

2015/9



Largest minimal inversion-complete and
pair-complete sets of permutations

Eric Balandraud, Maurice Queyranne and Fabio Tardella



CORE

The logo for CORE (Center for Operations Research and Econometrics) features the word "CORE" in a bold, black, sans-serif font. A light blue arc curves over the letters, starting from the top left of the 'C' and ending at the bottom right of the 'E'.

DISCUSSION PAPER

Center for Operations Research
and Econometrics

Voie du Roman Pays, 34
B-1348 Louvain-la-Neuve
Belgium

<http://www.uclouvain.be/core>

CORE

Voie du Roman Pays 34, L1.03.01

B-1348 Louvain-la-Neuve, Belgium.

Tel (32 10) 47 43 04

Fax (32 10) 47 43 01

E-mail: immaq-library@uclouvain.be

<http://www.uclouvain.be/en-44508.html>

CORE DISCUSSION PAPER
2015/9

**Largest minimal inversion-complete and pair-complete sets of
permutations**

Eric Balandraud¹, Maurice Queyranne^{2,3}
and Fabio Tardella⁴

February 25, 2015

Abstract

We solve two related extremal problems in the theory of permutations. A set Q of permutations of the integers 1 to n is inversion-complete (resp., pair-complete) if for every inversion (j, i) , where $1 \leq i < j \leq n$, (resp., for every pair (i, j) , where $i \neq j$) there exists a permutation in Q where j is before i . It is minimally inversion-complete if in addition no proper subset of Q is inversion-complete; and similarly for pair-completeness. The problems we consider are to determine the maximum cardinality of a minimal inversion-complete set of permutations, and that of a minimal pair-complete set of permutations. The latter problem arises in the determination of the Carathéodory numbers for certain abstract convexity structures on the $(n - 1)$ -dimensional real and integer vector spaces. Using Mantel's Theorem on the maximum number of edges in a triangle-free graph, we determine these two maximum cardinalities and we present a complete description of the optimal sets of permutations for each problem. Perhaps surprisingly (since there are twice as many pairs to cover as inversions), these two maximum cardinalities coincide whenever $n \geq 4$.

¹ Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie (Paris 6), Paris, France

² Sauder School of Business, University of British Columbia, Vancouver, B.C., Canada

³ Center for Operations Research and Econometrics (CORE), Université catholique de Louvain, Louvain-la-Neuve, Belgium

⁴ Dipartimento MEMOTEF, Sapienza University of Rome, Roma, Italy

E-mail addresses: eric.balandraud@imj-prg.fr, Maurice.Queyranne@sauder.ubc.ca, fabio.tardella@uniroma1.it.

1. INTRODUCTION

We consider the following extremal problems in the theory of permutations. Given integer $n \geq 2$, let S_n denote the symmetric group of all permutations of $[n] := \{1, 2, \dots, n\}$ (so $|S_n| = n!$), and $A_n = \{(i, j) : i, j \in [n], i \neq j\}$ the set of all (ordered) pairs from $[n]$ (so $|A_n| = n(n-1)$). A permutation $\pi = (\pi(1), \dots, \pi(n))$ covers the pair $(\pi(k), \pi(l)) \in A_n$ iff $k < l$. An *inversion* (see, e.g., [1, 4, 5]) is a pair $(j, i) \in A_n$ with $j > i$. Let $I_n \subset A_n$ denote the set of all inversions. A set $Q \subseteq S_n$ of permutations is *inversion-complete* (resp., *pair-complete*) if every inversion in I_n (resp., pair in A_n) is covered by at least one permutation in Q . An inversion-complete set Q is *minimally* inversion-complete if no proper subset of Q is inversion-complete; and similarly for pair-completeness. For example, the set $Q' = \{\text{rev}_n\}$, where (using compact notation for permutations) $\text{rev}_n = n(n-1)\dots 21$ is the reverse permutation, is minimally inversion-complete, and has minimum cardinality for this property; whereas the set $P' = \{\text{id}_n, \text{rev}_n\}$, where $\text{id}_n = 12\dots n$ is the identity permutation, is minimally pair-complete, and has minimum cardinality.

We determine the *maximum* cardinality $\gamma_I(n)$ of a minimal inversion-complete subset $Q \subseteq S_n$, as well as the maximum cardinality $\gamma_P(n)$, of a minimal pair-complete subset $P \subseteq S_n$. The latter problem arose in the determination of the Carathéodory numbers for the integral L^\natural convexity structures on the $(n-1)$ -dimensional real and integer vector spaces \mathbb{R}^{n-1} and \mathbb{Z}^{n-1} , see [7].¹ It was posed by the second author as ‘‘An Integer Programming Formulation Challenge’’ at the Integer Programming Workshop, Valparaiso, Chile, March 11-14, 2012.

Stimulated by personal communication of an early version of our results, Malvenuto et al. [2] determine the exact value of, or bounds on, the maximum cardinality of minimal inversion-complete sets in more general classes of finite reflection groups.

Perhaps unexpectedly (since there are twice as many pairs to cover as inversions), the maximum cardinalities $\gamma_I(n)$ and $\gamma_P(n)$ considered herein are equal for all $n \geq 4$ (and they only differ by one unit, viz., $\gamma_P(n) = \gamma_I(n) + 1$, for $n = 2$ and 3). Furthermore, for all $n \geq 4$ the family \mathcal{Q}_n^* of all maximum-cardinality minimal inversion-complete subsets of S_n is *strictly* contained in the family \mathcal{P}_n^* of all maximum-cardinality minimal pair-complete subsets. All our proofs are constructive and produce corresponding optimal sets of permutations.

In Section 2 we prove:

Theorem 1. (i) For every $n \geq 2$, the maximum cardinality of a minimal inversion-complete subset of S_n is $\gamma_I(n) = \lfloor n^2/4 \rfloor$.

(ii) For every even $n \geq 4$, the family \mathcal{Q}_n^* of all maximum-cardinality minimal inversion-complete subsets of S_n is the family of all transversals of a family of $n^2/4$ pairwise disjoint subsets of S_n , each of cardinality $\left[\left(\frac{n}{2} - 1\right)!\right]^2$, and thus $|\mathcal{Q}_n^*| = \left[\left(\frac{n}{2} - 1\right)!\right]^{n^2/2}$.

(iii) For every odd $n \geq 5$, \mathcal{Q}_n^* is the disjoint union of the families of all transversals of two families, each one of $\lfloor n^2/4 \rfloor$ pairwise disjoint subsets of S_n of cardinality $\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)! \left\lfloor \frac{n}{2} \right\rfloor!$, and thus $|\mathcal{Q}_n^*| = 2 \left[\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)! \left\lfloor \frac{n}{2} \right\rfloor!\right]^{\lfloor n^2/4 \rfloor}$.

¹We refer the curious reader to van de Vel’s monograph [9] for a general introduction to convexity structures and convexity invariants (such as the Carathéodory number), and to Murota’s monograph [6] on various models of discrete convexity, including L^\natural and related convexities.

To prove Theorem 1, we first establish the upper bound $\gamma_I(n) \leq \lfloor n^2/4 \rfloor$ by applying Mantel's Theorem (which states, [3, 8], that the maximum number of edges in an n -vertex triangle-free graph is $\lfloor n^2/4 \rfloor$) to certain "critical selection graphs" associated with the minimal inversion-complete subsets of S_n . We then show that this upper bound is attained by the families of transversals described in parts (ii)–(iii). We complete the proof by showing that, for $n \geq 4$, every $Q \in \mathcal{Q}_n^*$ must be such a transversal. Note that these results imply the asymptotic growth rate $|\mathcal{Q}_n^*| = 2^{\theta(n^3 \log n)}$ as n grows.

In Section 3 we prove:

Theorem 2. (i) For every integer $n \geq 2$, the maximum cardinality of a minimal pair-complete subset of S_n is $\gamma_P(n) = \max \{n, \lfloor n^2/4 \rfloor\}$.

(ii) For all $n \geq 5$ the set \mathcal{P}_n^* of maximum-cardinality minimal pair-complete subset of S_n is equal to the set $\tau \circ \mathcal{Q}_n^*$ resulting from applying every possible permutation $\tau \in S_n$ of the index set $[n]$ to each $Q \in \mathcal{Q}_n^*$.

(iii) For all $n \geq 5$ there is a one-to-one correspondence between \mathcal{P}_n^* and the Cartesian product $\binom{[n]}{\lfloor n/2 \rfloor} \times \mathcal{Q}_n^*$, where $\binom{[n]}{\lfloor n/2 \rfloor}$ is the family of all subsets $S \subset [n]$ with cardinality $|S| = \lfloor n/2 \rfloor$.

The intuition for the formula $\gamma_P(n) = \max \{n, \lfloor n^2/4 \rfloor\}$ in part (i) is that it suffices to consider two classes of minimal pair-complete subsets:

- (1) the subsets P (each of cardinality n) formed by the n circular shifts of any given permutation $\pi \in S_n$, i.e., $P = \{\pi, \pi \circ \sigma, \pi \circ \sigma^2, \dots, \pi \circ \sigma^{n-1}\}$, where the (forward) circular shift $\sigma \in S_n$ is defined by $\sigma(i) = (i \bmod n) + 1$ for all $i \in [n]$; and
- (2) the subsets $P = \tau \circ Q$ (each of cardinality $\lfloor n^2/4 \rfloor$) defined in part (ii) of Theorem 2.

The characterization in part (ii) of Theorem 2 only implies that $|\mathcal{P}_n^*| \leq n! |\mathcal{Q}_n^*|$, because different pairs (τ, Q) may give rise to the same set $\tau \circ Q$ (as will be seen, for example, in Remark 2 at the end of this paper, with the "class-(2) subsets" for the case $n = 4$ therein). Part (iii), on the other hand, refines the preceding result using a "canonical" permutation τ_W induced by a balanced partition $\{W, \overline{W}\}$ of the index set $[n]$ (i.e., with $|W|$ or $|\overline{W}| = \lfloor n/2 \rfloor$, a consequence of Mantel's Theorem). This implies that $|\mathcal{P}_n^*| = \binom{[n]}{\lfloor n/2 \rfloor} |\mathcal{Q}_n^*|$ for $n \geq 5$. Thus, although $|\mathcal{P}_n^*| > |\mathcal{Q}_n^*|$ for all $n \geq 5$, their asymptotic growth rate (as n grows) are similar, differing only in lower order terms in the exponent $\theta(n^3 \log n)$.

2. MINIMAL INVERSION-COMPLETE SETS OF PERMUTATIONS

In this Section we prove Theorem 1 and present a characterization of the family \mathcal{Q}_n^* of all maximum-cardinality minimal inversion-complete subsets of S_n . For $n = 2$, there is a single inversion $(2, 1)$, which is covered by the reverse permutation 21 , so part (i) of Theorem 1 trivially holds and $\mathcal{Q}_2^* = \{21\}$. Hence assume $n \geq 3$ in the rest of this Section.

Consider any minimal inversion-complete subset Q of S_n . Since Q is *minimally* inversion-complete, for every permutation $\pi \in Q$ there exists an inversion $(j, i) \in I_n$, called a *critical inversion*, which is covered by π and by no permutation in $Q \setminus \{\pi\}$ (for otherwise $Q \setminus \{\pi\}$ would also be inversion-complete, and thus Q would not be minimally inversion-complete). For every permutation $\pi \in Q$, select *one*

critical inversion that it covers (arbitrarily chosen if π covers more than one critical inversion). Let $q_{j,i}$ denote the unique permutation in Q that covers the selected critical inversion (j, i) . Consider a corresponding *critical selection graph* $G_Q = ([n], E_Q)$, where E_Q is the set of these $|Q|$ selected critical inversions (one for each permutation in Q), considered as undirected edges. Thus $|E_Q| = |Q|$.

Recall that a graph G is *triangle-free* if there are no three distinct vertices i, j and k such that all three edges $\{i, j\}$, $\{i, k\}$ and $\{j, k\}$ are in G .

Lemma 1. *If subset $Q \subseteq S_n$ is minimally inversion-complete, then every corresponding critical selection graph G_Q is triangle-free.*

Proof. Assume $Q \subseteq S_n$ is minimally inversion-complete, and let $G_Q = ([n], E_Q)$ be a corresponding critical selection graph. We need to show that, if E_Q contains two adjacent edges $\{i, j\}$ and $\{j, k\}$, then it cannot contain edge $\{i, k\}$. Thus assume that $\{i, j\}$ and $\{j, k\} \in E_Q$ and, without loss of generality, that $i < k$. We want to show that (k, i) cannot be a selected critical inversion. We consider the possible relative positions of index j relative to i and k :

- If $j < i < k$, i.e., both (i, j) and (k, j) are selected critical inversions, then $q_{k,j}$ cannot cover (i, j) and therefore we must have k before j before i in $q_{k,j}$ (that is, these three indices must be in positions $\pi^{-1}(i) < \pi^{-1}(j) < \pi^{-1}(k)$ in $\pi = q_{k,j}$). This implies that (k, i) cannot be a selected critical inversion.
- If $i < k < j$, i.e., both (j, i) and (j, k) are selected critical inversions, then this is dual (in the order-theoretic sense) to the previous case: $q_{j,i}$ cannot cover (j, k) and therefore we must have k before j before i in $q_{j,i}$, implying that (k, i) cannot be a selected critical inversion.
- Else $i < j < k$, i.e., both (j, i) and (k, j) are selected critical inversions. In every permutation $\pi \in Q \setminus \{q_{j,i}, q_{k,j}\}$ we must have i before j before k . But then (k, i) cannot be a selected critical inversion, since it can only be covered in Q by $q_{j,i}$ or $q_{k,j}$, for each of which another critical inversion has been selected.

Therefore, (k, i) cannot be a selected critical inversion. This implies that no three indices i, j and k can define a triangle in G_Q . \square

Since $|Q| = |E_Q|$, Mantel's Theorem implies

Corollary 2. *For every $n \geq 2$, the maximum cardinality $\gamma_I(n)$ of a minimal inversion-complete subset of S_n satisfies $\gamma_I(n) \leq \lfloor n^2/4 \rfloor$.*

We prove constructively that the upper bound in Corollary 2 is attained, i.e., that part (i) of Theorem 1 holds. For $n = 3$, we have 3 triangle-free graphs on vertex set $\{1, 2, 3\}$, each consisting of exactly two of the three possible edges. Consider the edge set $E' = \{\{1, 2\}, \{1, 3\}\}$: if it is the edge set of a critical selection graph $G_{Q'}$, then we must have $q'_{2,1} = 213 \in Q'$ (for otherwise, $q'_{2,1}$ would also cover the inversion $(3, 1)$, contradicting that $(3, 1)$ is also selected), and similarly $q'_{3,1} = 312 \in Q'$. Thus Q' must be the set $\{213, 312\}$, which is indeed inversion-complete, and thus a largest minimal inversion-complete subset of S_3 . This implies that $\gamma_I(n) = 3 = \lfloor \frac{n^2}{4} \rfloor$ holds for $n = 3$. Similarly, the edge sets $E'' = \{\{1, 2\}, \{2, 3\}\}$ and $E''' = \{\{1, 3\}, \{2, 3\}\}$ define the other two maximum-cardinality minimal inversion-complete subsets $Q'' = \{213, 123\}$ and $Q''' = \{231, 321\}$ of S_3 . Thus $\mathcal{Q}_3^* = \{Q', Q'', Q'''\}$ and $|\mathcal{Q}_3^*| = 3$.

Thus assume $n \geq 4$ in the rest of this Section. We now introduce certain subsets of S_n , which we will use to show that the upper bound in Corollary 2 is attained, and to construct the whole set \mathcal{Q}^* . For every triple (i, c, j) of integers such that $1 \leq i \leq c < j \leq n$, let $F_{i,c,j}$ denote the set of all permutations $\pi \in S_n$ such that:

- $\pi(h) \leq c$ for all $h < c$;
- $\pi(c) = j$;
- $\pi(c+1) = i$; and
- $\pi(k) \geq c+1$ for all $k > c+1$.

If $c > 1$ the first two conditions imply that $(\pi(1), \dots, \pi(c-1))$ is any permutation of $[c] \setminus \{i\}$; and if $c+1 < n$ the last two conditions imply that $(\pi(c+2), \dots, \pi(n))$ is any permutation of $\{c+1, \dots, n\} \setminus \{j\}$. Thus the cardinality of $F_{i,c,j}$ is $(c-1)!(n-c-1)!$. Note also that, for every $\pi \in F_{i,c,j}$, $(k, h) = (j, i)$ is the unique inversion (k, h) with $h \leq c < k$ that is covered by π . Thus for every fixed c the sets $F_{i,c,j}$ ($1 \leq i \leq c < j \leq n$) are pairwise disjoint ($F_{i,c,j} \cap F_{i',c,j'} = \emptyset$ whenever $(i, j) \neq (i', j')$). Recall that, given a collection \mathcal{F} of sets, a *transversal* is a set containing exactly one element from each member of \mathcal{F} .

Lemma 3. *For every integers $1 \leq c < n$, every transversal T of the family $\mathcal{F}_c = \{F_{i,c,j} : 1 \leq i \leq c < j \leq n\}$ is minimally inversion-complete.*

Proof. Given such a transversal T , let $t_{i,j}$ denote the permutation in $T \cap F_{i,c,j}$. For every inversion $(j, i) \in F_n$, we consider the relative positions of i and j with respect to c :

- If $i \leq c < j$, then $t_{i,j}$ is the unique permutation in T that covers the inversion (j, i) .
- If $i < j \leq c$, then the inversion (j, i) is covered by every $t_{i,j'} \in T$ with $j' > c$.
- Else, $c+1 \leq i < j$, then the inversion (j, i) is covered by every $t_{i',j} \in T$ with $i' \leq c$.

Therefore, T is inversion-complete and for every $i \leq c < j$ the inversion (j, i) , covered by $t_{i,j}$, is critical. This implies that T is minimally inversion-complete. \square

For a fixed c such that $1 \leq c < n$, there are $c(n-c)$ subsets $F_{i,c,j}$ (with $i \leq c < j$) (and these subsets are nonempty and pairwise disjoint). Hence the cardinality of every transversal T satisfies $|T| = |\mathcal{F}_c| = c(n-c) \leq \lfloor \frac{n^2}{4} \rfloor$, with equality iff $c \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$. Combining with Lemma 2, we obtain:

Corollary 4. *For every $n \geq 4$, $\gamma_I(n) = \lfloor n^2/4 \rfloor$ and, for every $c \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$, every transversal T of the family $\mathcal{F}_c = \{F_{i,c,j} : 1 \leq i \leq c < j \leq n\}$ is a maximum-cardinality minimal inversion-complete subset of S_n .*

Part (i) of Theorem 1 follows. To prove parts (ii) and (iii), we invoke the “strong form” of Mantel’s Theorem [3, 8]: an n -vertex triangle-free graph has the maximum number $\lfloor n^2/4 \rfloor$ of edges iff it is a *balanced* bipartite graph, i.e., with $\lfloor \frac{n}{2} \rfloor$ vertices on one side and $\lceil \frac{n}{2} \rceil$ on the other.

Lemma 5. *For $n \geq 4$, a subset of S_n is a maximum-cardinality minimal inversion-complete subset iff it is a transversal of the family \mathcal{F}_c for some $c \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$.*

Proof. Sufficiency was established by Lemma 3. To prove necessity, let $n \geq 4$ and consider any $Q \in \mathcal{Q}_n^*$ and a corresponding critical selection graph G_Q . By Lemma 1,

Corollary 4, and the strong form of Mantel's Theorem, G_Q is a balanced complete bipartite graph. We first claim that the side W of G_Q that contains index 1 must be $W = [c]$ with $c \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$. For this, consider (since $n \geq 4$) any three indices $i > 1$ in W and $j < k$ on the other side. Thus $(j, 1)$ and $(k, 1)$ are critical inversions. Furthermore, the edges $\{i, j\}$ and $\{i, k\}$ in G_Q are also defined by critical inversions, which depend on the position of index i relative to j and k :

- If $1 < j < i < k$, then (i, j) and (k, i) are critical inversions. Then every permutation $\pi \in Q \setminus \{q_{j,1}, q_{i,j}, q_{k,i}\}$ has 1 before j before i before k , and thus does not cover the inversion $(k, 1)$. Thus $(k, 1)$ cannot be a critical inversion, a contradiction.
- If $1 < j < k < i$, then (i, j) and (i, k) are critical inversions. On one hand, every permutation $\pi \in Q' = Q \setminus \{q_{j,1}, q_{i,j}\}$ has 1 before j before i , and thus does not cover the inversion $(i, 1)$. Similarly, every permutation $\pi \in Q'' = Q \setminus \{q_{k,1}, q_{i,k}\}$ has 1 before k before i , and thus does not cover the inversion $(i, 1)$ either. Therefore $Q = Q' \cup Q''$ does not cover the inversion $(i, 1)$, a contradiction.

This implies that we must have $1 < i < j < k$, i.e., that $i < j$ for every $i \in W$ and every $j \in [n] \setminus W$. This proves our claim that $W = [c]$ for some c which, by the strong form of Mantel's Theorem, must be $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$.

As a consequence, $Q = \{q_{j,i} : 1 \leq i \leq c < j \leq n\}$. Every $q_{j,i} \in Q$ must have j before i , and also h before j for every $h \in [c] \setminus \{i\}$ (for otherwise $q_{j,i}$ would also cover the inversion (j, h) , contradicting the fact that $q_{j,h}$ is the unique permutation in Q that covers (j, h)) and i before k for every $k \in \{c+1, \dots, n\} \setminus \{j\}$ (for otherwise $q_{j,i}$ would also cover the inversion (k, i)). Therefore $q_{j,i} \in F_{i,c,j}$, and thus Q is a transversal of \mathcal{F}_c . The proof is complete. \square

Parts (ii) and (iii) of Theorem 1 now follow, noting that: (1) \mathcal{F}_c consists of $c(n-c)$ pairwise disjoint subsets $F_{i,c,j}$; (2) $c \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$; and (3) for n odd, all subsets $F_{i,\lfloor n/2 \rfloor, j}$ and $F_{i', \lceil n/2 \rceil, j'}$ are pairwise disjoint (indeed, with n odd, every $\pi \in F_{i,\lfloor n/2 \rfloor, j}$ has $\pi(c+1) \in [c]$ while every $\pi \in F_{i', \lceil n/2 \rceil, j'}$ has $\pi(c+1) \in [n] \setminus [c]$).

Remark 1. Thus we have $\mathcal{Q}_2^* = 1$, $\mathcal{Q}_3^* = 3$, $\mathcal{Q}_4^* = 1$, $\mathcal{Q}_5^* = 128$ and, as noted in the Introduction, the asymptotic growth rate $|\mathcal{Q}_n^*| = 2^{\theta(n^3 \log n)}$.

3. MINIMAL PAIR-COMPLETE SETS OF PERMUTATIONS

In this Section we prove Theorem 2. To simplify the presentation, let $\mu(n) := \max\{n, \lfloor n^2/4 \rfloor\}$. For $n = 2$, the unique cover of the two pairs $(1, 2)$ and $(2, 1)$ is S_2 itself, hence $\gamma_P(2) = 2 = \mu(2)$ and $\mathcal{P}_2^* = \{S_2\}$.

Note that, as for inversions, given any minimal pair-complete subset P of S_n , for every permutation $\pi \in P$ there exists a *critical pair* $(i, j) \in F_n$ which is covered by π and by no other permutation in P . Observe however that, in contrast with inversion-completeness, the notion of pair-completeness does not assume any particular order of the indices. Thus, if $P \subseteq S_n$ is (minimally) pair-complete then for any permutation $\tau \in S_n$ of the index set $[n]$, the set $\tau \circ P = \{\tau \circ \pi : \pi \in P\}$ is also (minimally) pair-complete. (Indeed, π covers (i, j) iff $\tau \circ \pi$ covers $(\tau(i), \tau(j))$.)

For $n = 3$ consider the set $P_3 := \{123, 231, 312\}$. It is easily verified that P_3 is pair-complete and the pairs $(1, 3)$, $(2, 1)$ and $(3, 2)$ are critical pairs covered by the permutations 123, 231 and 312, respectively. Hence $P_3 \in \mathcal{P}_3^*$ and thus

$\gamma_P(3) \geq |P_3| = 3 = \mu(3)$. To verify the converse inequality, viz., $\gamma_P(3) \leq \mu(3)$, consider any $P \in \mathcal{P}_3^*$: by the preceding observation, we may assume, w.l.o.g., that P contains the identity permutation $\pi_1 = \text{id}_3$. This permutation π_1 covers all three pairs (i, j) with $i < j$. Then the permutation $\pi_2 \in P$ that covers the pair $(3, 1)$ must also (depending of the position of index 2) cover at least one of the pairs $(2, 1)$ or $(3, 2)$. Hence there is at most one pair which is not covered by $\{\pi_1, \pi_2\}$, and thus $\gamma_P(3) = |P| \leq 3 = \mu(3)$, implying $\gamma_P(3) = \mu(3)$. Therefore part (i) of Theorem 2 holds for $n \in \{2, 3\}$.

Lemma 6. *If $n \geq 4$, for every permutation $\tau \in S_n$ of the index set $[n]$ and every maximum-cardinality minimal inversion-complete set $Q \subset S_n$, the set $\tau \circ Q$ is minimally pair-complete.*

Proof. By a preceding observation, it suffices to prove that, for $n \geq 4$, every maximum-cardinality minimal inversion-complete set $Q \in S_n$ is minimally pair-complete. By Lemma 5, every such Q must be a transversal of \mathcal{F}_c for some $c \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$. Consider any pair $(i, j) \in A_n$ and, w.l.o.g., $i < j$:

- If $i \leq c < j$ then $q_{j,i} \in F_{i,c,j} \cap Q$ is the unique permutation in Q that covers the inversion (j, i) , and every other permutation in Q covers (i, j) .
- If $i < j \leq c$ then, for every $k \in \{c+1, \dots, n\}$, $q_{k,i}$ covers (j, i) and $q_{k,j}$ covers (i, j) .
- Else, $c < i < j$ and, dually, for every $h \in [c]$, $q_{i,h}$ covers (i, j) and $q_{j,h}$ covers (j, i) .

Therefore Q is pair-complete, and every pair (j, i) with $i \leq c < j$ is critical and covered by $q_{j,i} \in Q$. Since $|Q| = \lfloor \frac{n^2}{4} \rfloor = |\{(j, i) : 1 \leq i \leq c < j \leq n\}|$, Q is minimally pair-complete. \square

Corollary 7. *For every $n \geq 4$, the maximum cardinality $\gamma_P(n)$ of a minimal pair-complete subset of S_n satisfies $\gamma_P(n) \geq \gamma_I(n) = \lfloor n^2/4 \rfloor$.*

As we did for inversions, to every minimal pair-complete subset P of S_n and selection of a critical pair covered by each permutation in P , we associate a corresponding *critical selection graph* $G_P = ([n], E_P)$ where E_P is the set of $|P|$ selected critical pairs (one for each permutation in P), considered as undirected edges. Thus $|E_P| = |P|$. Let $p_{i,j}$ denote the unique permutation in P that covers the selected critical pair (i, j) .

Lemma 8. *If $n \geq 4$ and $P \subseteq S_n$ is minimally pair-complete, then every corresponding critical selection graph G_P is triangle-free.*

Proof. Assume $n \geq 4$ and $P \subseteq S_n$ is minimally pair-complete, and let $G_P = ([n], E_P)$ be a corresponding critical selection graph. By Corollary 7, $|P| \geq \lfloor n^2/4 \rfloor \geq 4$. We have to show that for any three indices i, j, k such that $\{i, j\}$ and $\{j, k\} \in E_P$, we must have $\{i, k\} \notin E_P$.

- First, consider the case where both pairs (i, j) and (j, k) are critical. In every permutation $\pi \in P \setminus \{p_{i,j}, p_{j,k}\}$ we must thus have k before j before i . Since k is before i in all these $|P| - 2 \geq 2$ permutations, (k, i) cannot be a critical pair. Furthermore (i, k) cannot be a selected critical pair, since it can only be covered by $p_{i,j}$ and $p_{j,k}$, for each of which another critical pair has been selected. Therefore, as claimed, we cannot have $\{i, k\}$ in E_P .

- A dual argument shows that if both (j, i) and (k, j) are selected critical pairs then $\{i, k\} \notin E_P$.
- Now consider the case where (j, i) and (j, k) are selected critical pairs. Since $p_{j,i}$ does not cover (j, k) , we have k before j before i in $p_{j,i}$, implying that (k, i) cannot be a selected critical pair. Similarly, $p_{j,k}$ does not cover (j, i) and therefore we must have i before j before k in $p_{j,k}$, implying that (i, k) cannot be a selected critical pair. Therefore, as claimed, we cannot have $\{i, k\}$ in E_P .
- A dual argument applies to the remaining case, showing that if both (i, j) and (k, j) are selected critical pairs then $\{i, k\} \notin E_P$.

Thus we must have $\{i, k\} \notin E_P$. This completes the proof that G_P is triangle-free. \square

These results and Mantel's Theorem imply part (i) of Theorem 2. They also imply that, for $n \geq 4$, all the sets $\tau \circ Q$ in Lemma 6 are in \mathcal{P}_n^* . To complete the proof of part (ii) it now suffices to prove the converse for $n \geq 5$.

Lemma 9. *If $n \geq 5$, a subset P of S_n is a maximum-cardinality minimal pair-complete subset iff $P = \tau \circ Q$ for some permutation τ of the index set $[n]$ and some maximum-cardinality minimal inversion-complete subset Q of S_n .*

Proof. Sufficiency was just established. To prove necessity, let $n \geq 5$ and consider any $P \in \mathcal{P}_n^*$ and a corresponding critical selection graph G_P . By Lemma 8, Theorem 2 (i), and the strong form of Mantel's Theorem, G_P is a balanced complete bipartite graph.

Now, also consider the associated critical selection *digraph* (directed graph) $D_P = ([n], A_P)$, wherein each edge $\{i, j\}$ is directed as arc $(i, j) \in A_P$ if the pair (i, j) is critical. If $(i, j) \in A_P$, then the reverse pair (j, i) must be covered by every permutation in $P \setminus \{p_{i,j}\}$; since $|P| - 1 = \lfloor n^2/4 \rfloor - 1 \geq 2$ when $n \geq 5$, (j, i) cannot be critical, i.e., $(j, i) \notin A_P$. Thus, every edge $\{i, j\}$ in the underlying graph G_P of D_P corresponds to exactly one arc, (i, j) or (j, i) , in D_P . We now prove that when $n \geq 5$ the digraph D_P is acyclic, i.e., it does not contain any (directed) circuit. Indeed, if D_P contains a directed path $(i(1), \dots, i(k))$, then every permutation $\pi \in Q \setminus \{p_{i(1),i(2)}, p_{i(2),i(3)} \dots, p_{i(k-1),i(k)}\}$ has $i(1)$ after $i(2)$ after $i(3)$, etc, after $i(k-1)$ after $i(k)$, and thus $i(k)$ before $i(1)$. Since $|P| - (k-1) \geq \lfloor \frac{n^2}{4} \rfloor - (n-1) \geq 2$ when $n \geq 5$, pair $(i(k), i(1))$ cannot be critical, and thus $(i(k), i(1)) \notin A_P$.

Since digraph D_P is acyclic, there is at least one vertex i with in-degree zero. Since the underlying graph G_P is a balanced complete bipartite graph, the side W of G_Q that contains index i has cardinality $|W| \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$. Consider any indices $j \in W$ and $k \neq l$ on the other side. Since vertex i has no entering arc, arcs (i, k) and (i, l) are in A_P , and we consider the possible orientations of the edges $\{j, k\}$ and $\{j, l\}$:

- If both edges are oriented into j , i.e., (k, j) and (l, j) in A_P , then, on one hand, every permutation $\pi \in P' = P \setminus \{p_{k,j}, p_{l,j}\}$ has j before k before i , and thus does not cover the pair (i, j) . Similarly, every permutation $\pi \in P'' = P \setminus \{p_{l,j}, p_{i,l}\}$ has j before l before i , and thus does not cover the pair (i, j) either. Therefore $P = P' \cup P''$ does not cover the pair (i, j) , a contradiction.
- If one of these two edges is oriented into j and the other one from j , w.l.o.g., (k, j) and (j, l) in A_P , then every permutation $\pi \in P \setminus \{p_{j,l}, p_{k,j}, p_{i,k}\}$ has

l before k before i , and thus does not cover the pair (i, l) . Thus (i, l) cannot be a critical inversion, a contradiction.

Thus we must have both (j, k) and (j, l) in A_P . This implies that all pairs (j, k) with $j \in W$ and $k \in \overline{W} := [n] \setminus W$ define arcs in A_P , i.e., are critical. Let $c := |\overline{W}| \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$ and consider any permutation τ that sends $[c]$ to \overline{W} (and thus $\overline{[c]}$ to W): every critical pair in $Q = \tau^{-1} \circ P$ is an inversion, hence Q is minimally inversion-complete. Since $|Q| = |P| = \gamma_P(n) = \gamma_I(n)$, it has maximum cardinality. This completes the proof. \square

It remains to prove:

Lemma 10. (Part (iii) of Theorem 2.) *For all $n \geq 5$ there is a one-to-one correspondence between \mathcal{P}_n^* and the Cartesian product $\binom{[n]}{\lfloor n/2 \rfloor} \times \mathcal{Q}_n^*$.*

Proof. Assume $n \geq 5$ and consider any $P \in \mathcal{P}_n^*$. In the proof of Lemma 9 we showed that there exists a unique balanced ordered partition (W, \overline{W}) of $[n]$ such that all critical pairs (j, k) of P have $j \in W$ and $k \in \overline{W}$. Let again $c := |\overline{W}| \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$, but now select the (unique) “canonical” permutation τ_W associated with this ordered partition, that *monotonically* maps $[c]$ to \overline{W} (i.e., such that $1 \leq i < j \leq c$ implies $\tau_W(i) \in \overline{W}$ and $\tau_W(i) < \tau_W(j) \in \overline{W}$) and monotonically maps $\overline{[c]} = \{c+1, \dots, n\}$ to W (i.e., such that $c+1 \leq k < l \leq n$ implies $\tau_W(k) \in W$ and $\tau_W(k) < \tau_W(l) \in W$). Then, as also noted in the proof of Lemma 9, every critical pair in $Q_P := \tau_W^{-1} \circ P$ is an inversion, and thus $Q_P \in \mathcal{Q}_n^*$.

Consider the mapping $\Phi : \mathcal{P}_n^* \mapsto \binom{[n]}{\lfloor n/2 \rfloor} \times \mathcal{Q}_n^*$ defined by $\Phi(P) = (\Phi(P)_1, \Phi(P)_2)$ with $\Phi(P)_1 = \overline{W}$ if Q_P is a transversal of $\mathcal{F}_{\lfloor n/2 \rfloor}$ (i.e., if $|\overline{W}| = \lfloor n/2 \rfloor$, where $\{W, \overline{W}\}$ is the balanced ordered partition associated with P , as defined in the preceding paragraph), and $\Phi(P)_1 = W$ otherwise (i.e., if n is odd and Q_P is a transversal of $\mathcal{F}_{\lceil n/2 \rceil}$, and thus $|W| = \lfloor n/2 \rfloor$); and with $\Phi(P)_2 = Q_P = \tau_W^{-1} \circ P$. To complete the proof, it suffices to show that every pair $(X, Q) \in \binom{[n]}{\lfloor n/2 \rfloor} \times \mathcal{Q}_n^*$ is the image $\Phi(P)$ of exactly one $P \in \mathcal{P}_n^*$.

Thus consider any $(X, Q) \in \binom{[n]}{\lfloor n/2 \rfloor} \times \mathcal{Q}_n^*$. Recall that the balanced ordered partition associated with any transversal Q of \mathcal{F}_c is $(\overline{[c]}, [c])$.

If Q is a transversal of $\mathcal{F}_{\lfloor n/2 \rfloor}$ then we use \overline{X} to play the role of W , that is, we let $P := \tau_{\overline{X}} \circ Q$, so $P \in \mathcal{P}_n^*$ and $\Phi(P)_2 = \tau_{\overline{X}}^{-1} \circ P = Q$. The balanced ordered partition associated with P is $(\tau_{\overline{X}} \circ \overline{[c]}, \tau_{\overline{X}} \circ [c]) = (\overline{X}, X)$, and therefore $\Phi(P)_1 = X$. This implies that $\Phi(P) = (X, Q)$, as desired. Furthermore, consider any $P' \in \mathcal{P}_n^*$ such that $\Phi(P') = (X, Q)$. Since $Q_{P'} = \Phi(P')_2 = Q$ is a transversal of $\mathcal{F}_{\lfloor n/2 \rfloor}$ and $\Phi(P')_1 = X$, the balanced ordered partition associated with P' is $(W', \overline{W}') = (\overline{X}, X)$. But then $P' = \tau_{W'} \circ Q_{P'} = \tau_{\overline{X}} \circ Q = P$. Therefore, for every $(X, Q) \in \binom{[n]}{\lfloor n/2 \rfloor} \times \mathcal{Q}_n^*$ such that Q is a transversal of $\mathcal{F}_{\lfloor n/2 \rfloor}$, there exists exactly one $P \in \mathcal{P}_n^*$ such that $\Phi(P) = (X, Q)$.

The proof for the remaining case, i.e., when n is odd and Q is a transversal of $\mathcal{F}_{\lceil n/2 \rceil}$, is similar, by simply exchanging the roles of X and \overline{X} . This shows that Φ is a one-to-one correspondence from \mathcal{P}_n^* to $\binom{[n]}{\lfloor n/2 \rfloor} \times \mathcal{Q}_n^*$. \square

The proof of Theorem 2 is complete

Remark 2. As seen at the beginning of Section 3, $\mathcal{P}_2^* = 1 = \mathcal{Q}_2^*$. For $n = 3$ it can be verified that \mathcal{P}_3^* consists of the two orbits $\{123, 312, 231\}$ and $\{132, 213, 321\}$ of the circular shift, thus $|\mathcal{P}_3^*| = 2$ (while $|\mathcal{Q}_3^*| = 3$).

For $n = 4$ we have two classes of maximum-cardinality minimal pair-complete subsets (mentioned in the introduction):

- (1) the $3! = 6$ orbits $P = \{\pi, \pi \circ \sigma, \pi \circ \sigma^2, \dots, \pi \circ \sigma^{n-1}\}$ of the circular shift σ , one for each permutation $\pi = \rho 4$ (permutation ρ followed by 4) defined by each $\rho \in S_3$; and
- (2) the $\binom{4}{2} = 6$ distinct sets $P = \tau \circ Q$ where Q is the (unique) maximum-cardinality minimal pair-complete subset of S_4 , namely, the sets $P_{i,j} = \{ijkl, ilkj, kjil, klij\}$ where $1 \leq i < j \leq 4$ and $\{k, l\} = [4] \setminus \{i, j\}$.

Thus $|\mathcal{P}_4^*| = 12$ (while $|\mathcal{Q}_4^*| = 1$).

For $n \geq 5$, part (iii) of Theorem 2 implies that $|\mathcal{P}_n^*| = \binom{n}{\lfloor n/2 \rfloor} |\mathcal{Q}_n^*|$. Thus, for example, $|\mathcal{P}_5^*| = 10 |\mathcal{Q}_5^*| = 128$, and so on, with the same asymptotic growth rate $|\mathcal{P}_n^*| = 2^{\theta(n^3 \log n)}$ as $|\mathcal{Q}_n^*|$.

REFERENCES

- [1] M. Bóna, *Combinatorics of permutations*, 2nd ed., CRC Press, Boca Raton, FL, 2012.
- [2] C. Malvenuto, P. Möseneder Frajria, L. Orsina, and P. Papi, “The maximum cardinality of minimal inversion complete sets in finite reflection groups”, *Journal of Algebra* **424** (2015) 330–356.
- [3] W. Mantel, “Vraagstuk XXVIII”, *Wiskundige Opgaven met de Oplossingen* **10** (1907) 60–61.
- [4] B.H. Margolius, “Permutations with inversions”, *Journal of Integer Sequences* **4** (2001), Article 01.2.4, 13 pp. (electronic)
- [5] G. Markowsky, “Permutation lattices revisited”, *Mathematical Social Sciences* **27** (1994) 59–72.
- [6] K. Murota, *Discrete convex analysis*, SIAM Monographs on Discrete Mathematics and Applications 10, Philadelphia, 2003.
- [7] M. Queyranne and F. Tardella, “Carathéodory, Helly and Radon Numbers for Sublattice Convexities in Euclidian, Integer and Boolean Spaces”, CORE Discussion Paper 2015/10, Université catholique de Louvain, 2015.
- [8] P. Turán, “Egy gráfelméleti szélsőértékfeladatrol”, *Mat. Fiz. Lapok* **48** (1941) 436–453.
- [9] M. L. J. van de Vel, *Theory of convex structures*, North-Holland Mathematical Library, vol. 50, North-Holland Publishing Co., Amsterdam, 1993.

Recent titles

CORE Discussion Papers

- 2014/38 Diane PIERRET. Systemic risk and the solvency-liquidity nexus of banks.
- 2014/39 Paul BELLEFLAMME and Julien JACQMIN. An economic appraisal of MOOC platforms: business models and impacts on higher education.
- 2014/40 Marie-Louise LEROUX, Pierre PESTIEAU and Grégory PONTIERE. Longévité différentielle et redistribution: enjeux théoriques et empiriques.
- 2014/41 Chiara CANTA, Pierre PESTIEAU and Emmanuel THIBAUT. Long term care and capital accumulation: the impact of the State, the market and the family.
- 2014/42 Gilles GRANDJEAN, Marco MANTOVANI, Ana MAULEON and Vincent VANNETELBOSCH. Whom are you talking with ? An experiment on credibility and communication structure.
- 2014/43 Julio DAVILA. The rationality of expectations formation.
- 2014/44 Florian MAYNERIS, Sandra PONCET and Tao ZHANG. The cleaning effect of minimum wages. Minimum wages, firm dynamics and aggregate productivity in China.
- 2014/45 Thierry BRECHET, Natali HRITONENKOVA and Yuri YATSENKO. Domestic environmental policy and international cooperation for global commons.
- 2014/46 Mathieu PARENTI, Philip USHCHEV and Jacques-François THISSE. Toward a theory of monopolistic competition.
- 2014/47 Takatoshi TABUCHI, Jacques-François THISSE and Xiwei ZHU. Does technological progress affect the location of economic activity?
- 2014/48 Paul CASTANEDA DOWER, Victor GINSBURGH and Shlomo WEBER. Colonial legacy, linguistic disenfranchisement and the civil conflict in Sri Lanka.
- 2014/49 Victor GINSBURGH, Jacques MELITZ and Farid TOUBAL. Foreign language learnings: An econometric analysis.
- 2014/50 Koen DECANQ and Dirk NEUMANN. Does the choice of well-being measure matter empirically? An illustration with German data.
- 2014/51 François MANIQUET. Social ordering functions.
- 2014/52 Ivar EKELAND and Maurice QUEYRANNE. Optimal pits and optimal transportation.
- 2014/53 Luc BAUWENS, Manuela BRAIONE and Giuseppe STORTI. Forecasting comparison of long term component dynamic models for realized covariance matrices.
- 2014/54 François MANIQUET and Philippe MONGIN. Judgment aggregation theory can entail new social choice results.
- 2014/55 Pasquale AVELLA, Maurizio BOCCIA and Laurence A. WOLSEY. Single-period cutting planes for inventory routing problems.
- 2014/56 Jean-Pierre FLORENS and Sébastien VAN BELLEGEM. Instrumental variable estimation in functional linear models.
- 2014/57 Abdelrahman ALY and Mathieu VAN VYVE. Securely solving classical networks flow problems.
- 2014/58 Henry TULKENS. Internal vs. core coalitional stability in the environmental externality game: A reconciliation.
- 2014/59 Manuela BRAIONE and Nicolas K. SCHOLTES. Construction of Value-at-Risk forecasts under different distributional assumptions within a BEKK framework.
- 2014/60 Jörg BREITUNG and Christian M. HAFNER. A simple model for now-casting volatility series.
- 2014/61 Timo TERASVIRTA and Yukai YANG. Linearity and misspecification tests for vector smooth transition regression models.
- 2014/62 Timo TERASVIRTA and Yukai YANG. Specification, estimation and evaluation of vector smooth transition autoregressive models with applications.
- 2014/63 Axel GAUTIER and Nicolas PETIT. Optimal enforcement of competition policy: the commitments procedure under uncertainty.
- 2014/64 Sébastien BROOS and Axel GAUTIER. Competing one-way essential complements: the forgotten side of net neutrality.

Recent titles

CORE Discussion Papers - continued

- 2014/65 Jean HINDRIKS and Yukihiko NISHIMURA. On the timing of tax and investment in fiscal competition models.
- 2014/66 Jean HINDRIKS and Guillaume LAMY. Back to school, back to segregation?
- 2014/67 François MANIQUET et Dirk NEUMANN. Echelles d'équivalence du temps de travail: évaluation de l'impôt sur le revenu en Belgique à la lumière de l'éthique de la responsabilité.
- 2015/1 Yurii NESTEROV and Vladimir SHIKHMAN. Algorithm of Price Adjustment for Market Equilibrium.
- 2015/2 Claude d'ASPREMONT and Rodolphe DOS SANTOS FERREIRA. Oligopolistic vs. monopolistic competition: Do intersectoral effects matter?
- 2015/3 Yuuri NESTEROV. Complexity bounds for primal-dual methods minimizing the model of objective function.
- 2015/4 Hassène AISSI, A. Ridha MAHJOUB, S. Thomas MCCORMICK and Maurice QUEYRANNE. Strongly polynomial bounds for multiobjective and parametric global minimum cuts in graphs and hypergraphs.
- 2015/5 Marc FLEURBAEY and François MANIQUET. Optimal taxation theory and principles of fairness.
- 2015/6 Arnaud VANDAELE, Nicolas GILLIS, François GLINEUR and Daniel TUYTTENS. Heuristics for exact nonnegative matrix factorization.
- 2015/7 Luc BAUWENS, Jean-François CARPANTIER and Arnaud DUFAYS. Autoregressive moving average infinite hidden Markov-switching models.
- 2015/8 Koen DECANCO, Marc FLEURBAEY and François MANIQUET. Multidimensional poverty measurement with individual preferences
- 2015/9 Eric BALANDRAUD, Maurice QUEYRANNE, and Fabio TARDELLA. Largest minimally inversion-complete and pair-complete sets of permutations.

Books

- W. GAERTNER and E. SCHOKKAERT (2012), *Empirical Social Choice*. Cambridge University Press.
- L. BAUWENS, Ch. HAFNER and S. LAURENT (2012), *Handbook of Volatility Models and their Applications*. Wiley.
- J-C. PRAGER and J. THISSE (2012), *Economic Geography and the Unequal Development of Regions*. Routledge.
- M. FLEURBAEY and F. MANIQUET (2012), *Equality of Opportunity: The Economics of Responsibility*. World Scientific.
- J. HINDRIKS (2012), *Gestion publique*. De Boeck.
- M. FUJITA and J.F. THISSE (2013), *Economics of Agglomeration: Cities, Industrial Location, and Globalization*. (2nd edition). Cambridge University Press.
- J. HINDRIKS and G.D. MYLES (2013). *Intermediate Public Economics*. (2nd edition). MIT Press.
- J. HINDRIKS, G.D. MYLES and N. HASHIMZADE (2013). *Solutions Manual to Accompany Intermediate Public Economics*. (2nd edition). MIT Press.
- J. HINDRIKS (2015). *Quel avenir pour nos pensions ? Les grands défis de la réforme des pensions*. De Boeck.

CORE Lecture Series

- R. AMIR (2002), Supermodularity and Complementarity in Economics.
- R. WEISMANTEL (2006), Lectures on Mixed Nonlinear Programming.
- A. SHAPIRO (2010), Stochastic Programming: Modeling and Theory.