and obtain a pair  $(g_1, g_2)$  of the form desired. Suppose now that both these precedence relations are false. Then, the definition of paths and the fact that  $C(M_3 \rightarrow M_2)$  contains a  $\text{start}^-$  implies that:

$$\pi_1(1) \prec C(M_3 \rightarrow M_1) \preceq \pi_2(1),$$

which is in contradiction with the relation  $\pi_2(1) \prec \pi_1(1)$  obtained above.

Therefore, one of the two choices of  $M_i$ ,  $M_j$ ,  $M_k$ ,  $S_j$  and  $S_k$  we proposed can be adopted.

A similar reasoning has been carried out for the eight remaining problematic cases. So, we found adequate choices for  $M_i$ ,  $M_j$ ,  $M_k$ ,  $S_j$  and  $S_k$  for any GDH-Protocol executed by four or five principals. ■

#### 4.5 Attacking GDH-Protocols with more than five participants

We now try to extend the result we just obtained to an unbounded number of protocol participants. This unbounded number of participants implies that we have to check the five sufficient conditions of Prop. 4.8 and Prop. 4.9 for an unbounded number of contributions and histories. We will see however that the specific choices of  $S_j$  and  $S_k$  we made will make this difficulty much easier to solve. The following theorem is the same as Theorem 4.10, except that it claims the insecurity of any GDH-Protocols with at least four participants.

**Theorem 4.11** *For any GDH-Protocol with  $n \geq 4$  participants, it is possible for an active attacker to obtain a pair  $(g_1, g_2)$  of elements of  $\mathbb{G}$  such that  $g_2 = g_1^p$  where*

$$\begin{aligned} p = & C^{-1}(M_i \rightarrow M_i) \cdot C(M_i \rightarrow M_j) \\ & \cdot [S_j \setminus M_I : C^{-1}(M_i \rightarrow M_j) \cdot C(M_i \rightarrow M_k)] \\ & \cdot \prod_{M_l \in S_k} [S_j \setminus M_I : C^{-1}(M_l \rightarrow M_i) \cdot C(M_l \rightarrow M_k)] \\ & \cdot \prod_{M_l \in S_j} [S_k \setminus M_I : C^{-1}(M_l \rightarrow M_i) \cdot C(M_l \rightarrow M_j)] \\ & \cdot \prod_{l \in 1 \dots n} K_{Il}^{e_l} \end{aligned}$$

for some choice of  $M_i, M_j, M_k, S_j, S_k$  and  $e_l$ ; where  $M_i, M_j$  and  $M_k$  are three different members of the group  $M$  while  $S_j$  and  $S_k$  are two disjoint sets of users defined either as

- $S_j = \{M_k\}$  and  $S_k = M - \{M_i, M_k\}$  or
- $S_j = M - \{M_i, M_j\}$  and  $S_k = \{M_j\}$ .

Furthermore, it is possible to select  $M_i$  and  $M_j$  in such a way that  $\text{split}(\pi_i, \pi_j)$  does not belong to  $s_i$ .

*Proof.* We already proved this theorem for four and five participants. We now prove that the four and five-party case implies the validity of the result for any larger number of protocol participants.

Consider a GDH-Protocol  $GDH_1$  with  $n > 5$  participants and consider its first four histories  $\pi_1, \pi_2, \pi_3$  and  $\pi_4$ .

Consider now a five-party GDH-Protocol  $GDH_2$ , the first four histories of which split and start on the same strands as the  $GDH_1$  protocol, except that all histories splitting or starting on strands in the set  $\{s_6, \dots, s_n\}$  in the  $GDH_1$  protocol rather split or start on  $s_5$ .

For this protocol, we know from Theorem 4.10 that there is a choice of  $M_i, M_j$  and  $M_k$  as disjoint users in the set  $\{M_1, M_2, M_3, M_4\}$  and a choice of  $S_j$  and  $S_k$  such that

- $S_j = \{M_k\}$  and  $S_k = M - \{M_i, M_k\}$  or
- $S_j = M - \{M_i, M_j\}$  and  $S_k = \{M_j\}$

respecting one of the sufficient conditions of Prop. 4.8 and 4.9 and such that  $\text{split}(\pi_i, \pi_j)$  does not belong to  $s_i$ . We call  $p'$  the expression of the product  $p$  in the  $GDH_2$  protocol, call  $S$  the set in  $\{S_j, S_k\}$  containing more than one protocol participant and  $M_s$  the user of the set in  $\{S_j, S_k\}$  containing one protocol participant.

We now observe that the product

$$\prod_{M_l \in \{M_5, \dots, M_n\}} [S \setminus M_l : C^{-1}(M_l \rightarrow M_i) \cdot C(M_l \rightarrow M_s)]$$

in the  $GDH_1$  protocol contains at most one  $\text{split}$ , one  $\text{start}^+$  and one  $\text{start}^-$ . Furthermore, these numbers of splitting and starting points are equal to those in the product

$$[M_5 \setminus M_l : C^{-1}(M_5 \rightarrow M_i) \cdot C(M_5 \rightarrow M_s)]$$

in the  $GDH_2$  protocol, and the splitting and starting points occur in contributions intended to the same users in the two cases.

So, we know that the products  $p$  and  $p'$  contain the same number of splitting and starting points, and that these points are part of contributions intended to the same users. This implies that if one of the conditions of Prop. 4.8 is verified in  $p'$ , it is also verified in  $p$  and the selection of  $M_i$ ,

$M_j, M_k, S_j$  and  $S_k$  made for the  $GDH_2$  protocol is also valid for the  $GDH_1$  protocol.

This is only valid for protocols for which we found a choice of  $M_i, M_j, M_k, S_j$  and  $S_k$  respecting one of the conditions of Prop. 4.8. On the other hand, the protocols which required to use the condition of Prop. 4.9 are such that all splitting and starting points of the first four histories belong to the first four strands and, so, the arguments we used in the proof for four and five participants remain valid. Therefore, at least one of the two possible choices of  $M_i, M_j, M_k, S_j$  and  $S_k$  we suggest in Appendix D is also valid for the  $GDH_1$  protocol. ■

This concludes our proof that it is never possible to guarantee the IKA property for all participants of a GDH-Protocol executed by at least four parties.

## 5 Concluding Remarks

### 5.1 Summary

In this paper, we analyzed a family of authenticated group key agreement protocols defined as a generalization of the A-GDH protocols proposed in the context of the Cliques project [1, 2].

Our main result is the proof that it is impossible to define a protocol of this family providing implicit key authentication for all group members if it is executed by at least four participants. As we established this proof throughout all the paper, we gather its main points here.

We prove our result by providing a systematic way to set up a scenario that undermines the implicit key authentication property. The process is as follows.

Consider a GDH-Protocol executed by a group  $M$  of  $n$  users such that  $n \geq 4$  and  $M_I \notin M$ . If  $M_I$  wants to undermine the IKA property in that session, he can select:

- three members of the group  $M$ :  $M_i, M_j$  and  $M_k$  and
- two disjoint sets of users  $S_j$  and  $S_k$  such that  $M_k \in S_j, M_j \in S_k, M_i \notin S_j \cup S_k$  and  $S_j \cup S_k \cup \{M_i\} = M$ .

This selection must also respect the two following conditions:

- the product

$$\begin{aligned}
p &= C^{-1}(M_i \rightarrow M_i) \cdot C(M_i \rightarrow M_j) \\
&\cdot [S_j \setminus M_I : C^{-1}(M_i \rightarrow M_j) \cdot C(M_i \rightarrow M_k)] \\
&\cdot \prod_{M_l \in S_k} [S_j \setminus M_I : C^{-1}(M_l \rightarrow M_i) \cdot C(M_l \rightarrow M_k)] \\
&\cdot \prod_{M_l \in S_j} [S_k \setminus M_I : C^{-1}(M_l \rightarrow M_i) \cdot C(M_l \rightarrow M_j)] \\
&\cdot \prod_{M_l \in M} K_{Il}^{e_l}
\end{aligned}$$

respects at least one of the conditions described in Propositions 4.8 and 4.9.

- $split(\pi_i, \pi_j)$  does not belong to  $s_i$

Theorem 4.11 guarantee that the choice of such  $M_i, M_j, M_k, S_j$  and  $S_k$  is always possible.

After having selected these values,  $M_I$  can build a pair  $(g_1, g_2)$  such that  $g_2 = g_1^p$  by exploiting a procedure similar to the one described in Algorithm 1 and described in the proof of Prop. 4.8 and 4.9, and replace the value that  $M_i$  will use to compute the group key with  $g_1$ .

At this time, and given that  $p = P(\pi_i(\bar{0}))$  as we proved in Theorem 3.10,  $M_i$  will compute  $g_2$  as his view of the group key, which is in contradiction with the implicit key authentication property.

A detailed example of this attack process is given in Appendix C.

## 5.2 Cardinality of the group

Unexpectedly, and even though the three-party version of the Cliques A-GDH.2 and SA-GDH.2 protocols have been shown to be flawed, our result is found to be only valid for protocols executed by at least four users. This shows that the attacks we discovered are really attacks against group protocols and emphasizes the need to consider these protocols differently than simple extensions of two-party ones.

We think this limit is minimal: we are not able to find any attack against the implicit key authentication property for the two-party version of the A-GDH.2 protocol, nor against our Tri-GDH protocol defined in Section 4.3. Our method fails to find attacks against these two protocols for two different reasons: we are not able to break the two-party version of the A-GDH.2 protocol because we are not able to find services which could be exploited in order to build a pair of the form desired. This is not the case for the Tri-GDH protocol as Theorem 3.10 provides several choices for such services. However, for this last protocol, we are not able to combine these services

in a useful way, as we have always need to use the starting point of three histories.

### 5.3 Conclusion

We think our contribution in this paper has two main aspects.

A first aspect is that we now know that the A-GDH protocols cannot be corrected without changing the design assumptions at their basis. One possible direction to solve this problem would consist in considering the use of a signature scheme or of message authentication codes, what would to separate the key generation part of the protocol (i.e. the sending of the partial Diffie-Hellman values) from the authentication mechanisms. This would allow to make the authentication more explicit, by including the identifiers of the protocol participants and freshness guarantees such as nonces for instance. Such a method has already been exploited in [16] for instance, or by Katz and Yung in [10] for an extension of the Burmester-Desmedt protocol [3] and studied from a more theoretical point of view by Datta & al. in [4] for instance.

A more theoretical aspect concerns the form of our result. While several papers (such as [7, 8, 11, 20]) describe systematic ways to analyze well-defined families of authentication protocols, we do not know any other general impossibility result for such families. It would be interesting to investigate in which measure our result could be transposed to other families of protocols. As our attacks are based on the absence of explicitness of the messages in GDH-Protocols, it would be interesting to see which degree of explicitness would be necessary to obtain secure authenticated group key agreement protocols.

Probably the most closely related results are those concerning the security of ping-pong protocols [7, 8]: as ping-pong protocols, GDH-Protocols are executed by successively applying well-defined transformations on the messages the different users receive (without checking anything about their content). In that sense, we could have used a method similar as their one, but only for obtaining the results of Section 3, i.e. for expressing the secrets of the different users as products of contributions and keys the intruder knows. On the other hand, the routing problems we considered in Section 4 have no correspondence in ping-pong protocols: these protocols consider only one history, and so do not raise the problems we encountered with splitting and starting points.

Our developments rely on several particularities which are only present in Dolev-Yao-type analysis of security protocols (as opposed to computational approaches): we consider a highly restricted set of actions which the intruder can perform, and our analysis method indicates attack scenarios for incorrect protocols rather than leading the analyst to the impossibility of finding a proof. Therefore, we think that our result emphasizes the advantages of

using high-level models in the analysis of security protocols.

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## A Strand Spaces and Bundles

The following definitions and lemma are taken from [21], Definitions 2.1-2.6 and Lemma 2.7.

**Definition A.1** *A signed GDH-Term is a pair  $\langle \sigma, t \rangle$  with  $t \in \mathbf{G}$  and  $\sigma$  is one of the symbols  $+$ ,  $-$ . We will write a signed GDH-Term as  $+t$  or  $-t$ .*

$(\pm\mathbf{G})^*$  is the set of finite sequences of signed GDH-Terms. We will denote a typical element of  $(\pm\mathbf{G})^*$  by  $\langle\langle\sigma_1, t_1\rangle, \dots, \langle\sigma_n, t_n\rangle\rangle$  or in a shorter way by  $\langle\sigma_1 t_1, \dots, \sigma_n t_n\rangle$ .

**Definition A.2** A strand space over  $\mathbf{G}$  is a set  $\Sigma$  with a trace mapping  $\text{tr} : \Sigma \rightarrow (\pm\mathbf{G})^*$ .

By abuse of language, we will still treat signed GDH-Terms as ordinary GDH-Terms. For instance, we shall refer to subterms of signed GDH-Terms. We will also usually refer to GDH-Terms simply as terms.

A strand space will usually be represented by its underlying set of strands  $\Sigma$ .

**Definition A.3** Fix a strand space  $\Sigma$ .

1. A node is a pair  $\langle s, i \rangle$ , with  $s \in \Sigma$  and  $i$  an integer satisfying  $1 \leq i \leq \text{length}(\text{tr}(s))$ . The set of nodes is denoted  $\mathcal{N}$ . We will say the node  $\langle s, i \rangle$  belongs to strand  $s$ . Clearly, every node belongs to a unique strand.
2. If  $n = \langle s, i \rangle \in \mathcal{N}$  then  $\text{index}(n) = i$  and  $\text{strand}(n) = s$ . Define  $\text{term}(n)$  to be  $(\text{tr}(s))(i)$ , i.e. the  $i$ -th signed term in the trace of  $s$ . Similarly,  $\text{uns\_term}(n)$  is  $((\text{tr}(s))(i))_2$ , i.e. the unsigned part of the  $i$ -th signed term in the trace of  $s$ .
3. There is an edge  $n_1 \rightarrow n_2$  if and only if  $\text{term}(n_1) = +t$  and  $\text{term}(n_2) = -t$  for some  $t \in \mathbf{G}$ . Intuitively, the edge means that  $n_1$  sends the message  $t$ , which is received by  $n_2$ , recording a potential causal link between those strands.
4. When  $n_1 = \langle s, i \rangle$  and  $n_2 = \langle s, i + 1 \rangle$  are members of  $\mathcal{N}$ , there is an edge  $n_1 \Rightarrow n_2$ . Intuitively, the edge expresses that  $n_1$  is an immediate causal predecessor of  $n_2$  on the strand  $s$ . We write  $n' \Rightarrow^+ n$  to mean that  $n'$  precedes  $n$  (not necessarily immediately) on the same strand.

$\mathcal{N}$  together with both sets of edges  $n_1 \rightarrow n_2$  and  $n_1 \Rightarrow n_2$  is a directed graph  $\langle \mathcal{N}, (\rightarrow \cup \Rightarrow) \rangle$ .

A bundle is a finite subgraph of  $\langle \mathcal{N}, (\rightarrow \cup \Rightarrow) \rangle$  for which we can regard the edges as expressing the causal dependencies of the nodes.

**Definition A.4** Suppose  $\rightarrow_{\mathcal{C}} \subset \rightarrow$ ; suppose  $\Rightarrow_{\mathcal{C}} \subset \Rightarrow$ ; and suppose that  $\mathcal{C} = \langle \mathcal{N}_{\mathcal{C}}, (\rightarrow_{\mathcal{C}} \cup \Rightarrow_{\mathcal{C}}) \rangle$  is a subgraph of  $\langle \mathcal{N}, (\rightarrow \cup \Rightarrow) \rangle$ .  $\mathcal{C}$  is a bundle if:

1.  $\mathcal{N}_{\mathcal{C}}$  and  $\rightarrow_{\mathcal{C}} \cup \Rightarrow_{\mathcal{C}}$  are finite;
2. if  $n_2 \in \mathcal{N}_{\mathcal{C}}$  and  $\text{term}(n_2)$  is negative, then there is a unique  $n_1$  such that  $n_1 \rightarrow_{\mathcal{C}} n_2$ ;
3. if  $n_2 \in \mathcal{N}_{\mathcal{C}}$  and  $n_1 \Rightarrow n_2$  then  $n_1 \Rightarrow_{\mathcal{C}} n_2$ ;

4.  $\mathcal{C}$  is acyclic.

In conditions (2) and (3), it follows that  $n_1 \in \mathcal{N}_{\mathcal{C}}$ , because  $\mathcal{C}$  is a graph.

**Definition A.5** A node  $n$  is in a bundle  $\mathcal{C} = \langle \mathcal{N}_{\mathcal{C}}, (\rightarrow_{\mathcal{C}} \cup \Rightarrow_{\mathcal{C}}) \rangle$ , written  $n \in \mathcal{C}$ , if  $n \in \mathcal{N}_{\mathcal{C}}$ ; a strand  $s$  is in  $\mathcal{C}$  if all of its nodes are in  $\mathcal{N}_{\mathcal{C}}$ .

If  $\mathcal{C}$  is a bundle, then the  $\mathcal{C}$ -height of a strand  $s$  is the largest  $i$  such that  $\langle s, i \rangle \in \mathcal{C}$ .

**Example A.6** The scheme of Example 2.7 represents a bundle  $\mathcal{C}$  and it remains a bundle if we suppress  $\langle s_1, 4 \rangle$  from  $\mathcal{N}_{\mathcal{C}}$  as well as the arrows leading to this node from  $\rightarrow_{\mathcal{C}}$  and  $\Rightarrow_{\mathcal{C}}$ . However, it is not a bundle anymore if  $\langle s_2, 1 \rangle$  and the arrows leading to and starting from this node are suppressed from  $\mathcal{N}_{\mathcal{C}}$ ,  $\rightarrow_{\mathcal{C}}$  and  $\Rightarrow_{\mathcal{C}}$  since  $\langle s_2, 2 \rangle \in \mathcal{C}$  and  $\langle s_2, 1 \rangle \Rightarrow \langle s_2, 2 \rangle$ .

**Definition A.7** If  $\mathcal{S}$  is a set of edges, i.e.  $\mathcal{S} \subset \rightarrow \cup \Rightarrow$ , then  $\prec_{\mathcal{S}}$  is the transitive closure of  $\mathcal{S}$  and  $\preceq_{\mathcal{S}}$  is the reflexive, transitive closure of  $\mathcal{S}$ .

The relations  $\prec_{\mathcal{S}}$  and  $\preceq_{\mathcal{S}}$  are each subsets of  $\mathcal{N}_{\mathcal{S}} \times \mathcal{N}_{\mathcal{S}}$ , where  $\mathcal{N}_{\mathcal{S}}$  is the set of nodes incident with any edge in  $\mathcal{S}$ .

**Lemma A.8** Suppose  $\mathcal{C}$  is a bundle. Then  $\preceq_{\mathcal{C}}$  is a partial order, i.e. a reflexive, antisymmetric, transitive relation. Every non-empty subset of the nodes in  $\mathcal{C}$  has  $\preceq_{\mathcal{C}}$ -minimal members.

We regard  $\preceq_{\mathcal{C}}$  as expressing causal precedence, because  $n \preceq_{\mathcal{C}} n'$  holds only when  $n$ 's occurrence causally contributes to the occurrence of  $n'$ . When a bundle  $\mathcal{C}$  is understood, we will simply write  $\preceq$ . Similarly, we will say that a node  $n$  precedes a node  $n'$  if  $n \preceq n'$ .

## B Proof of Theorem 3.10

Several lemmas will be useful to prove Theorem 3.10, which comes down to proving that, for any GDH-Protocol,

$$\begin{aligned}
P(\pi_i(\bar{0})) &= C^{-1}(M_i \rightarrow M_i) \cdot C(M_i \rightarrow M_j) \\
&\quad \cdot [\mathcal{S}_j \setminus M_I : C^{-1}(M_i \rightarrow M_j) \cdot C(M_i \rightarrow M_k)] \\
&\quad \cdot \prod_{M_l \in \mathcal{S}_k} [\mathcal{S}_j \setminus M_I : C^{-1}(M_l \rightarrow M_i) \cdot C(M_l \rightarrow M_k)] \\
&\quad \cdot \prod_{M_l \in \mathcal{S}_j} [\mathcal{S}_k \setminus M_I : C^{-1}(M_l \rightarrow M_i) \cdot C(M_l \rightarrow M_j)] \\
&\quad \cdot \prod_{M_l \in \mathcal{M}} K_{I_l}^{e_l}
\end{aligned} \tag{2}$$

if  $\mathcal{S}_j$  and  $\mathcal{S}_k$  are two disjoint sets of users such that  $M_k \in \mathcal{S}_j$ ,  $M_j \in \mathcal{S}_k$ ,  $M_i \notin \mathcal{S}_j \cup \mathcal{S}_k$  and  $\mathcal{S}_j \cup \mathcal{S}_k \cup \{M_i\} = \mathcal{M}$ .

Our first lemma says that the key part of the contribution of  $M_i$  to  $M_j$  in a session executed by the group of users  $\mathbf{M}$  is a product of keys that  $M_i$  shares with the other group members, and only of such keys (that is, it does not contain keys which  $M_i$  shares with users outside the group).

**Lemma B.1** *For any GDH-protocol, if  $i \neq j$ ,*

$$C_{\mathbf{K}}(M_i \rightarrow M_j) = \prod_{k=1 \dots n, k \neq i} C_{K_{ik}}(M_i \rightarrow M_j).$$

*Proof.*  $C_{\mathbf{K}}(M_i \rightarrow M_j)$  can only contain keys of the form  $K_{ix} \in K_i$  since these are the only known on  $s_i$ . From Proposition 3.5, we can observe that  $C_{K_{ii}}(M_i \rightarrow M_j) = 1$ . Furthermore, the same proposition guarantees that  $C_{K_{ik}}(M_i \rightarrow M_j) = 1$  when  $k \notin \{1 \dots n\}$  (that is, that  $C_{\mathbf{K}}(M_i \rightarrow M_j)$  does not contain keys shared between  $M_i$  and users which are not expected to take part to the protocol execution) since  $C_{K_{ik}}(M_k \rightarrow M_i)$  is undefined when  $M_k \notin \mathbf{M}$ . ■

Our second lemma provides a relation which will be very useful when we will have to prove Lemma B.3. This third lemma will provide an expression of  $P_{\mathbf{K}}(\pi_i(\bar{0}))$  as a product of contributions which are a subset of the contributions included in the expression of  $p$  given in Equation (2). So, our goal after having proved that lemma will be to prove that the product of the key part of all other contributions included in the product of Equation 2 is known by  $M_I$ .

**Lemma B.2** *Consider a GDH-Protocol executed by a group of users  $\mathbf{M} = \{M_1, \dots, M_n\}$  where  $n \geq 3$  and let  $\mathbf{S}_j$  and  $\mathbf{S}_k$  be two disjoint sets of users such that  $M_i \notin \mathbf{S}_j \cup \mathbf{S}_k$  and  $\mathbf{S}_j \cup \mathbf{S}_k \cup \{M_i\} = \mathbf{M}$ . Then*

$$\prod_{M_l \in \mathbf{S}_k} [\mathbf{S}_j \setminus M_I : C_{\mathbf{K}}(M_l \rightarrow M_i)] = \prod_{M_l \in \mathbf{S}_k} C_{K_{li}}(M_l \rightarrow M_i) \cdot \prod_{l \in 1 \dots n} K_{II}^{e_l}$$

*Proof.* Let  $M_l \in \mathbf{S}_k$ . From Lemma B.1,  $[\mathbf{S}_j \setminus M_I : C_{\mathbf{K}}(M_l \rightarrow M_i)] = [\mathbf{S}_j \setminus M_I : \prod_{M_m \in \mathbf{M} - \{M_l\}} C_{K_{lm}}(M_l \rightarrow M_i)]$ . If  $M_m \in \mathbf{S}_j$ , then  $[\mathbf{S}_j \setminus M_I : C_{K_{lm}}(M_l \rightarrow M_i)]$  is a term of the form  $K_{II}^{e_l}$ , and is therefore known by  $M_I$ . So,  $[\mathbf{S}_j \setminus M_I : \prod_{M_m \in \mathbf{M} - \{M_l\}} C_{K_{lm}}(M_l \rightarrow M_i)] = [\mathbf{S}_j \setminus M_I : \prod_{m \in \mathbf{S}_k \cup \{M_i\} \setminus \{M_l\}} C_{K_{lm}}(M_l \rightarrow M_i)] \cdot K_{II}^{e_l}$ .

Finally, if we simplify the terms in

$$\prod_{M_l \in \mathbf{S}_k} [\mathbf{S}_j \setminus M_I : \prod_{M_m \in \mathbf{S}_k \cup \{M_i\} \setminus \{M_l\}} C_{K_{lm}}(M_l \rightarrow M_i)]$$

by using Proposition 3.5, we can observe that the only remaining terms are those of the form  $C_{K_{li}}(M_l \rightarrow M_i)$ , which proves our identity. ■

**Lemma B.3** Consider a GDH-Protocol executed by a group of users  $\mathbf{M} = \{M_1, \dots, M_n\}$  where  $n \geq 3$ . Let  $M_i, M_j$  and  $M_k$  be three different members of the group  $\mathbf{M}$ . Let  $\mathbf{S}_j$  and  $\mathbf{S}_k$  be two disjoint sets of users such that  $M_k \in \mathbf{S}_j$ ,  $M_j \in \mathbf{S}_k$ ,  $M_i \notin \mathbf{S}_j$ ,  $M_i \notin \mathbf{S}_k$  and  $\mathbf{S}_j \cup \mathbf{S}_k \cup \{M_i\} = \mathbf{M}$ . Then

$$\begin{aligned} P_{\mathbf{K}}(\pi_i(\bar{0}))^{-1} &= C_{\mathbf{K}}(M_i \rightarrow M_i) \\ &\cdot \prod_{M_l \in \mathbf{S}_k} [\mathbf{S}_j \setminus M_l : C_{\mathbf{K}}(M_l \rightarrow M_i)] \\ &\cdot \prod_{M_l \in \mathbf{S}_j} [\mathbf{S}_k \setminus M_l : C_{\mathbf{K}}(M_l \rightarrow M_i)] \cdot \prod_{m \in 1 \dots n} K_{Im}^{e_m} \end{aligned}$$

*Proof.* Observation 3.2 tells us that

$$P_{\mathbf{K}}(\pi_i(\bar{0}))^{-1} = \prod_{j=1 \dots n} C_{\mathbf{K}}(M_j \rightarrow M_i)$$

Then we observe that

$$\prod_{M_j \in \mathbf{M} - \{M_i\}} C_{\mathbf{K}}(M_j \rightarrow M_i) = \prod_{M_j \in \mathbf{M} - \{M_i\}} C_{K_{ij}}(M_j \rightarrow M_i)$$

Actually, from Lemma B.1,  $C_{\mathbf{K}}(M_j \rightarrow M_i) = \prod_{M_m \in \mathbf{M} - \{M_j\}} C_{K_{jm}}(M_j \rightarrow M_i)$  and, from Prop. 3.5,  $C_{K_{jm}}(M_j \rightarrow M_i) = C_{K_{jm}}^{-1}(M_m \rightarrow M_i)$ . So, the only terms that are not inverted in  $\prod_{M_j \in \mathbf{M} - \{M_i\}} \prod_{M_m \in \mathbf{M} - \{M_j\}} C_{K_{jm}}(M_j \rightarrow M_i)$  are those of the form  $C_{K_{ji}}(M_j \rightarrow M_i)$  (for  $M_j \in \mathbf{M} - \{M_i\}$ ), what proves our identity.

The last step of our proof consists in observing that

$$\prod_{M_j \in \mathbf{M} - \{M_i\}} C_{K_{ij}}(M_j \rightarrow M_i) = \prod_{M_j \in \mathbf{S}_j} C_{K_{ij}}(M_j \rightarrow M_i) \cdot \prod_{M_j \in \mathbf{S}_k} C_{K_{ij}}(M_j \rightarrow M_i)$$

and in using Lemma B.2 on the two terms of the product on the right of this equation.  $\blacksquare$

As we proved that

$$\begin{aligned} P_{\mathbf{K}}(\pi_i(\bar{0}))^{-1} &= C_{\mathbf{K}}(M_i \rightarrow M_i) \\ &\cdot \prod_{M_l \in \mathbf{S}_k} [\mathbf{S}_j \setminus M_l : C_{\mathbf{K}}(M_l \rightarrow M_i)] \\ &\cdot \prod_{M_l \in \mathbf{S}_j} [\mathbf{S}_k \setminus M_l : C_{\mathbf{K}}(M_l \rightarrow M_i)] \cdot \prod_{m \in 1 \dots n} K_{Im}^{e_m} \end{aligned}$$

we now have to prove that the product

$$\begin{aligned} &C_{\mathbf{K}}(M_i \rightarrow M_j) \cdot [\mathbf{S}_j \setminus M_l : C_{\mathbf{K}}^{-1}(M_i \rightarrow M_j)] \\ &\cdot \prod_{M_l \in (\mathbf{M} - \mathbf{S}_j)} [\mathbf{S}_j \setminus M_l : C_{\mathbf{K}}(M_l \rightarrow M_k)] \\ &\cdot \prod_{M_l \in \mathbf{S}_j} [\mathbf{S}_k \setminus M_l : C_{\mathbf{K}}(M_l \rightarrow M_j)] \end{aligned}$$

containing the remaining contributions in the expression of  $P(\pi_i(\bar{0}))$  we try to prove, is a product of keys that  $M_I$  knows.

To this purpose, we prove one last expression, which is very close from the one we proved in Lemma B.2.

**Lemma B.4** *Consider a GDH-Protocol executed by a group of users  $\mathbb{M} = \{M_1, \dots, M_n\}$  where  $n \geq 3$  and let  $\mathbb{S} \subset \mathbb{M}$  be a subgroup of  $\mathbb{M}$  such that  $M_i \in \mathbb{S}$ . Then*

$$\prod_{M_j \in \mathbb{M} \setminus \mathbb{S}} [\mathbb{S} \setminus M_I : C_{\mathbb{K}}(M_j \rightarrow M_i)] = \prod_{j \in 1 \dots n} K_{I_j}^{e_j}$$

*Proof.* We first note that

$$\prod_{M_j \in \mathbb{M} \setminus \mathbb{S}} [\mathbb{S} \setminus M_I : C_{\mathbb{K}}(M_j \rightarrow M_i)] = \prod_{M_j \in \mathbb{M}} [\mathbb{S} \setminus M_I : C_{\mathbb{K}}(M_j \rightarrow M_i)] \cdot \prod_{j \in 1 \dots n} K_{I_j}^{e'_j}$$

as, if  $M_j \in \mathbb{S}$ , we can be sure that  $[\mathbb{S} \setminus M_I : C_{\mathbb{K}}(M_j \rightarrow M_i)]$  is a product of keys known by  $M_I$ .

Then, we note that

$$\prod_{M_j \in \mathbb{M}} [\mathbb{S} \setminus M_I : C_{\mathbb{K}}(M_j \rightarrow M_i)] = [\mathbb{S} \setminus M_I : P_{\mathbb{K}}(\pi_i(\bar{0}))^{-1}]$$

which proves our lemma since  $M_i \in \mathbb{S}$ . ■

This lemma proves that the term  $\prod_{M_l \in (\mathbb{M} - \mathbb{S}_j)} [\mathbb{S}_j \setminus M_I : C_{\mathbb{K}}(M_l \rightarrow M_k)]$  in the product above is a product of keys the adversary knows.

We now provide a proof of Theorem 3.10.

**Theorem B.5** *For any GDH-Protocol executed by a group of users  $\mathbb{M} = \{M_1, \dots, M_n\}$  where  $n \geq 3$ , if  $\mathbb{S}_j$  and  $\mathbb{S}_k$  are two disjoint sets of users such that  $M_k \in \mathbb{S}_j$ ,  $M_j \in \mathbb{S}_k$ ,  $M_i \notin \mathbb{S}_j \cup \mathbb{S}_k$  and  $\mathbb{S}_j \cup \mathbb{S}_k \cup \{M_i\} = \mathbb{M}$ . Then,*

$$\begin{aligned} P(\pi_i(\bar{0})) &= C^{-1}(M_i \rightarrow M_i) \cdot C(M_i \rightarrow M_j) \\ &\cdot [\mathbb{S}_j \setminus M_I : C^{-1}(M_i \rightarrow M_j) \cdot C(M_i \rightarrow M_k)] \\ &\cdot \prod_{M_l \in \mathbb{S}_k} [\mathbb{S}_j \setminus M_I : C^{-1}(M_l \rightarrow M_i) \cdot C(M_l \rightarrow M_k)] \\ &\cdot \prod_{M_l \in \mathbb{S}_j} [\mathbb{S}_k \setminus M_I : C^{-1}(M_l \rightarrow M_i) \cdot C(M_l \rightarrow M_j)] \\ &\cdot \prod_{M_l \in \mathbb{M}} K_{I_l}^{e_l} \end{aligned}$$

*Proof.* We proceed in two steps. In the first one, we prove that the expression above is true for the random part of the product (i.e.  $P_{\mathbb{R}}(\pi_i(\bar{0}))$ ), then we prove that it is correct for the key part of the product (i.e.  $P_{\mathbb{K}}(\pi_i(\bar{0}))$ ).

### 1. R-part

If we except the last line of the expression of  $P(\pi_i(\bar{0}))$  above which is a product of keys, this expression only contains products of pairs of contributions. Proposition 3.3 guarantees us that the random part of the last three lines of products of contributions must be equal to one. Finally, combining the expressions proved in Propositions 3.3 and 3.4 for the product  $C_R^{-1}(M_i \rightarrow M_j) \cdot C_R(M_i \rightarrow M_j)$  guarantees us that it is equal to  $P_R(\pi_i(\bar{0}))$ .

### 1. K-part

The use of Lemma B.3 combined with the use of Lemma B.4 says us that our result is proved if the product

$$C_K(M_i \rightarrow M_j) \cdot [S_j \setminus M_I : C_K^{-1}(M_i \rightarrow M_j)] \cdot \prod_{M_l \in S_j} [S_k \setminus M_I : C_K(M_l \rightarrow M_j)] \quad (3)$$

only contains keys known by  $M_I$ .

Using Lemma B.1 with the first two terms of this product provides

$$\begin{aligned} & C_K(M_i \rightarrow M_j) \cdot [S_j \setminus M_I : C_K^{-1}(M_i \rightarrow M_j)] \\ &= \prod_{k=1 \dots n, k \neq i} C_{K_{ik}}(M_i \rightarrow M_j) \cdot [S_j \setminus M_I : \prod_{k=1 \dots n, k \neq i} C_{K_{ik}}^{-1}(M_i \rightarrow M_j)] \\ &= \prod_{M_k \in S_j} C_{K_{ik}}(M_i \rightarrow M_j) \cdot \prod_{M_l \in M} K_{Il}^{e'_l} \\ &= [S_k \setminus M_I : C_K(M_i \rightarrow M_j)] \cdot \prod_{M_l \in M} K_{Il}^{e''_l} \end{aligned}$$

Inserting this last expression in 3 provides:

$$\begin{aligned} (3) &= [S_k \setminus M_I : C_K(M_i \rightarrow M_j)] \\ &\quad \cdot \prod_{M_l \in S_j} [S_k \setminus M_I : C_K(M_l \rightarrow M_j)] \cdot \prod_{M_l \in M} K_{Il}^{e''_l} \\ &= \prod_{M_l \in M - S_k} [S_k \setminus M_I : C_K(M_l \rightarrow M_j)] \cdot \prod_{M_l \in M} K_{Il}^{e''_l} \end{aligned}$$

and this last product is a product of keys  $M_I$  knows, as we proved in Lemma B.4.  $\blacksquare$

## C Illustration of the Attack Process

We illustrate the full attack construction process developed all along this paper. To this purpose, we define a deliberately intricate protocol, the Int-GDH protocol, which will allow us to illustrate our attack construction process more completely than if we considered a simple, regular protocol.

A typical execution of the Int-GDH protocol is represented in the strand space of Fig. 9, where, in order to keep our figure in a reasonable dimension, we represented the transmission of a sequence of elements of  $\mathcal{G}$  as a single arrow between two nodes instead of splitting these transmissions into sequences of transmissions of a single element of  $\mathcal{G}$ .

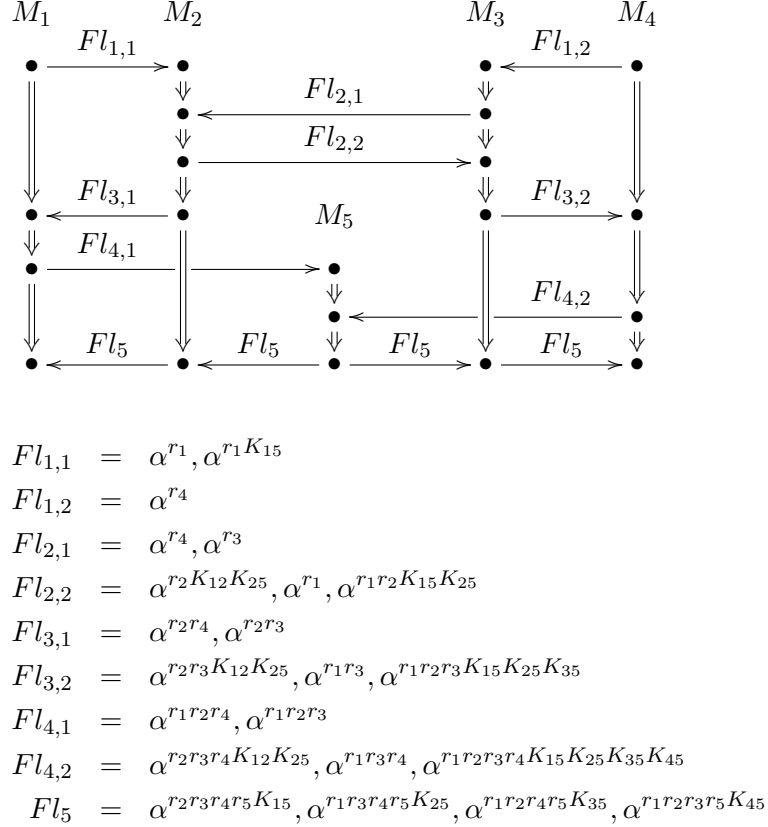


Figure 9: A run of the Int-GDH protocol

Even though the five corresponding histories can be easily deduced from the strand definitions, we provide them in Table 2, where the notation  $(\langle s_i, j \rangle, k)$  refers to the  $k$ -th element of  $\mathcal{G}$  transmitted on  $\langle s_i, j \rangle$ .

We now have a complete definition of the Int-GDH protocol: strands inform us about the way messages are (normally) exchanged, while histories indicate us how they are computed.

We will now build an attack against this protocol.

We first have to select:

- three group members:  $M_i$ ,  $M_j$  and  $M_k$
- two disjoint sets of users  $S_j$  and  $S_k$  such that  $M_k \in S_j$ ,  $M_j \in S_k$ ,

Table 2: Histories in the Int-GDH Protocol

$\pi_1$	$\pi_2$	$\pi_3$	$\pi_4$	$\pi_5$
$(\langle s_2, 3 \rangle, 1)$	$(\langle s_1, 1 \rangle, 1)$	$(\langle s_4, 1 \rangle, 1)$	$(\langle s_3, 2 \rangle, 2)$	$(\langle s_1, 1 \rangle, 2)$
$(\langle s_3, 3 \rangle, 1)$	$(\langle s_2, 1 \rangle, 2)$	$(\langle s_3, 1 \rangle, 1)$	$(\langle s_2, 2 \rangle, 2)$	$(\langle s_2, 1 \rangle, 2)$
$(\langle s_3, 4 \rangle, 1)$	$(\langle s_2, 3 \rangle, 2)$	$(\langle s_3, 2 \rangle, 1)$	$(\langle s_2, 4 \rangle, 2)$	$(\langle s_2, 3 \rangle, 3)$
$(\langle s_4, 2 \rangle, 1)$	$(\langle s_3, 3 \rangle, 2)$	$(\langle s_2, 2 \rangle, 1)$	$(\langle s_1, 2 \rangle, 2)$	$(\langle s_3, 3 \rangle, 3)$
$(\langle s_4, 3 \rangle, 1)$	$(\langle s_3, 4 \rangle, 2)$	$(\langle s_2, 4 \rangle, 1)$	$(\langle s_1, 3 \rangle, 2)$	$(\langle s_3, 4 \rangle, 3)$
$(\langle s_5, 2 \rangle, 1)$	$(\langle s_4, 2 \rangle, 2)$	$(\langle s_1, 2 \rangle, 1)$	$(\langle s_5, 1 \rangle, 2)$	$(\langle s_4, 2 \rangle, 3)$
$(\langle s_5, 3 \rangle, 1)$	$(\langle s_4, 3 \rangle, 2)$	$(\langle s_1, 3 \rangle, 1)$	$(\langle s_5, 3 \rangle, 4)$	$(\langle s_4, 3 \rangle, 3)$
$(\langle s_1, 4 \rangle, 1)$	$(\langle s_5, 2 \rangle, 2)$	$(\langle s_5, 1 \rangle, 1)$	$(\langle s_4, 4 \rangle, 4)$	$(\langle s_5, 2 \rangle, 3)$
	$(\langle s_5, 3 \rangle, 2)$	$(\langle s_5, 3 \rangle, 3)$		
	$(\langle s_2, 5 \rangle, 2)$	$(\langle s_3, 5 \rangle, 3)$		

$$M_i \notin S_j \cup S_k, S_j \cup S_k \cup \{M_i\} = M.$$

This selection must also respect the two following conditions:

- $split(\pi_i, \pi_j)$  does not belong to  $s_i$
- the product

$$\begin{aligned}
 p &= C^{-1}(M_i \rightarrow M_i) \cdot C(M_i \rightarrow M_j) \\
 &\cdot [S_j \setminus M_I : C^{-1}(M_i \rightarrow M_j) \cdot C(M_i \rightarrow M_k)] \\
 &\cdot \prod_{M_l \in S_k} [S_j \setminus M_I : C^{-1}(M_l \rightarrow M_i) \cdot C(M_l \rightarrow M_k)] \\
 &\cdot \prod_{M_l \in S_j} [S_k \setminus M_I : C^{-1}(M_l \rightarrow M_i) \cdot C(M_l \rightarrow M_j)] \cdot \prod_{M_l \in M} K_{II}^{e_l}
 \end{aligned}$$

respects at least one of the sufficient conditions described in Proposition 4.8.

We first observe that the five histories of the Int-GDH protocol have no common part, so that there are no splitting point.

As a first try, we consider the choice  $M_i = M_1$ ,  $M_j = M_2$  and  $M_k = M_3$ . Whatever choice we do for  $S_j$  and  $S_k$ , we can verify that the product  $p$  will contain at least three starting points:  $C(M_1 \rightarrow M_2)$ ,  $[S_j \setminus M_I : C^{-1}(M_1 \rightarrow M_2)]$  and  $[S_j \setminus M_I : C^{-1}(M_2 \rightarrow M_1)]$ . These values of  $M_i$ ,  $M_j$  and  $M_k$  are therefore not admissible.

As a second attempt, we consider the choice  $M_i = M_1$ ,  $M_j = M_3$ ,  $M_k = M_2$  while  $S_j = \{M_2\}$  and  $S_k = \{M_3, M_4, M_5\}$ . This solution implies that  $p$  contains one  $start^+$ :  $[S_j \setminus M_I : C(M_1 \rightarrow M_2)]$  and one  $start^-$ :  $[S_k \setminus M_I : C^{-1}(M_2 \rightarrow M_1)]$ . As expressed in our fifth condition, this is acceptable

only if  $C(M_1 \rightarrow M_2) \prec C(M_1 \rightarrow M_3)$  or  $C(M_2 \rightarrow M_1) \prec C(M_2 \rightarrow M_3)$ . Checking these conditions in Table 2 shows that  $C(M_2 \rightarrow M_1) \not\prec C(M_2 \rightarrow M_3)$  because  $\langle s_2, 2 \rangle \prec \langle s_2, 3 \rangle$ . However, we can verify that  $C(M_1 \rightarrow M_2) \prec C(M_1 \rightarrow M_3)$  since  $\langle s_1, 1 \rangle$  strictly precedes all nodes of  $\alpha_3$  belonging to  $s_1$ . We are therefore able to build an attack for this selection of values.

A simple way to construct our attack consist in following the procedure explained in Proposition 4.9's proof.

The first step in this procedure consists in defining  $\hat{z}$  as the index of the starting point of  $\alpha_2$  in  $s_1$  (given that  $C(M_1 \rightarrow M_2) \prec C(M_1 \rightarrow M_3)$ ). A simple examination shows that  $\hat{z} = 1$ .

We now have to execute Algorithm 1 for the product of contributions  $[M_2 \setminus M_I : C^{-1}(M_1 \rightarrow M_3) \cdot C(M_1 \rightarrow M_2)]$  and for values of  $z$  ranging from 1 to  $\hat{z}$ , what means that we will execute this algorithm for only one step of the **for** loop. The values  $g_1$  and  $g_2$  are initialized to  $\alpha$  and, for the simplicity of the writings, we always select  $\alpha^x$  as random element of  $\mathbf{G}$ . The random contribution of  $M_i$  during the session we are attacking will be written  $r_i$ , while his contribution during the session where the intruder replaces the users included in  $\mathbf{S}_j$  will be denoted  $r'_i$ , and we will use the letter  $r''_i$  to write the contribution  $M_i$  generated during the session where the intruder replaces the users included in  $\mathbf{S}_k$ . The strand space resulting from this partial execution of Algorithm 1 is represented in Fig. 10. The current values of  $g_1$  and  $g_2$  are indicated as well.

$$M_1 \xrightarrow{\alpha^{r'_1}, \alpha^{r'_1 K_{15}}} M_I$$

$$g_1 = \alpha, g_2 = \alpha^{r'_1}$$

Figure 10: First Step

Always following the procedure indicated in the fifth part of the proof of Proposition 4.8, we now have to execute Algorithm 1 for the product  $[\{M_3, M_4, M_5\} \setminus M_I : C^{-1}(M_2 \rightarrow M_1) \cdot C(M_2 \rightarrow M_3)]$ , keeping the current values of  $g_1$  and  $g_2$  as initial values. The resulting strand space is represented in Fig. 11.

The next step in the procedure described in the fifth part of the proof of Proposition 4.8 consists in completing the execution of Algorithm 1 for the product  $[M_2 \setminus M_I : C^{-1}(M_1 \rightarrow M_3) \cdot C(M_1 \rightarrow M_2)]$ . The result of this execution (with the updated values of  $g_1$  and  $g_2$ ) is represented in Fig. 12.

We now have to execute Algorithm 1 for the remaining products of pairs

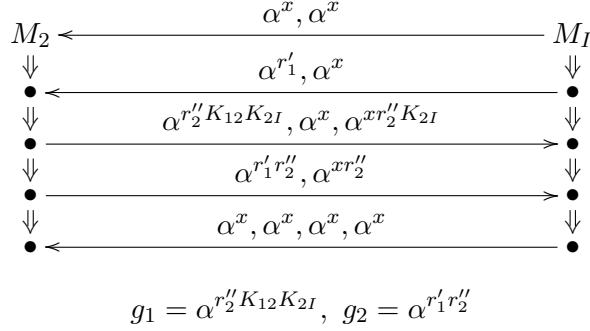


Figure 11: Second Step

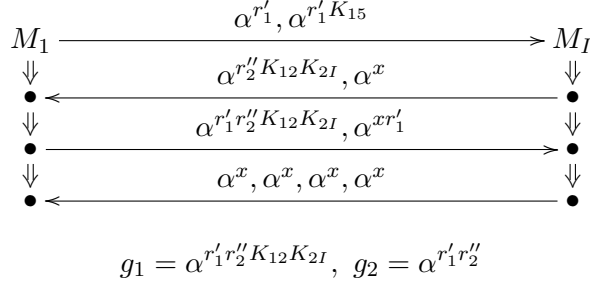


Figure 12: Third Step

of contributions in

$$\begin{aligned}
p &= C^{-1}(M_1 \rightarrow M_1) \cdot C(M_1 \rightarrow M_3) \\
&\cdot [M_2 \setminus M_I : C^{-1}(M_1 \rightarrow M_3) \cdot C(M_1 \rightarrow M_2)] \\
&\cdot \prod_{M_l \in \{M_3, M_4, M_5\}} [M_2 \setminus M_I : C^{-1}(M_l \rightarrow M_1) \cdot C(M_l \rightarrow M_2)] \\
&\cdot [\{M_3, M_4, M_5\} \setminus M_I : C^{-1}(M_2 \rightarrow M_1) \cdot C(M_2 \rightarrow M_3)] \cdot \prod_{M_l \in M} K_{ll}^{e_l}
\end{aligned}$$

A direct observation however shows that

$$[M_2 \setminus M_I : C^{-1}(M_3 \rightarrow M_1) \cdot C(M_3 \rightarrow M_2)] = 1$$

$$[M_2 \setminus M_I : C^{-1}(M_4 \rightarrow M_1) \cdot C(M_4 \rightarrow M_2)] = 1$$

so we do not need to apply Algorithm 1 for these products in our protocol.

We however have to collect the products  $[M_2 \setminus M_I : C^{-1}(M_5 \rightarrow M_1) \cdot C(M_5 \rightarrow M_2)]$  and  $C^{-1}(M_1 \rightarrow M_1) \cdot C(M_1 \rightarrow M_3)$  by applying the same process as before. Always keeping the current values of  $g_1$  and  $g_2$ , the strand space obtained for the product  $[M_2 \setminus M_I : C^{-1}(M_5 \rightarrow M_1) \cdot C(M_5 \rightarrow M_2)]$  is represented in Fig. 13.

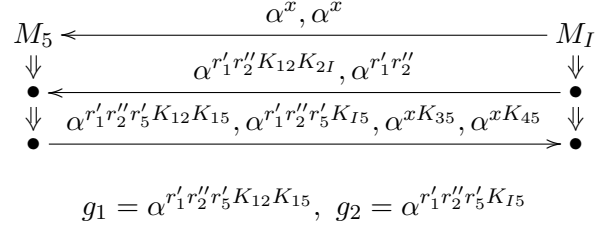


Figure 13: Fourth Step

In order to complete our attack, we still have to execute Algorithm 1 for the product  $C^{-1}(M_1 \rightarrow M_1) \cdot C(M_1 \rightarrow M_3)$  and to send  $g_1$  as the value  $M_1$  will use to compute the group key. This is represented in Fig. 14.

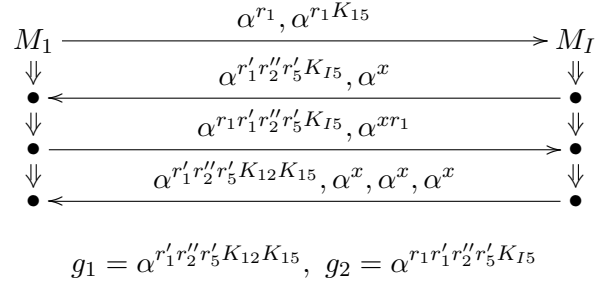


Figure 14: Last Step

At the end of this process, when  $M_1$  will compute his view of the group key, he will exponentiate  $g_1 = \alpha^{r'_1 r''_2 r'_5 K_{12} K_{15}}$  with  $r_1 K_{12}^{-1} K_{15}^{-1}$  and obtain  $\alpha^{r_1 r'_1 r''_2 r'_5} = g_2^{K_{I5}^{-1}}$ . The intruder can therefore compute a key that would normally have to be kept secret.

## D Possible choices for $M_i, M_j, M_k, S_j$ and $S_k$

The following table provides adequate choices for  $M_i, M_j, M_k, S_j$  and  $S_k$  in the nine problematic cases discussed in the proof of Theorem 4.10. The first column explains on which strands the four histories start, while the second column provides the corresponding choices of  $M_i, M_j, M_k, S_j$  and  $S_k$ .

Table 3: Possible choices for  $M_i, M_j, M_k, S_j$  and  $S_k$

	$\pi_1$	$\pi_2$	$\pi_3$	$\pi_4$	$M_i$	$M_j$	$M_k$	$S_j$	$S_k$
1)	$s_2$	$s_1$	$s_4$	$s_3$	$M_1$ $M_3$	$M_3$ $M_1$	$M_2$ $M_4$	$\{M_2\}$ $\{M_4\}$	$\mathbb{M} \setminus \{M_1, M_2\}$ $\mathbb{M} \setminus \{M_3, M_4\}$
2)	$s_2$	$s_3$	$s_4$	$s_1$	$M_3$ $M_3$	$M_1$ $M_4$	$M_4$ $M_1$	$\{M_4\}$ $\mathbb{M} \setminus \{M_3, M_4\}$	$\mathbb{M} \setminus \{M_3, M_4\}$ $\{M_4\}$
3)	$s_2$	$s_4$	$s_1$	$s_3$	$M_1$ $M_1$	$M_2$ $M_4$	$M_4$ $M_2$	$\mathbb{M} \setminus \{M_1, M_2\}$ $\{M_2\}$	$\{M_2\}$ $\mathbb{M} \setminus \{M_1, M_2\}$
4)	$s_3$	$s_1$	$s_4$	$s_2$	$M_1$ $M_1$	$M_4$ $M_3$	$M_3$ $M_4$	$\{M_3\}$ $\mathbb{M} \setminus \{M_1, M_3\}$	$\mathbb{M} \setminus \{M_1, M_3\}$ $\{M_3\}$
5)	$s_3$	$s_4$	$s_1$	$s_2$	$M_1$ $M_2$	$M_2$ $M_1$	$M_3$ $M_4$	$\{M_3\}$ $\{M_4\}$	$\mathbb{M} \setminus \{M_1, M_3\}$ $\mathbb{M} \setminus \{M_2, M_4\}$
6)	$s_3$	$s_4$	$s_2$	$s_1$	$M_1$ $M_1$	$M_2$ $M_3$	$M_3$ $M_2$	$\{M_3\}$ $\mathbb{M} \setminus \{M_1, M_3\}$	$\mathbb{M} \setminus \{M_1, M_3\}$ $\{M_3\}$
7)	$s_4$	$s_1$	$s_2$	$s_3$	$M_1$ $M_1$	$M_3$ $M_4$	$M_4$ $M_3$	$\{M_4\}$ $\mathbb{M} \setminus \{M_1, M_4\}$	$\mathbb{M} \setminus \{M_1, M_4\}$ $\{M_4\}$
8)	$s_4$	$s_3$	$s_1$	$s_2$	$M_1$ $M_1$	$M_2$ $M_4$	$M_4$ $M_2$	$\{M_4\}$ $\mathbb{M} \setminus \{M_1, M_4\}$	$\mathbb{M} \setminus \{M_1, M_4\}$ $\{M_4\}$
9)	$s_4$	$s_3$	$s_2$	$s_1$	$M_1$ $M_2$	$M_2$ $M_1$	$M_4$ $M_3$	$\{M_4\}$ $\{M_3\}$	$\mathbb{M} \setminus \{M_1, M_4\}$ $\mathbb{M} \setminus \{M_2, M_3\}$